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LIE ALGEBROIDS FRAMEWORK FOR DISTRIBUTIONAL SYSTEMS

BY

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Abstract. In this paper we study the distributional systems (driftless control affine systems) with positive homogeneous cost, using the framework of Lie algebroids. The method consists of applying the Pontryagin Maximum Principle at the level of Lie algebroid built for both holonomic and nonholonomic distributions. This simplified the approaches and reveals certain important connections.

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1. Preliminaries. The geometric methods have been applied by many authors (see e.g. [3, 2, 7]) in control theory. One of the most important issues in the geometric approach is the analysis of the solution to the optimal control problem as provided by Pontryagin's Maximum Principle; that is, the curve $c(t) = (x(t), u(t))$ is an optimal trajectory if there exists a lifting of $x(t)$ to the dual space $(x(t), p(t))$ satisfying the Hamilton equations together with a maximization condition for the Hamiltonian with respect to the control variables $u(t)$.

A Lie algebroid over the manifold M is a vector bundle E over the manifold whose properties are very similar to those of a tangent bundle TM [11]. Weinstein [15] developed a generalized theory of Lagrangian Mechanics on Lie algebroids and obtained the equations of motion using the Poisson structure on the dual and the Legendre transformation. In the paper [10] Martinez presents the Pontryagin Maximum Principle on Lie algebroids using the prolongation (in sense of Higgins and Mackenzie [5]) of the Lie algebroid over the vector bundle projection of a dual bundle. In [4] Cortes and MARTINEZ study mechanical control systems on Lie algebroids whereas MESTDAG and LANGEROCK [12] investigate the Lie algebroids framework for nonholonomic mechanical systems.

The purpose of the present paper is to study the distributional systems with positive homogeneous cost using the Pontryagin Maximum Principle on the prolongation of Lie algebroids over the vector bundle projection of a dual bundle. We prove that the framework of a Lie algebroid is better than cotangent bundle in order to solve some problems of control systems. The paper is organized as follows. In the first section the known results on optimal control systems and Lie algebroids are recalled. In section 2 the distributional systems are presented and the relation between the Hamiltonians on E^* and T^*M is given. We investigate the cases of holonomic and nonholonomic distributions. In the holonomic case, we will consider the Lie algebroid being just the distribution whereas in the nonholonomic case (i.e., strong bracket generating distribution) the Lie algebroid is the tangent bundle with the basis given by vectors of distribution completed by the Lie brackets. In the both cases illustrative examples are presented.

1.1. Optimal control systems. Let M be a smooth n-dimensional manifold. We consider the control system

$$\frac{dx^i}{dt} = f^i(x,u),$$

where $x \in M$ and the control u takes values in an open subset Ω of R^m. Let x_0 and x_1 be two points of M. An optimal control problem consists of finding the trajectories of our control system which connects x_0 and x_1 and minimizing the cost

$$\min \int_0^T L(x(t),u(t))dt, \quad x(0) = x_0,\ x(T) = x_1,$$

where L is the *Lagrangian* or *running cost*.

Necessary conditions for a trajectory to be an extremal are given by Pontryagin Maximum Principle. The Hamiltonian reads as

$$H(x,p,u) = \langle p, f(x,u)\rangle - L(x,u), \quad p \in T^*M$$

while the maximization condition with respect to the control variables u, namely $H(x(t),p(t),u(t)) = \max_v H(x(t),p(t),v)$, leads to $\frac{\partial H}{\partial u} = 0$. The extremal trajectories satisfy the Hamilton's equations

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}. \tag{1}$$

From a geometric viewpoint the pair (x^i, u^a), where $i = \overline{1, n}$ and $a = \overline{1, m}$, can be understood as a local coordinate pair of a manifold E that is fibred over M by the projection $\pi : E \to M$. The functions $f^i(x, u)$ are the components of a vector field $X = f^i(x, u)\frac{\partial}{\partial x^i}$ along π, that is, of a fibered mapping $X : E \to TM$ from the bundle (E, π, M) to the tangent bundle (TM, τ, M) such that $\tau \circ X = \pi$. The admissible curves of the control system are the curves $\gamma : I \subset R \to E$ such that

$$X(\gamma(t)) = \frac{d}{dt}(\pi(\gamma(t)).$$

The optimal control problem consists of obtaining the admissible curves that minimize the cost

$$\min \int_I L(\gamma(t))dt, \quad L \in C^\infty(E),$$

and satisfy certain boundary conditions not to be considered here.

The Hamiltonian H is a real-valued function defined on the fibred product $T^*M \times_M E$ that is given by $H(\mu, v) = \langle \mu, X(v)\rangle - L(v)$ for any $(\mu, v) \in T^*M \times_M E$.

