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A VIABILITY RESULT FOR DIFFERENTIAL INCLUSIONS ON GRAPHS*

BY

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Abstract. Using a tangency condition expressed with a set of integrals, we establish several necessary and sufficient conditions for viability referring to differential inclusions on graphs.

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1. Introduction. Let X be a real Banach space, let $I \subseteq \mathrm{R}$ be a nonempty and bounded interval and let $K : I \rightsquigarrow X$ and $F : \mathcal{K} \rightsquigarrow X$ be two multi-functions with nonempty values, where $\mathcal{K} := \mathrm{graph}(K)$.

Our aim here is to prove some new necessary and sufficient conditions in order that $\mathcal{K}$ be viable with respect to F. This paper is an extension of the results established by Necula-Popescu-Vrabie [6].

Let us recall that $W^{1,1}([\tau, T]; X)$ denotes the space of all functions $u : [\tau, T] \to X$ which are a.e. differentiable on $[\tau, T]$ with $u' \in L^1(\tau, T; X)$ and

$$u(t) = u(\tau) + \int_\tau^t u'(s)\, ds$$

for each $t \in [\tau, T]$. Next, let us consider the Cauchy Problem

$$\begin{cases} u'(t) \in F(t, u(t)) \\ u(\tau) = \xi. \end{cases} \tag{1.1}$$

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Definition 1.1. By a *solution* of (1.1) on $[\tau, T] \subseteq I$, we mean a function $u \in W^{1,1}([\tau, T]; X)$ satisfying $(t, u(t)) \in \mathcal{K}$, $u(\tau) = \xi$ and $u'(t) \in F(t, u(t))$ a.e. for $t \in [\tau, T]$.

Definition 1.2. We say that the graph, $\mathcal{K}$, of $K : I \rightsquigarrow X$, is *viable* with respect to $F : \mathcal{K} \rightsquigarrow X$ if for each $(\tau, \xi) \in \mathcal{K}$, there exists $T > \tau$, such that $[\tau, T] \subseteq I$ and (1.1) has at least one solution $u : [\tau, T] \to X$. If $T \in (\tau, \sup I)$ can be taken arbitrary, we say that $\mathcal{K}$ is *globally viable* with respect to F.

The first two sections of the paper are concerned with some prerequisites and basic concepts and results needed in the sequel. In Section 3 we prove the main necessary condition of viability, in Section 4 we give a relationship between two tangency conditions, Section 5 contains the statement of the two sufficient conditions for viability and the statement and proof of a technical approximation lemma, while in Sections 6 and 7, we give the proofs of Theorems 5.1 and 5.2.

2. Preliminaries. If (Y, d) is a metric space, $y \in Y$ and $r > 0$, $D(y, r)$ denotes the closed ball with center y and radius $r > 0$, i.e. $D(y, r) = \{x \in Y;\ d(y, x) \le r\}$, while $S(y, r)$ denotes the open ball with center y and radius $r > 0$, i.e. $S(y, r) = \{x \in Y;\ d(y, x) < r\}$. If $B \subseteq Y$ and $C \subseteq Y$, we denote by

$$\operatorname{dist}(y, C) := \inf\{d(y, z);\ z \in C\}$$

and by

$$\operatorname{dist}(B, C) := \inf\{d(x, y);\ x \in B,\ y \in C\}.$$

Definition 2.1. Let (Y, d) be a metric space and let $F : Y \rightsquigarrow X$ a multi-function. We say that F is *u.s.c. at* $y \in Y$ if for each open set $U \subseteq X$ with $F(y) \subseteq U$ there exists an open set $V \subseteq Y$ with $y \in V$ and $F(V) \subseteq U$. We say that F is *u.s.c. on* Y if it is u.s.c. at each $y \in Y$.

Definition 2.2. Let $Y \subseteq X$ be nonempty. The function $\beta_Y : \mathcal{B}(X) \to \mathbb{R}_+$, defined by

$$\beta_Y(B) := \inf\left\{\varepsilon > 0; \exists x_1, x_2, \ldots, x_{n(\varepsilon)} \in Y,\ B \subseteq \bigcup_{i=1}^{n(\varepsilon)} D(x_i, \varepsilon)\right\},$$

is called the *Hausdorff measure of noncompactness on* X *subordinated to* Y. If $Y = X$, we simply denote β_X by β, and we simply call it the *Hausdorff measure of noncompactness on* X.