The critical equations follow from asking a vector field X_H defined along a map $pr_1 : T^*M \times_M E \to T^*M$ to satisfy the symplectic equations

$$i_{X_H}\omega = dH,$$

where $\omega = dx^i \wedge dp_i$ is the canonical symplectic form on T^*M . Since

$$dH = \frac{\partial H}{\partial x^i}dx^i + \frac{\partial H}{\partial p_i}dp_i + \frac{\partial H}{\partial u^a}du^a,$$

we obtain that the solution of the above equations is the vector field (see [7])

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial}{\partial p_i}$$

defined on the subset $\frac{\partial H}{\partial u^a} = 0$ of $T^*M \times_M E$, and therefore, the critical trajectories are the integral curves of the above vector field, namely

$$\frac{\partial H}{\partial u^a} = 0, \quad \dot{x^i} = \frac{\partial H}{\partial p_i}, \quad \dot{p_i} = -\frac{\partial H}{\partial x^i}. \tag{2}$$

1.2. Lie algebroids. Let M be a differentiable, n-dimensional manifold and (TM,τ,M) its tangent bundle. Let (E,π,M) be a vector bundle with the dimension of fibre type m. A Lie algebroid over a manifold M (see [11]) is a vector bundle (E,π,M) equipped with a Lie algebra structure $[\cdot,\cdot]$ on its space of sections, denoted $\Gamma(E)$, and with a map $\sigma:E\to TM$ (called the anchor) which induces a Lie algebra homomorphism (also denoted σ) from sections of E to the vector fields on M satisfying the Leibniz rule

$$[s_1,fs_2]=f[s_1,s_2]+(\sigma(s_1)f)s_2 \tag{3}$$

for $f\in C^\infty(M)$ and s_1, $s_2\in\Gamma(E)$. Therefore, we have

$$[\sigma(s_1),\sigma(s_2)]=\sigma[s_1,s_2],\quad [s_1,[s_2,s_3]]+[s_2,[s_3,s_1]]+[s_3,[s_1,s_2]]=0$$

and the triple $(E,[\cdot,\cdot],\sigma)$ is called a Lie algebroid over M . The Lie algebroid can be regarded as a substitute of the tangent bundle of M. Therefore, we regard any element g of E as a generalized velocity and so the actual velocity v is obtained by applying the anchor to g, i.e., $v=\sigma(g)$.

The image of the anchor map $\sigma(E)\subseteq TM$ defines an integrable smooth distribution on M. Therefore, the manifold M is foliated by the integral leaves of $\sigma(E)$, which are called the leaves of the Lie algebroid. A curve $u:[t_0,t_1]\to E$ is called admissible if $\sigma(u(t))=\dot{c}(t)$, where $c(t)=\pi(u(t))$ is the base curve. It follows that $u(t)$ is admissible if and only if the base curve $c(t)$ lies on a leaf of the Lie algebroid whereas two points can be joint by an admissible curve if and only if they are situated on the same leaf.

If ω is a k-form, $\omega\in\bigwedge^k(E)=\Gamma((E^*)^k\to M$ then the exterior derivative $d\omega\in\bigwedge^{k+1}(E)$ is given by the formula

$$\begin{aligned}d\omega(s_1,...,s_{k+1})&=\sum_{i=1}^{k+1}(-1)^{i+1}\sigma(s_i)\omega(s_1,...,\widehat{s_i},...,s_{k+1})\\&+\sum_{1\le i<j\le k+1}(-1)^{i+j}\omega([s_i,s_j],s_1,...,\widehat{s_i},...,\widehat{s_j},...s_{k+1}).\end{aligned}$$

Given $\xi\in\Gamma(E)$, we can define the *Lie derivative* with respect to ξ by $\mathcal{L}_\xi=i_\xi\circ d+d\circ i_\xi$. If we consider the local coordinates (x^i) on an open $U\subset M$ and the local basis $\{s_\alpha\}$ of sections of the bundle $\pi^{-1}(U)\to U$, then (x^i,y^α) are the local coordinates on E. These coordinates determine the *structure functions* $\sigma^i_\alpha(x)$, $L^\gamma_{\alpha\beta}(x)$ on M, namely

$$\sigma(s_\alpha)=\sigma^i_\alpha\frac{\partial}{\partial x^i},\quad [s_\alpha,s_\beta]=L^\gamma_{\alpha\beta}s_\gamma,\quad i=\overline{1,n},\quad \alpha,\beta,\gamma=\overline{1,m}, \tag{4}$$

which satisfy the relations

$$(5)\qquad \sigma^j_\alpha \frac{\partial \sigma^i_\beta}{\partial x^j} - \sigma^j_\beta \frac{\partial \sigma^i_\alpha}{\partial x^j} = \sigma^i_\gamma L^\gamma_{\alpha\beta}, \qquad \sum_{(\alpha,\beta,\gamma)} \left(\sigma^i_\alpha \frac{\partial L^\delta_{\beta\gamma}}{\partial x^i} + L^\delta_{\alpha\eta} L^\eta_{\beta\gamma} \right) = 0.$$

The relations (5) are called the *structure equations* of the Lie algebroid.

The basic example of a Lie algebroid is the tangent bundle TM itself with the identity mapping as anchor. The structure functions with respect to the natural basis are $L^\gamma_{\alpha\beta} = 0$ and $\sigma^i_j = \delta^i_j$. However, if we use another basis for the vector fields then the structure functions are nonzero. Also, any integrable subbundle of TM is a Lie algebroid with the inclusion as anchor and the induced bracket.