Remark 2.1. We have the following properties:

(i) for each $B \in \mathcal{B}(X)$ and $r > 0$ with $B \subseteq D(0, r)$, we have $\beta(B) \le r$;

(ii) $\beta(B) = 0$ if and only if B is relatively compact;

(iii) the restriction of β_Y to $\mathcal{B}(Y)$ coincides with the Hausdorff measure of noncompactness on Y;

(iv) for each $B \in \mathcal{B}(Y)$ we have $\beta(B) \le \beta_Y(B) \le 2\,\beta(B)$.

The next lemma is due to Mönch [4].

Lemma 2.1. *Let X be a separable Banach space and $\{f_m;\ m \in \mathbb{N}\}$ a subset in $L^1(\tau, T; X)$ for which there exists $\ell \in L^1(\tau, T; \mathbb{R}_+)$ such that*

$$\|f_m(s)\| \le \ell(s)$$

for each $m \in \mathbb{N}$ and a.e. for $s \in [\tau, T]$. Then the mapping

$$s \mapsto \beta(\{f_m(s);\ m \in \mathbb{N}\})$$

is integrable on $[\tau, T]$ and, for each $t \in [\tau, T]$, we have

$$\beta\left(\left\{\int_\tau^t f_m(s)\,ds;\ m \in \mathbb{N}\right\}\right) \le \int_\tau^t \beta(\{f_m(s);\ m \in \mathbb{N}\})\,ds. \tag{2.1}$$

For further details on the Haussdorf measure of noncompactness see Cârjă, Necula, Vrabie [3], Section 2.7, pp. 48∼53.

Let X be a real Banach space, $I \subseteq \mathrm{R}$ a nonempty and bounded interval, $K : I \rightsquigarrow X$ a multi-function with nonempty values and let $\mathcal{K} := \mathrm{graph}(K)$.

Let $(\tau, \xi) \in \mathcal{K}$ and let $E \in \mathcal{B}(X)$.

Definition 2.3. We say that E is *right-tangent* to $\mathcal{K}$ at $(\tau, \xi) \in \mathcal{K}$ if

$$\liminf_{h \downarrow 0} \frac{1}{h} \mathrm{dist}(\xi + hE, K(\tau + h)) = 0. \tag{2.2}$$

Throughout, we denote by $\mathcal{TS}_{\mathcal{K}}(\tau, \xi)$ the set of all right-tangent sets to $\mathcal{K}$ at (τ, ξ). If K is constant, E is right-tangent to $\mathcal{K}$ at (τ, ξ) if and only if it is tangent to K at $\xi \in K$ in the sense of Cârjă, Necula, Vrabie [2], [3], i.e., if $E \in \mathcal{TS}_K(\xi)$, where

$$\mathcal{TS}_K(\xi) = \left\{E \in \mathcal{B}(X);\ \liminf_{h \downarrow 0} \frac{1}{h} \mathrm{dist}(\xi + hE, K) = 0\right\}. \tag{2.3}$$

Here and thereafter, $\mathcal{K}$ is conceived as a metric space, whose metric, d, is defined by $d((\tau, \xi), (\theta, \mu)) = \max\{|\tau - \theta|, \|\xi - \mu\|\}$, for all $(\tau, \xi), (\theta, \mu) \in \mathcal{K}$.

Definition 2.4. The multi-function $F : \mathcal{K} \rightsquigarrow X$ is called *almost u.s.c.* if for each $\varepsilon > 0$ there exists an open set $\mathcal{O}_\varepsilon \subseteq I$ such that $\lambda(\mathcal{O}_\varepsilon) \le \varepsilon$ and $F_{|[(I\setminus\mathcal{O}_\varepsilon)\times X]\cap\mathcal{K}}$ is u.s.c. from $[(I \setminus \mathcal{O}_\varepsilon) \times X] \cap \mathcal{K}$ endowed with the strong topology into X endowed with the strong topology.

We say that F is *strongly-weakly almost u.s.c.* if the domain $[(I \setminus \mathcal{O}_\varepsilon) \times X] \cap \mathcal{K}$ is endowed with the strong topology while the range X is equipeped with the weak topology

Definition 2.5. The multi-function $F : \mathcal{K} \rightsquigarrow X$ is called *integrally-bounded* if for each $(\tau, \xi) \in \mathcal{K}$ there exist $\rho > 0$, $\ell_1 \in L^1(I; \mathbb{R})$ and a negligible set $N_1 \subseteq I$ satisfying: for each $(t, u) \in [(I \setminus N_1) \times S(\xi, \rho)] \cap \mathcal{K}$, we have $\|F(t, u)\| \le \ell_1(t)$.

Remark 2.2. (i) If X is separable we can choose N_1 in Definition 2.5 the same for all $(\tau, \xi) \in \mathcal{K}$ and in this case for each $(\tau, \xi) \in [(I \setminus N_1) \times X] \cap \mathcal{K}$, $F(\tau, \xi)$ is bounded.

(ii) Moreover, if, in addition, F is closed valued and almost u.s.c., then, for each locally absolutely continuous function $u : I \to X$ with $(t, u(t)) \in \mathcal{K}$ for each $t \in I$, the multi-function $t \mapsto F(t, u(t))$ has at least one locally integrable selection on I. The same conclusion holds true if F is closed valued,strongly-weakly almost u.s.c. and has separable range. The latter assertion follows from Pettis' Measurability Theorem 1.1.3, p. 3, in Vrabie [8].

The next special class of graphs was considered for the first time by Necula [5].

Definition 2.6. The graph, $\mathcal{K}$, of $K : I \rightsquigarrow X$ is said to be *viable by itself* if for each $(\tau, \xi) \in \mathcal{K}$, there exist $T > \tau$, $\rho > 0$ and $\ell_2 \in L^1(I; \mathbb{R})$, so that for each $(\widetilde{\tau}, \widetilde{\xi}) \in ([\tau, T) \times S(\xi, \rho)) \cap \mathcal{K}$, there exist $\widetilde{T} \in (\widetilde{\tau}, T]$ and a function $v \in W^{1,1}([\widetilde{\tau}, \widetilde{T}]; X)$ satisfying:

(v_1) $v(\widetilde{\tau}) = \widetilde{\xi}$;

(v_2) $(t, v(t)) \in ([\widetilde{\tau}, \widetilde{T}]) \times S(\xi, \rho)) \cap \mathcal{K}$ for each $t \in [\widetilde{\tau}, \widetilde{T}]$;

(v_3) $\|v'(s)\| \le \ell_2(s)$ a.e $s \in [\widetilde{\tau}, \widetilde{T}]$.

A function v satisfying $(v_1) \sim (v_3)$ is called *simple solution issuing from* $(\widetilde{\tau}, \widetilde{\xi}) \in ([\tau, T) \times S(\xi, \rho)) \cap \mathcal{K}$.

Remark 2.3. In other words, the graph, $\mathcal{K}$, of $K : I \rightsquigarrow X$ is viable by itself if and only if, for each $(\tau, \xi) \in \mathcal{K}$, there exist $T > \tau$, $\rho > 0$ and $\ell_2 \in L^1(I; \mathbb{R})$, so that $([\tau, T) \times S(\xi, \rho)) \cap \mathcal{K}$ is viable with respect to the multi-function $G : ([\tau, T) \times X) \cap \mathcal{K} \rightsquigarrow X$, defined by $G(t, \xi) := \{v \in X;\ \|v\| \leq \ell_2(t)\}$, for each $(t, \xi) \in ([\tau, T) \times X) \cap \mathcal{K}$

Remark 2.4. (i) Clearly, if $K : I \rightsquigarrow X$ is constant, then $\mathcal{K}$ is viable by itself. Indeed, in this case, $\ell \equiv 0$ and $G(t, \xi) \equiv \{0\}$ satisfy all the requirements in Definition 2.6.

(ii) If $\mathcal{K}$ viable with respect to some integrally-bounded multi-function $F : \mathcal{K} \rightsquigarrow X$ then, one may easily check out that, for each $(\tau, \xi) \in \mathcal{K}$, the function G, defined as in Remark 2.3, with $\rho > 0$ given by Definition 2.5, and $\ell_2 = \ell_1$, where ℓ_1 are given by Definition 2.5, satisfies the conditions in Remark 2.4, and thus $\mathcal{K}$ is viable by itself.

Let $\mathcal{K}$ be viable by itself and let $F : \mathcal{K} \rightsquigarrow X$ be integrally bounded. Let $(\tau, \xi) \in \mathcal{K}$ and let $\rho > 0$, $T > \tau$ and ℓ_1, ℓ_2 given by Definition 2.5 and 2.6. Let $\ell \in L^1(I, \mathbb{R})$ such that $\ell(t) \geq \max\{\ell_1(t), \ell_2(t)\}$ a.e. $t \in I$ and $0 < h \leq T - \tau$

Let us denote by $\mathcal{ACS}_h(\tau, \xi, \ell)$ the set of all functions $v \in W^{1,1}([\tau, \tau + h]; S(\xi, \rho))$, with $v(\tau) = \xi$, $(t, v(t)) \in \mathcal{K}$ for each $t \in [\tau, \tau + h]$ and$\|v'(s)\| \leq \ell(s)$ a.e. $s \in [\tau, \tau + h]$. Since $\mathcal{K}$ is viable by itself, we easily deduce that $\mathcal{ACS}_h(\tau, \xi, \ell)$ is nonempty.