1.3. Prolongation of Lie algebroid. Let $\tau^* : E^* \to M$ be the dual of the Lie algebroid. Using [5, 8, 9], we get that the dual of a Lie algebroid induces a vector bundle $\tau_1 : \mathcal{T}E^* \to E^*$ with a structure of Lie algebroid. The manifold $\mathcal{T}E^*$ is

$$\mathcal{T}E^* = \{(u, w) \in E \times_{TM} TE^* \mid \ w \in T_u E^*, \ \sigma(u) = T_\tau(w)\}.$$

Here, $T_\tau : TE^* \to TM$ is the tangent application and the projection $\tau_1 : \mathcal{T}E^* \to E^*$ is given by $\tau_1(u, w) = \pi_{E^*}(w)$, where $\pi_{E^*} : TE^* \to E^*$ is the tangent projection. We have also the canonical projection $\tau_2 : \mathcal{T}E^* \to E$ given by $\tau_2(u, w) = u$. The projection onto the second factor $\sigma^1 : \mathcal{T}E^* \to TE^*$, $\sigma^1(u, w) = w$ is the anchor of a new Lie algebroid over the manifold E^*. An element of $\mathcal{T}E^*$ is said to be vertical if it belongs to the kernel of the projection τ_2. We will denote $Ker\tau_2 = V\mathcal{T}E^*$ and $(V\mathcal{T}E^*, \tau_{1|_{V\mathcal{T}E^*}}, E^*)$ is the vertical subbundle of $(\mathcal{T}E^*, \tau_1, E^*)$. A basis of sections of $\mathcal{T}E^*$ is given by

$$\mathcal{X}_\alpha(u^*) = \left(s_\alpha(\tau(u^*)), \sigma^i_\alpha \frac{\partial}{\partial x^i}\bigg|_{u^*} \right), \quad \mathcal{P}^\alpha(u^*) = \left(0, \frac{\partial}{\partial \mu_\alpha}\bigg|_{u^*} \right)$$

for $u^* \in {\tau^*}^{-1}(U)$, where $(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial \mu_\alpha})$ is the local basis on TE^* , and the associated coordinates are denoted $(x^i, \mu_\alpha, z^\alpha, \omega_\alpha)$. The dual basis is $(\mathcal{X}^\alpha, \mathcal{P}_\alpha)$ and the structure functions of $\mathcal{T}E^*$ are given by the following formulas

$$(6)\qquad \sigma^1(\mathcal{X}_\alpha) = \sigma^i_\alpha \frac{\partial}{\partial x^i}, \ \sigma^1(\mathcal{P}^\alpha) = \frac{\partial}{\partial \mu_\alpha},$$

(7) $$[\mathcal{X}_\alpha, \mathcal{X}_\beta] = L^\gamma_{\alpha\beta}\mathcal{X}_\gamma, \quad [\mathcal{X}_\alpha, \mathcal{P}^\alpha] = 0, \quad [\mathcal{P}^\alpha, \mathcal{P}^\beta] = 0.$$

In the Lie algebroid $\mathcal{T}E^*$ there exists the canonical symplectic structure given by $\omega = -d\theta$, where $\theta = \mu_\alpha \mathcal{X}^\alpha$ is the Liouville section. In local coordinates we get

$$\omega = \mathcal{X}^\alpha \wedge \mathcal{P}_\alpha + \frac{1}{2}\mu_\alpha L^\alpha_{\beta\gamma}\mathcal{X}^\beta \wedge \mathcal{X}^\gamma.$$

By a control system on the Lie algebroid (see [10]) $\pi : E \to M$ with the control space $\tau : A \to M$ we mean a section ρ of E along τ. A trajectory of the system ρ is an integral curve of the vector field $\sigma(\rho)$. Given the cost function $\mathcal{L} \in C^\infty(A)$, we have to minimize the integral of $\mathcal{L}$ over the set of those system trajectories which satisfy certain boundary conditions. The Hamiltonian function $\mathcal{H} \in C^\infty(E^* \times_M A)$ is defined by $\mathcal{H}(\mu, u) = \langle \mu, \rho(u)\rangle - \mathcal{L}(u)$ whereas the associated Hamiltonian control system ρ_H is given by the symplectic equation $i_{\rho_H}\omega = d\mathcal{H}$. In local coordinates, the solution of the previous equation reads as

$$\rho_H = \frac{\partial \mathcal{H}}{\partial \mu_\alpha}\mathcal{X}_\alpha - \left(\sigma^i_\alpha \frac{\partial \mathcal{H}}{\partial x^i} + \mu_\gamma L^\gamma_{\alpha\beta}\frac{\partial \mathcal{H}}{\partial \mu_\beta}\right)\mathcal{P}^\alpha$$

on the subset where $\frac{\partial \mathcal{H}}{\partial u^A} = 0$. Therefore, the critical trajectories are given by

(8) $$\frac{\partial \mathcal{H}}{\partial u^A} = 0, \quad \frac{dx^i}{dt} = \sigma^i_\alpha \frac{\partial \mathcal{H}}{\partial \mu_\alpha}, \quad \frac{d\mu_\alpha}{dt} = -\sigma^i_\alpha \frac{\partial \mathcal{H}}{\partial x^i} - \mu_\gamma L^\gamma_{\alpha\beta}\frac{\partial \mathcal{H}}{\partial \mu_\beta}.$$

2. Distributional systems. Let M be a smooth n-dimensional manifold. We consider the distributional system (driftless control-affine system)

(9) $$\dot{x} = \sum_{i=1}^{m} u_i X_i(x),$$

where $x \in M$, $X_1, X_2, ..., X_m$ are smooth vector fields on M and the control $u = (u_1, u_2, ..., u_m)$ takes values in an open subset Ω of R^m. The vector fields $X_i, i = \overline{1, m}$, generate a distribution $D \subset TM$ such that the rank of D is constant. Let x_0 and x_1 be two points of M. An optimal control problem consists of finding those trajectories of the distributional system which connect x_0 and x_1 while minimizing the cost

$$\min_{u(\cdot)} \int_I \mathcal{F}(u(t)dt, \tag{10}$$

where $\mathcal{F}$ is a Minkowski norm (positive homogeneous) on D.