Let $v \in \mathcal{ACS}_h(\tau, \xi, \ell)$. We denote by $\mathcal{S}_{F(\cdot, v(\cdot))}$ the set of all measurable a.e. selections of the multi-function $t \mapsto F(t, v(t))$ on $[\tau, \tau + h]$. If F is closed valued and almost u.s.c. , in view of (ii) in Remark 2.2, $\mathcal{S}_{F(\cdot, v(\cdot))}$ is nonempty too, and, more than this, it contains only integrable functions. Finally, let us denote by

$$E_{\tau,\xi,\ell,h} := \{f \in \mathcal{S}_{F(\cdot, v(\cdot))};\ v \in \mathcal{ACS}_h(\tau, \xi, \ell)\}, \tag{2.4}$$

set which, in view of the remarks above, is nonempty and included in $L^1(\tau, \tau + h; X)$.

We consider the generalized tangency condition

$$\liminf_{h \downarrow 0} \frac{1}{h} \operatorname{dist}\left(\xi + \int_\tau^{\tau+h} E_{\tau,\xi,\ell,h}\, ds, K(\tau + h)\right) = 0. \tag{2.5}$$

At this point, let us observe that (2.5) makes sense whenever $E_{\tau,\xi,\ell,h}$ is nonempty. As we already pointed out, in order for the above set to be nonempty it is sufficient that $\mathcal{K}$ be viable by itself and $F : \mathcal{K} \rightsquigarrow X$ be integrally bounded, closed valued and almost u.s.c. Here and thereafter, when we say that (2.5) takes place, we understand that $\mathcal{K}$ is viable by itself,

F is integrally bounded and $E_{\tau,\xi,\ell,h} \neq \emptyset$ for h small enough (sufficiently for a certain h). The fact that (2.5) can take place even in the absence of continuity or measurability conditions for F is illustrated by the first very simple necessary condition for viability in the next section.

3. Necessary conditions for viability. The hypotheses we will use in the sequel are listed below. Throughout, λ denotes the Lebesgue measure on $\mathbb{R}$.

(H_1) the graph $\mathcal{K}$ is viable by itself;

(H_2) F has nonempty and closed values and is integrally bounded;

(H_3) $F : \mathcal{K} \rightsquigarrow X$ is almost u.s.c.;

(H_4) $F : \mathcal{K} \rightsquigarrow X$ is strongly-weakly almost u.s.c.;

(H_5) there exists a set $N \subseteq I$, with $\lambda(N) = 0$, and such that for each $(\tau, \xi) \in ((I \setminus N) \times X) \cap \mathcal{K}$, we have $F(\tau, \xi) \in \mathcal{TS}_{\mathcal{K}}(\tau, \xi)$.

(H_6) there exists a set $N \subseteq I$, with $\lambda(N) = 0$, and such that for each $(\tau, \xi) \in ((I \setminus N) \times X) \cap \mathcal{K}$, we have (2.5)

(H_7) for each $(\tau, \xi) \in \mathcal{K}$, we have (2.5).

Theorem 3.1. *If $\mathcal{K}$ is viable with respect to an integrally bounded multifunction F, then (H_7) holds true.*

Proof. First let us observe that even if F is not closed valued and almost u.s.c. the set $\mathcal{ACS}_h(\tau, \xi, \ell)$ is nonempty, $\mathcal{S}_{F(\cdot,u(\cdot))} \cap L^1(\tau, \tau + h; X) \neq \emptyset$, and thus $E_{\tau,\xi,\ell,h} \cap L^1(\tau, \tau + h; X) \neq \emptyset$ too for h small enough. Indeed, let ρ from the Definition 2.5 and $u : [\tau, T] \to S(\xi, \rho)$ be any solution of (1.1). Then for each $h \in (0, T - \tau]$ we have $u \in \mathcal{ACS}_h(\tau, \xi, \ell)$, $u' \in \mathcal{S}_{F(\cdot,u(\cdot))} \cap L^1(\tau, \tau + h; X)$ and therefore $u' \in E_{\tau,\xi,\ell,h}$. Hence

$$\operatorname{dist}\left(\xi + \int_\tau^{\tau+h} E_{\tau,\xi,\ell,h}\, ds, K(\tau + h)\right) \leq \operatorname{dist}\left(\xi + \int_\tau^{\tau+h} u'(s)\, ds, K(\tau + h)\right)$$

$$= \operatorname{dist}\left(u(\tau + h), K(\tau + h)\right) = 0$$

for each $h \in (0, T - \tau]$ and this completes the proof.