Remark 1. We can associate to any positive homogeneous cost $\mathcal{F}$ on the Lie algebroid E a cost F on $Im\sigma \subset TM$ defined by

$$F(v) = \{\mathcal{F}(u) \,|u \in E_x, \quad \sigma(u) = v\}, \tag{11}$$

where $v \in (Im\sigma)_x \subset T_xM$, $x \in M$.

A piecewise smooth curve $c : I \subset R \to M$ is called horizontal if the tangent vectors are in D, i.e. $\dot{c}(t) \in D_{c(t)} \subset TM$ for almost every $t \in I$. Let $u : I \to E$ be an admissible curve projected by π onto the horizontal curve $c : I \to M$. The length of the horizontal curve c is defined by

$$length(c) = \int_I \mathcal{F}(u(t))dt = \int_I F(\dot{c}(t))dt,$$

and the distance is given by $d(a, b) = \inf length(c)$ where the infimum is taken over all the horizontal curves connecting a and b. The distance is infinite if there is no admissible curve that connects these two points.

Remark 2. The energy of a horizontal curve is

$$E(c) = \frac{1}{2}\int_I F^2(\dot{c}(t))dt$$

and it can easily be proved that if a curve is parametrized to a constant speed, then it minimizes the length integral if and only if it minimizes the energy integral.

For the 2-homogeneous Lagrangians $L = \frac{1}{2}F^2$ and $\mathcal{L} = \frac{1}{2}\mathcal{F}^2$ we have $\mathcal{L} = L \circ \sigma$. Further, L on TM is a Lagrangian with constraints. According to the Pontryagin Maximum Principle, the Hamiltonian is recast as

$$H(x, p, u) = \langle p, \dot{x} \rangle - L(x, u).$$

If the equations $\frac{\partial H(x,p,u)}{\partial u} = 0$ permit us to find in a unique way u as a smooth function of (x, p) then we can write the Hamiltonian system without any dependence on the control. This nice situation happens always for distributional systems with quadratic cost

$$\min \int_I \sum_{i=1}^{m} u_i^2 dt.$$

If the cost is not quadratic, then we cannot quarantee that the Hamiltonian can be calculated without dependence on the control. However, there exist several situations when the Hamiltonian can still be found.

Proposition 1. *The relation between the Hamiltonian H on cotangent bundle T^*M and the Hamiltonian $\mathcal{H}$ on dual bundle E^* is given by*

$$H(p) = \mathcal{H}(\mu), \quad \mu = \sigma^\star(p), \quad p \in T_x^*M, \quad \mu \in E_x^*. \tag{12}$$

Proof. The Fenchel-Legendre dual of Lagrangian L is the Hamiltonian H given by

$$\begin{aligned} H(p) &= \sup_v \{\langle p, v\rangle - L(v)\} = \sup_v \{\langle p, v\rangle - \mathcal{L}(u); \sigma(u) = v\} \\ &= \sup_u \{\langle p, \sigma(u)\rangle - \mathcal{L}(u)\} = \sup_u \{\langle \sigma^\star(p), u\rangle - \mathcal{L}(u)\} = \mathcal{H}(\sigma^\star(p)), \end{aligned}$$

and we get $H(p) = \mathcal{H}(\mu)$, $\mu = \sigma^\star(p)$, $p \in T_x^*M$, $\mu \in E_x^*$, or locally $\mu_\alpha = \sigma_\alpha^i p_i$, where the Hamiltonian $H(p)$ is degenerate on $Ker\sigma^\star$. □

2.1. Holonomic distribution. We assume for the beginning that the distribution $D = \langle X_1, X_2, ..., X_m\rangle$ is holonomic, which means that $[X_i, X_j] \in D$ for every $i, j = \overline{1, m}$, $i \neq j$. In order to apply the theory of Lie algebroids we consider $E = D$ with the inclusion as anchor $\sigma : E \to TM$. From the Frobenius theorem, the distribution D is integrable, it determines a foliation on M and two points can be joined if and only if they are situated on the same leaf.

We consider the following distributional system with positive homogeneous cost:

$$\dot{x} = u_1 X_1 + u_2 X_2, \quad x = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \in R^3, \ X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ X_2 = \begin{pmatrix} x^1 \\ x^2 \\ 1 \end{pmatrix}$$
$$\min_{u(\cdot)} \int_0^T \mathcal{F}(u(t))dt, \ \mathcal{F}(u) = \sqrt{(u^1)^2 + (u^2)^2} + \varepsilon u^1, \quad 0 \leq \varepsilon < 1$$

We are looking for the trajectories starting from the point $(1,1,0)^t$ and parameterized by arclength. The associated distribution $D = \langle X_1, X_2\rangle$ is holonomic, because $X_1 = \frac{\partial}{\partial x^1}$, $X_2 = x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3}$, and therefore $[X_1, X_2] = X_1$. In the case of the Lie algebroid, we consider $E = \langle X_1, X_2\rangle$ and the anchor $\sigma : E \to T\mathbb{R}^3$ has the components

$$\sigma^i_\alpha = \begin{pmatrix} 1 & x^1 \\ 0 & x^2 \\ 0 & 1 \end{pmatrix} \tag{13}$$

and we get the Lagrangian $\mathcal{L} = \frac{1}{2}\left(\sqrt{(u_1)^2 + (u_2)^2} + \varepsilon u_1\right)^2$. Using ([13, p. 191]) we can find the Hamiltonian on E^* given by

$$\mathcal{H}(\mu) = \frac{1}{2}\left(\sqrt{\frac{(\mu_1)^2}{(1-\varepsilon^2)^2} + \frac{(\mu_2)^2}{1-\varepsilon^2}} - \frac{\varepsilon\mu_1}{1-\varepsilon^2}\right)^2. \tag{14}$$