Let us remark that we have proved that for h sufficiently small

$$\{\xi + \int_{\tau}^{\tau+h} E_{\tau,\xi,\ell,h}\, ds\} \cap K(\tau + h) \neq \emptyset.\square$$

So, under more general hypotheses on F, (H_6) is necessary in order for $\mathcal{K}$ be viable with respect to F. In that follows, we shall see that, under some additional natural assumptions on F, the converse statement is also true.

The next result was proved in Necula, Popescu, Vrabie [6].

Theorem 3.2. *Let X be separable, and let $F : \mathcal{K} \rightsquigarrow X$ be a multi-function with convex values satisfying (H_2) and (H_3). If $\mathcal{K}$ is viable with respect to (1.1), then (H_1) and (H_5) hold true.*

4. The relationship between (H_5) and (H_6)

Definition 4.1. We say that the multi-function $F : \mathcal{K} \rightsquigarrow X$ is almost ε–δ *l.s.c.* if for each $\gamma > 0$, there exists an open set $\mathcal{O}_\gamma \subset I$, with $\lambda(\mathcal{O}_\gamma) \leq \gamma$, and such that the mapping $(t, \xi) \mapsto F(t, \xi)$ is ε–δ l.s.c. on $((I \setminus \mathcal{O}_\gamma) \times X) \cap \mathcal{K}$.

Theorem 4.1. *Let X be separable and let $\mathcal{K}$ and F satisfy (H_1) and (H_2).*

(i) If F is almost ε–δ l.s.c., then (H_5) implies (H_6).

(ii) If F has convex values and is almost u.s.c., then (H_6) implies (H_5).

Proof. We begin with the proof of (i). From (H_2) and the fact that X is separable, it follows that there exist a finite or at most countable set Γ, $(\xi_i)_{i\in\Gamma} \subset K$, $(\rho_i)_{i\in\Gamma} \subset (0, \infty)$, $(\ell_i)_{i\in\Gamma} \subset L^1_{\text{loc}}(I)$ and a negligible set $N \subset I$ such that $\mathcal{K} \subseteq I \times \cup_{i\in\Gamma} S(\xi_i, \rho_i)$ and, for all $i \in \Gamma$, and all $(t, u) \in ((I \setminus N) \times S(\xi_i, \rho_i)) \cap \mathcal{K}$, we have $\|F(t, u)\| \leq \ell_i(t)$.

Since F is ε–δ l.s.c., it follows that, for each $n \in \mathbb{N}$, $n \geq 1$ there exists $I_n \subset I$, with $\lambda(I \setminus I_n) < \frac{1}{n}$, and such that the mapping $(t, \xi) \mapsto F(t, \xi)$ is ε–δ l.s.c. on $(I_n \times X) \cap \mathcal{K}$.

Let $A_n \subset I_n$ the set of all density points of I_n which are Lebesgue points too for ℓ_i for all $i \in \Gamma$. Let $A = (\cup_{n\geq 1} A_n) \cap (I \setminus N)$, where N is the negligible set in (H_4). Obviously, $\lambda(I \setminus A) = 0$.

Let $(\tau, \xi) \in (A \times X) \cap \mathcal{K}$. We will show that

$$\liminf_{h\downarrow 0} \frac{1}{h} \text{dist}\left(\xi + \int_{\tau}^{\tau+h} E_{\tau,\xi,\ell,h}\, ds, K(\tau + h)\right) = 0,$$

where $E_{\tau,\xi,\ell,h}$ is given by (2.4). Let us observe that there exist $i_0 \in \Gamma$ and $n_0 \in \mathbb{N}$ such that $\tau \in A_{n_0}$ and $\xi \in S(\xi_{i_0}, \rho_{i_0})$. Let $\varepsilon > 0$ be arbitrary but fixed and let $\delta > 0$ be such that $F(\tau, \xi) \subset F(s, \mu) + D(0, \varepsilon)$, for all $(s, \mu) \in [([\tau, \tau + \delta] \cap A_{n_0}) \times D(\xi, \delta)] \cap \mathcal{K}$. From (H_5), it follows that there exists $h_n \downarrow 0$, $\eta_n \in F(\tau, \xi)$ and $p_n \in X$, with $\|p_n\| \to 0$, and such that $\xi + h_n\eta_n + h_n p_n \in K(\tau + h_n)$ for all $n \in \mathbb{N}$, $n \geq 1$. Consequently $\eta_n \in F(s, \mu) + D(0, \varepsilon)$, for all $(s, \mu) \in [([\tau, \tau + \delta] \cap A_{n_0}) \times D(\xi, \delta)] \cap \mathcal{K}$ and for all $n \in \mathbb{N}$, $n \geq 1$.

Since $\mathcal{K}$ is viable by itself there exists $u \in W^{1,1}([\tau, T]; X)$ such that $u(\tau) = \xi, (t, u(t)) \in \mathcal{K}$ for all $t \in [\tau, T]$ and $\|u'(t)\| \leq \ell(t)$ a.e. $t \in [\tau, T]$. Diminishing T if necessary we may assume that $T < \delta$ and $u(t) \in S(\xi, \rho) \cap S(\xi, \delta)$ for all $t \in [\tau, T]$.

At this point, let us observe that, for each $n \in \mathbb{N}$, $n \geq 1$, the multi-function $s \mapsto F(s, u(s)) \cap D(\eta_n, \varepsilon)$ is measurable, nonempty and closed valued from $[\tau, T] \cap A_{n_0}$ to X. Since X is separable, from Kuratowski and Ryll-Nardzewski Theorem 3.1.1, p. 86 in Vrabie [7], it follows that the multi-function above has at least one measurable selection. Let us denote by $f_n : [\tau, T] \cap A_{n_0} \to X$ such a selection. Next, let us extend f_n to a measurable selection of $F(\cdot, u(\cdot))$ on $[\tau, T]$, extension denoted, for simplicity, again by f_n. So, for each $n \in \mathbb{N}$, $n \geq 1$, and $s \in [\tau, T]$, we have $f_n(s) \in F(s, u(s))$. Also, for each $n \in \mathbb{N}$, $n \geq 1$, and $s \in [\tau, T] \cap A_{n_0}$, we have $\|f_n(s) - \eta_n\| \leq \varepsilon$. Let $E_{\tau,\xi,\ell,h}$ be as in (2.4). Then, for each $h_n \in (0, T)$ (we recall that $T \leq \delta$), we deduce

$$\begin{aligned}
&\frac{1}{h_n}\mathrm{dist}\left(\xi + \int_\tau^{\tau+h_n} E_{\tau,\xi,\ell,h}\, ds, K(\tau + h_n)\right) \\
&\leq \frac{1}{h_n}\mathrm{dist}\left(\xi + \int_\tau^{\tau+h_n} f_n(s)\, ds, \xi + h_n\eta_n + h_n p_n\right) \\
&\leq \left\| \frac{1}{h_n}\int_\tau^{\tau+h_n} f_n(s)\, ds - \eta_n \right\| + \|p_n\| \\
&\leq \frac{1}{h_n}\int_{[\tau,\tau+h_n]\cap A_{n_0}} \|f_n(s) - \eta_n\|\, ds \qquad (4.1)\\
&+ \frac{1}{h_n}\int_{[\tau,\tau+h_n]\setminus A_{n_0}} \|f_n(s) - \eta_n\|\, ds + \|p_n\| \\
&\leq \varepsilon + \frac{1}{h_n}\int_{[\tau,\tau+h_n]\setminus A_{n_0}} \|f_n(s) - \eta_n\|\, ds + \|p_n\|.
\end{aligned}$$

But

$$\frac{1}{h_n}\int_{[\tau,\tau+h_n]\setminus A_{n_0}} \|f_n(s)-\eta_n\|\,ds$$
$$\leq \frac{1}{h_n}\int_{[\tau,\tau+h_n]\setminus A_{n_0}} |\ell_{i_0}(s)-\ell_{i_0}(\tau)|\,ds + \frac{2}{h_n}\int_{[\tau,\tau+h_n]\setminus A_{n_0}} \ell_{i_0}(\tau)\,ds$$
$$\leq \frac{1}{h_n}\int_{\tau}^{\tau+h_n} \mid \ell_{i_0}(s)-\ell_{i_0}(\tau)\mid\; ds + 2\ell_{i_0}(\tau)\frac{\lambda([\tau,\tau+h_n]\setminus A_{n_0})}{h_n}.$$

Passing to the lim sup in (4.1), we get

$$\limsup_{n\to\infty}\frac{1}{h_n}d\left(\xi+\int_{\tau}^{\tau+h_n} E_{\tau,\xi,\ell,h}\,ds, K(\tau+h_n)\right)\leq\varepsilon$$

and therefore (H_5) holds true and this completes the proof of the first part of Theorem 4.1.

Now let us prove (ii). From (H_3), it follows that for each $n\in\mathbb{N}$, $n\geq 1$ there exists $I_n\subset I$, with $\lambda(I\setminus I_n)<\frac{1}{n}$, such that the mapping $(t,\xi)\mapsto F(t,\xi)$ is u.s.c. on $(I_n\times X)\cap\mathcal{K}$.