Remark 3. Using relation (12) we can calculate the Hamiltonian H on T^*M given by $H(x,p) = \mathcal{H}(\mu)$, $\mu = \sigma^\star(p)$, where

$$\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ x^1 & x^2 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}.$$

We get that

$$H(x,p) = \frac{1}{2}\left(\sqrt{\frac{(p_1)^2}{(1-\varepsilon^2)^2} + \frac{(p_1x^1 + p_2x^2 + p_3)^2}{1-\varepsilon^2}} - \frac{\varepsilon p_1}{1-\varepsilon^2}\right)^2. \tag{15}$$

Unfortunately, with $H(x,p)$ from (15) the Hamilton equations on T^*M lead to a complicated system of implicit differential equations.

We will use the geometric model of a Lie algebroid. From the relation $[X_\alpha, X_\beta] = L^\gamma_{\alpha\beta}X_\gamma$ we obtain the non-zero components $L^1_{12} = 1$, $L^1_{21} = -1$ while from (8) we deduce that

$$\begin{aligned} &\dot{x}^1 = \frac{\partial\mathcal{H}}{\partial\mu_1} + x^1\frac{\partial\mathcal{H}}{\partial\mu_2},\ \dot{x}^2 = x^2\frac{\partial\mathcal{H}}{\partial\mu_2},\ \dot{x}^3 = \frac{\partial\mathcal{H}}{\partial\mu_2}, \\ &\dot{\mu}_1 = -\mu_1\frac{\partial\mathcal{H}}{\partial\mu_2},\ \dot{\mu}_2 = \mu_1\frac{\partial\mathcal{H}}{\partial\mu_1}, \end{aligned} \tag{16}$$

where

$$\frac{\partial \mathcal{H}}{\partial \mu_1} = \frac{\left(1+\varepsilon^2\right)\mu_1}{(1-\varepsilon^2)^2} - \frac{\varepsilon\sqrt{\frac{(\mu_1)^2}{(1-\varepsilon^2)^2}+\frac{(\mu_2)^2}{1-\varepsilon^2}}}{1-\varepsilon^2} - \frac{\varepsilon\mu_1^2}{(1-\varepsilon^2)^3\sqrt{\frac{(\mu_1)^2}{(1-\varepsilon^2)^2}+\frac{(\mu_2)^2}{1-\varepsilon^2}}}$$

$$\frac{\partial \mathcal{H}}{\partial \mu_2} = \frac{\mu_2}{1-\varepsilon^2} - \frac{\varepsilon\mu_1\mu_2}{(1-\varepsilon^2)^2\sqrt{\frac{(\mu_1)^2}{(1-\varepsilon^2)^2}+\frac{(\mu_2)^2}{1-\varepsilon^2}}}. \tag{17}$$

The form of the last relations leads to the following change of variables

$$\mu_1(t) = (1-\varepsilon^2)r(t)\sec h\theta(t), \quad \mu_2(t) = \sqrt{1-\varepsilon^2}r(t)\tanh\theta(t).$$

In these circumstances we have $\sqrt{\frac{(\mu_1)^2}{(1-\varepsilon^2)^2}+\frac{(\mu_2)^2}{1-\varepsilon^2}} = |r|$ whereas $\dot{\mu}_1 = -\mu_1\frac{\partial\mathcal{H}}{\partial\mu_2}$ yields

$$\sqrt{1-\varepsilon^2}\left(\frac{\dot{r}}{r} - \dot{\theta}\tanh\theta\right) = r(-\tanh\theta + \varepsilon\sec h\theta\tanh\theta). \tag{18}$$

From $\dot{\mu}_2 = \mu_1\frac{\partial\mathcal{H}}{\partial\mu_1}$ we get

$$\sqrt{1-\varepsilon^2}\left(\frac{\dot{r}}{r}\tanh\theta + \dot{\theta}\sec h^2\theta\right) = r((1+\varepsilon)^2\sec h^2\theta - \varepsilon\sec h\theta - \varepsilon\sec h^3\theta). \tag{19}$$

Now, reducing $\dot{\theta}$ and $\frac{\dot{r}}{r}$ from the relations (18) and (19), we obtain

$$\sqrt{1-\varepsilon^2}\dot{r} = r^2\varepsilon\sec h\theta\tanh\theta(\varepsilon\sec h\theta - 1)$$

and

$$\sqrt{1-\varepsilon^2}\dot{\theta} = r(\varepsilon\sec h\theta - 1)^2. \tag{20}$$

The last two relations lead to

$$\frac{\dot{r}}{\dot{\theta}} = \frac{r\varepsilon\sec h\theta\tanh\theta}{\varepsilon\sec h\theta - 1}$$

and respectively to

$$\frac{1}{r}dr = \frac{\varepsilon\sec h\theta\tanh\theta}{\varepsilon\sec h\theta - 1}d\theta,$$

with the solution $\ln|r| = -\ln(\varepsilon \sec h\theta - 1) - \ln c$. Therefore

$$|r| = \frac{1}{c\,(\varepsilon \sec h\theta - 1)}.$$

Since the geodesics are parameterized by arclength, the conclusion corresponds exactly to the $1/2$ level of the Hamiltonian and so we have

$$\mathcal{H} = \frac{r^2}{2}(1 - \varepsilon \sec h\theta)^2 = \frac{1}{2c^2}.$$