Let $A_n\subset I_n$ the set of all density points of I_n which are Lebesgue points too for ℓ_i, for all $i\in\Gamma$. Let $A=(\cup_{n\geq 1}A_n)\cap(I\setminus N)$, where N is the negligible set in (H_4). Obviously, $\lambda(I\setminus A)=0$.

Let $\tau\in A$ and $\xi\in K(\tau)$. We will show that

$$\liminf_{h\downarrow 0}\frac{1}{h}\text{dist}(\xi+hF(\tau,\xi),K(\tau+h))=0.$$

Since $\tau\in A$, there exists $n_0\in\mathbb{N}$ such that $\tau\in A_{n_0}$. Analogously, since $\mathcal{K}\subseteq I\times\cup_{i\in\Gamma}S(\xi_i,\rho_i)$, there exists $i_0\in\Gamma$ such that $\xi\in S(\xi_{i_0},\rho_{i_0})$. Let $\varepsilon>0$ be arbitrary but fixed and let $\delta>0$ be such that $F(s,\mu)\subset F(\tau,\xi)+D(0,\varepsilon)$, for all $(s,\mu)\in[([\tau,\tau+\delta]\cap A_{n_0}\times D(\xi,\delta)]\cap\mathcal{K}$.

From (H_1), (H_6) and the remark above, it follows that there exist $h_n\downarrow 0$, $u_n\in W^{1,1}([\tau,\tau+h_n],X)$, with $u_n(\tau)=\xi$, $(t,u_n(t))\in\mathcal{K}$, for each $t\in[\tau,\tau+h_n]$, and there exist $f_n\in L^1(\tau,\tau+h_n;X)$, with $f_n(s)\in F(s,u_n(s))$ a.e. for $s\in[\tau,\tau+h_n]$, and $p_n\in X$, with $\|p_n\|\to 0$, such that

$$\xi+\int_{\tau}^{\tau+h_n} f_n(s)\,ds+h_np_n\in K(\tau+h_n),$$

for all $n\in\mathbb{N}$, $n\geq 1$.

Since for every $u \in ACS_h(\tau,\xi,\ell)$ we have $\|u(t)-\xi\| \leq \int_\tau^h \ell(s)ds$ for all $t \in [\tau,\tau+h]$ we may suppose $u_n(t) \in S(\xi,\rho)\cap S(\xi,\delta)$ for all $t \in [\tau,\tau+h]$, where ρ is from Definition 2.5.

Let us observe that, for each $h_n \in (0,\delta)$, we have

$$\begin{aligned}
&\frac{1}{h_n}\operatorname{dist}(\xi + h_n F(\tau,\xi), K(\tau+h_n))\\
&\leq \frac{1}{h_n}\operatorname{dist}\left(\xi + h_n F(\tau,\xi), \xi + \int_\tau^{\tau+h_n} f_n(s)\,ds + h_n p_n\right)\\
&\leq \operatorname{dist}\left(F(\tau,\xi), \frac{1}{h_n}\int_\tau^{\tau+h_n} f_n(s)\,ds\right) + \|p_n\| \qquad (4.2)\\
&\leq \operatorname{dist}\left(F(\tau,\xi), \frac{1}{h_n}\int_{[\tau,\tau+h_n]\cap A_{n_0}} f_n(s)\,ds\right)\\
&+\frac{1}{h_n}\int_{[\tau,\tau+h_n]\setminus A_{n_0}} \|f_n(s)\|\,ds + \|p_n\|.
\end{aligned}$$

As in the first part of the proof, one shows that

$$\lim_{n\to\infty}\frac{1}{h_n}\int_{[\tau,\tau+h_n]\setminus A_{n_0}} \|f_n(s)\|\,ds = 0.$$

Since, for each $n \in \mathbb{N}$, $n \geq 1$, with $h_n \in (0,\delta)$, we have

$$\begin{aligned}
&\frac{1}{\widetilde{h}_n}\int_{[\tau,\tau+h_n]\cap A_{n_0}} f_n(s)\,ds \in \overline{\operatorname{conv}} \bigcup_{s\in[\tau,\tau+\delta]\cap A_{n_0}} F(s,u_n(s))\\
&\subset \overline{\operatorname{conv}}(F(\tau,\xi)+D(0,\varepsilon)) = \overline{F(\tau,\xi)+D(0,\varepsilon)},
\end{aligned}$$

where $\widetilde{h}_n = \lambda\left([\tau,\tau+h_n]\cap A_{n_0}\right)$, it follows that there exist $\eta_n \in F(\tau,\xi)$ and $r_n \in D(0,2\varepsilon)$ such that

$$\frac{1}{\widetilde{h}_n}\int_{[\tau,\tau+h_n]\cap A_{n_0}} f_n(s)\,ds = \eta_n + r_n.$$