Now, $c = \pm 1$ and

$$r = \pm\frac{1}{\varepsilon \sec h\theta - 1}.$$

From the relation (20) we have

$$\frac{d\theta}{dt} = \frac{\sqrt{1-\varepsilon^2}}{1 - \varepsilon \sec h\theta}$$

and respectively

$$t = \sqrt{1-\varepsilon^2}\int \frac{1}{1 - \varepsilon \sec h\theta}d\theta.$$

The relation $\dot{\mu}_1 = -\mu_1 \dot{x}^3$ implies that

$$x^3(\theta) = \ln\frac{c_1(1 - \varepsilon \sec h\theta)}{(1-\varepsilon^2)\sec h\theta}, \quad c_1 \in R.$$

Since we are looking for the trajectories starting from the point $(1,1,0)^t$, we have

$$\ln\frac{c_1}{1+\varepsilon} = 0 \Rightarrow c_1 = 1 + \varepsilon$$

and so

$$x^3(\theta) = \ln\frac{1 - \varepsilon \sec h\theta}{(1-\varepsilon)\sec h\theta} = \ln\frac{\cosh\theta - \varepsilon}{1-\varepsilon}.$$

The relation

$$\frac{\dot{x}^2}{x^2} = -\frac{\dot{\mu}_1}{\mu_1}$$

leads to

$$x^2(\theta) = \frac{c_2(1 - \varepsilon \sec h\theta)}{(1 - \varepsilon^2) \sec h\theta}$$

whereas from $x^2(0) = 1$ we get $c_2 = 1 + \varepsilon$. These lead to

$$x^2(\theta) = \frac{\cosh\theta - \varepsilon}{1 - \varepsilon}.$$

We obtain also that

$$\dot{\mu}_2 = \mu_1(\dot{x}^1 - x^1 \frac{\partial \mathcal{H}}{\partial \mu_2}) = \mu_1 \dot{x}^1 + x^1 \dot{\mu}_1$$

and, consequently, $\mu_2 = \mu_1 x^1 + c_3$. Further,

$$x^1(\theta) = \frac{\sinh\theta}{\sqrt{1 - \varepsilon^2}} \pm \frac{c_3(1 - \varepsilon \sec h\theta)}{(1 - \varepsilon^2) \sec h\theta}.$$

From $x^1(0) = 1$ we obtain that $c_3 = 1 + \varepsilon$ and this yields

$$x^1(\theta) = \frac{\sinh\theta}{\sqrt{1 - \varepsilon^2}} + \frac{\cosh\theta - \varepsilon}{1 - \varepsilon}.$$

Remark 4. If $\varepsilon = 0$ we regain the case of distributional systems with quadratic cost with the solution $x^1(t) = \sinh t + \cosh t$, $x^2(t) = \cosh t$, $x^3(t) = \ln \cosh t$.

2.2. Nonholonomic distribution. We assume that the distribution $D = \langle X_1, X_2, ..., X_m \rangle$ is nonholonomic and is strong bracket generating (see [1], [6], [14]), i.e. sections of D and first iterated brackets span the entire tangent space TM. By a well-known theorem of Chow, the system is controllable, that is any two points are connected through a horizontal curve (M is assumed to be connected). We also suppose that the vectors $\mathcal{B} = \{X_1, X_2, ..., X_m, [X_i, X_j]\}$ determine a base in TM. The space $E = TM$ with the base $\mathcal{B}$ is a Lie algebroid over M with at least one structural function nonzero. The anchor $\sigma : E \to TM$ is just the identity and the matrix corresponding to σ is determined by the base vectors.

The control system can be written as

$$\dot{x} = \sum_{i=1}^{m} u_i X_i(x) + 0X_{m+1} + \dots 0X_n$$

with

$$\min_{u(\cdot)} \int_0^T \mathcal{L}(u(t))dt.$$

To solve this minimization problem we consider the Lagrangian

$$\widetilde{\mathcal{L}} = \mathcal{L} + \sum_{k=m+1}^{n} \lambda_k u^k$$

(λ_k are the Lagrange multipliers) and still work via the maximum principle but at the level of Lie algebroids. We set $\mu_i = \frac{\partial \widetilde{\mathcal{L}}}{\partial u^i}$ and

$$\mathcal{H} = \sum_{i=1}^{n} \mu_i u^i - \widetilde{\mathcal{L}}$$

thus obtaining $\mu_j = \frac{\partial \mathcal{L}}{\partial u^j}$, $\mu_k = \lambda_k$ with $j = \overline{1,m}$ and $k = \overline{m+1,n}$. Since $\mathcal{L}$ is 2-homogeneous with respect to u^i, $i = \overline{1,m}$, we get

$$\mathcal{H} = \sum_{i=1}^{m} \frac{\partial \mathcal{L}}{\partial u^i} u^i - \mathcal{L} = \mathcal{L}$$

with the constrains $\mu_k = \lambda_k$, $k = \overline{m+1,n}$.

We consider the following distributional system with positive homogeneous cost:

$$\dot{x} = u^1 X_1 + u^2 X_2, \quad x = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \in R^3, \ X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ X_2 = \begin{pmatrix} 0 \\ 1 \\ x^1 \end{pmatrix}$$

and

$$\min_{u(\cdot)} \int_0^T \mathcal{F}(u(t))dt, \quad \mathcal{F}(u) = \sqrt{(u^1)^2 + (u^2)^2} + \varepsilon u^1, \quad 0 \le \varepsilon < 1.$$

We are looking for the trajectories starting from the origin and parametrized by arclength. We have $X_1 = \frac{\partial}{\partial x^1}$, $X_2 = \frac{\partial}{\partial x^2} + x^1 \frac{\partial}{\partial x^3}$, $X_3 = [X_1, X_2] = \frac{\partial}{\partial x^3}$ and $\langle X_1, X_2, X_3 \rangle \equiv TR^3$, hence the distribution $D = < X_1, X_2 >$ of constant rank is strong bracket generating.

Remark 5. We can work, as in classical case, directly on the cotangent bundle by computing the Hamiltonian $H(x,p) = \mathcal{H}(x,\mu)$, $\mu = \sigma^*(p)$. Since

$$\mathcal{H} = \frac{1}{2}\left(\sqrt{\frac{(\mu_1)^2}{(1-\varepsilon^2)^2} + \frac{(\mu_2)^2}{1-\varepsilon^2}} - \frac{\varepsilon\mu_1}{1-\varepsilon^2}\right)^2, \quad \mu_3 = \lambda,$$

$$\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x^1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix},$$

we obtain

$$H = \frac{1}{2}\left(\sqrt{\frac{(p_1)^2}{(1-\varepsilon^2)^2} + \frac{(p_2 + p_3 x^1)^2}{1-\varepsilon^2}} - \frac{\varepsilon p_1}{1-\varepsilon^2}\right)^2.$$

Unfortunately, from the Hamilton equations on T^*M a very complicated system of implicit differential equations is obtained.

We will use a different approach. Let us take $M = R^3$ and $E = TM$ with the basis $\{X_1, X_2, X_3\}$. E is a Lie algebroid over M with at least one structural function nonzero. The anchor $\sigma : E \to TM$ is just the identity and the matrix of σ with respect to the basis of E and TM basis is

$$\sigma_\alpha^i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x^1 & 1 \end{pmatrix}.$$

From the structure equations of Lie algebroids and the relation $[X_\alpha, X_\beta] = L_{\alpha\beta}^\gamma X_\gamma$ we obtain the non-zero structural functions $L_{12}^3 = 1$, $L_{21}^3 = -1$. Now, using the Hamilton equations on Lie algebroids (8) we get the following systems of differential equations

$$\dot{x}^1 = \frac{\partial \mathcal{H}}{\partial \mu_1}, \quad \dot{x}^2 = \frac{\partial \mathcal{H}}{\partial \mu_2}, \quad \dot{x}^3 = x^1 \frac{\partial \mathcal{H}}{\partial \mu_2} \tag{21}$$

and

$$\begin{cases} \dot{\mu}_1 = -L_{12}^3 \mu_3 \dfrac{\partial \mathcal{H}}{\partial \mu_2} = -\lambda \dfrac{\partial \mathcal{H}}{\partial \mu_2} \\ \dot{\mu}_2 = -L_{21}^3 \mu_3 \dfrac{\partial \mathcal{H}}{\partial \mu_1} = \lambda \dfrac{\partial \mathcal{H}}{\partial \mu_1} \\ \dot{\mu}_3 = 0 \Rightarrow \lambda = ct, \end{cases} \tag{22}$$

where

$$\frac{\partial \mathcal{H}}{\partial \mu_1} = \frac{\left(1+\varepsilon^2\right)\mu_1}{(1-\varepsilon^2)^2} - \frac{\varepsilon\sqrt{\frac{(\mu_1)^2}{(1-\varepsilon^2)^2}+\frac{(\mu_2)^2}{1-\varepsilon^2}}}{1-\varepsilon^2} - \frac{\varepsilon\mu_1^2}{(1-\varepsilon^2)^3\sqrt{\frac{(\mu_1)^2}{(1-\varepsilon^2)^2}+\frac{(\mu_2)^2}{1-\varepsilon^2}}}$$

$$\frac{\partial \mathcal{H}}{\partial \mu_2} = \frac{\mu_2}{1-\varepsilon^2} - \frac{\varepsilon\mu_1\mu_2}{(1-\varepsilon^2)^2\sqrt{\frac{(\mu_1)^2}{(1-\varepsilon^2)^2}+\frac{(\mu_2)^2}{1-\varepsilon^2}}}.$$

We may use to the following transformations

$$\begin{aligned} \mu_1(t) &= (1-\varepsilon^2)r(t)(a\cos A\theta(t) - b\sin A\theta(t)) \\ \mu_2(t) &= \sqrt{1-\varepsilon^2}r(t)(a\sin A\theta(t) + b\cos A\theta(t)) \end{aligned} \tag{23}$$

with $a^2+b^2=1$. We also have $\sqrt{\frac{(\mu_1)^2}{(1-\varepsilon^2)^2}+\frac{(\mu_2)^2}{(1-\varepsilon^2)}} = |r|$. Further, (22) yields

$$\begin{aligned} & c_1\left(\tfrac{\dot r}{r}(a\sin A\theta + b\cos A\theta) + A\dot\theta(a\cos A\theta - b\sin A\theta)\right) \\ & = (1+\varepsilon^2)(a\cos A\theta - b\sin A\theta) - \varepsilon(1 + (a\cos A\theta - b\sin A\theta)^2) \end{aligned} \tag{24}$$

and

(25)

$$\begin{aligned} & c_1\left(\tfrac{\dot r}{r}(a\cos A\theta - b\sin A\theta) - A\dot\theta(a\sin A\theta + b\cos A\theta)\right) = \\ & = -(a\sin A\theta + b\cos A\theta) + \varepsilon(a\cos A\theta - b\sin A\theta)(a\sin A\theta + b\cos A\theta), \end{aligned}$$

where $c_1 = \frac{(1-\varepsilon^2)\sqrt{1-\varepsilon^2}}{\lambda}$. Reducing $\dot\theta$ and $\frac{\dot r}{r}$ from (24) and (25), we get

$$c_1\frac{\dot r}{r} = \varepsilon\,(a\sin A\theta + b\cos A\theta)(\varepsilon a\cos A\theta - \varepsilon b\sin A\theta - 1) \tag{26}$$

and

$$c_1 A\dot\theta = (1 - \varepsilon(a\cos A\theta - b\sin A\theta))^2\,. \tag{27}$$

These lead to

$$t = \frac{(1-\varepsilon^2)\sqrt{1-\varepsilon^2}A}{\lambda}\int \frac{d\theta}{\left(1-\varepsilon\left(a\cos A\theta - b\sin A\theta\right)\right)^2}.$$

From (26) and (27) we obtain

$$\frac{dr}{r} = \frac{A\varepsilon(a\sin A\theta + b\cos A\theta)}{\varepsilon(a\cos A\theta - b\sin A\theta) - 1}d\theta$$

and

$$r=\frac{1}{c\left(1-\varepsilon\left(a\cos A\theta-b\sin A\theta\right)\right)}.$$

Since the geodesics are parametrized by arclength this corresponds exactly to the 1/2 level of the Hamiltonian and we have

$$\mathcal{H}=\frac{r^2}{2}\left(1-\varepsilon\left(a\cos A\theta-b\sin A\theta\right)\right)^2=\frac{1}{2c^2}.$$

So, $c=\pm1$ and

$$r=\pm\frac{1}{1-\varepsilon\left(a\cos A\theta-b\sin A\theta\right)}.$$

From (23) we obtain

$$\begin{aligned}\mu_1(t)&=\pm\frac{(1-\varepsilon^2)(a\cos A\theta-b\sin A\theta)}{1-\varepsilon\left(a\cos A\theta-b\sin A\theta\right)}\\ \mu_2(t)&=\frac{\sqrt{1-\varepsilon^2}(a\sin A\theta+b\cos A\theta)}{1-\varepsilon\left(a\cos A\theta-b\sin A\theta\right)}.\end{aligned}$$

Since $\dot{\mu}_2=\lambda\dot{x}^1$, we also have $x^1(\theta)=\frac{\mu_2}{\lambda}-a_1$. As we are looking for geodesics with start from the origin, we have $a_1=\frac{\sqrt{1-\varepsilon^2}b}{\lambda(1-\varepsilon a)}$ and therefore

$$x^1(\theta)=\frac{\sqrt{1-\varepsilon^2}(a\sin A\theta+b\cos A\theta)}{\lambda\left(1-\varepsilon\left(a\cos A\theta-b\sin A\theta\right)\right)}-\frac{\sqrt{1-\varepsilon^2}b}{\lambda(1-\varepsilon a)}.$$

From $\dot{\mu}_1=-\lambda\dot{x}^2$ we get

$$x^2(\theta)=\frac{(1-\varepsilon^2)(b\sin A\theta-a\cos A\theta)}{\lambda\left(1-\varepsilon\left(a\cos A\theta-b\sin A\theta\right)\right)}+\frac{(1-\varepsilon^2)a}{\lambda(1-\varepsilon a)}.$$

Finally, because $\dot{x}^3=x^1\dot{x}^2$ a straightforward computation leads to

$$\begin{aligned}x^3(\theta)&=\frac{(1-\varepsilon^2)\sqrt{1-\varepsilon^2}A}{\lambda^2}\int\frac{(a\sin A\theta+b\cos A\theta)^2}{\left(1-\varepsilon\left(a\cos A\theta-b\sin A\theta\right)\right)^3}d\theta\\ &\quad-\frac{(1-\varepsilon^2)Ab}{\lambda^2}\int\frac{a\sin A\theta+b\cos A\theta}{\left(1-\varepsilon\left(a\cos A\theta-b\sin A\theta\right)\right)^2}d\theta.\end{aligned}$$

Remark 6. For $\varepsilon = 0$ we obtain the sub-Riemannian case (distributional systems with quadratic cost) with the solution

$$x^1(t) = \frac{a\sin\lambda t - b(1-\cos\lambda\theta)}{\lambda},$$
$$x^2(t) = \frac{b\sin\lambda t + a(1-\cos\lambda t)}{\lambda},$$
$$x^3(t) = \frac{t}{2\lambda} + \frac{b^2-a^2}{4\lambda^2}\sin 2\lambda t - \frac{ab}{\lambda^2}\cos^2\lambda t + \frac{ab}{\lambda^2}\cos\lambda t - \frac{b^2}{a^2}\sin\lambda t,$$
$$a^2+b^2=1.$$

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