It follows that

$$\operatorname{dist}\left(F(\tau,\xi), \frac{1}{\widetilde{h}_n}\int_{[\tau,\tau+h_n]\cap A_{n_0}} f_n(s)\,ds\right) \leq \operatorname{dist}(F(\tau,\xi),\eta_n) + \|r_n\| \leq 2\varepsilon.$$

Passing to lim sup in (4.2), we obtain

$$\limsup_{n\to\infty} \frac{1}{h_n} \operatorname{dist}(\xi + h_n F(\tau,\xi), K) \leq 2\varepsilon.$$

But this shows that (H_5) holds true, and this completes the proof of Theorem 4.1. □

5. Sufficient conditions for viability

Definition 5.1. We say that the graph $\mathcal{K}$ is:

(i) *locally closed from the left* if for each $(\tau,\xi) \in \mathcal{K}$ there exist $T > \tau$ and $\rho > 0$ such that, for each $(\tau_n,\xi_n) \in ([\tau,T] \times D(\xi,\rho)) \cap \mathcal{K}$, with $(\tau_n)_n$ nondecreasing, $\lim_n \tau_n = \widetilde{\tau}$ and $\lim_n \xi_n = \widetilde{\xi}$, we have $(\widetilde{\tau},\widetilde{\xi}) \in \mathcal{K}$;

(ii) *closed from the left* if for each $(\tau_n,\xi_n) \in \mathcal{K}$, with $(\tau_n)_n$ nondecreasing, $\lim_n \tau_n = \widetilde{\tau}$ and $\lim_n \xi_n = \widetilde{\xi}$, we have $(\widetilde{\tau},\widetilde{\xi}) \in \mathcal{K}$;

(iii) *locally compact from the left* if, it is locally closed from the left and, for each $(\tau,\xi) \in \mathcal{K}$ there exist $T > \tau$ and $\rho > 0$ such that, for each $(\tau_n,\xi_n) \in ([\tau,T] \times D(\xi,\rho)) \cap \mathcal{K}$, with $(\tau_n)_n$ nondecreasing, and $\lim_n \tau_n = \widetilde{\tau}$, there exists a convergent subsequence $(\xi_{n_k})_k$ of $(\xi_n)_n$;

(iv) *compact from the left* if, it is closed from the left and, for each $(\tau_n,\xi_n) \in \mathcal{K}$ with $(\tau_n)_n$ nondecreasing, $\lim_n \tau_n = \widetilde{\tau}$, and $(\xi_n)_n$ bounded, there exists a convergent subsequence $(\xi_{n_k})_k$ of $(\xi_n)_n$.

Remark 5.1. Let $(\xi_{n_k})_k$ be the subsequence of $(\xi_n)_n$ whose existence is ensured by (ii) in Definition 5.1 and let $\xi = \lim_k \xi_{n_k}$. Then $(\tau,\xi) \in \mathcal{K}$.

Definition 5.2. By a *Carathéodory uniqueness function* we mean a function $\omega : I \times R_+ \to R_+$ such that:

(i) for each $x \in \mathbb{R}_+$, $t \mapsto \omega(t,x)$ is locally integrable;

(ii) for a.e. $t \in I$, $x \mapsto \omega(t,x)$ is continuous, nondecreasing;

(iii) for each $\tau \in I$, the only absolutely continuous solution of the Cauchy problem

$$\begin{cases} x'(t) = \omega(t,x(t)) \\ x(\tau) = 0 \end{cases}$$

is $x \equiv 0$.

Definition 5.3. The multi-function $F : \mathcal{K} \rightsquigarrow X$ is called β-compact if for all $(\tau, \xi) \in \mathcal{K}$ there exists $\rho > 0$, a Carathéodory uniqueness function, $\omega : I \times R_+ \to R_+$, and a negligible set $N \subseteq I$, such that, for all $B \subseteq X \cap D(\xi, \rho)$ and all $t \in I \setminus N$, we have $\beta(F([\{t\} \times B] \cap \mathcal{K})) \leq \omega(t, \beta(B))$.

Remark 5.2.

(i) If F is β-compact, then for each $(\tau, \xi) \in \mathcal{K}$, there exists $\rho > 0$ and a negligible set $N \subseteq I$ such that for all $(t, u) \in [(I \setminus N) \times D(\xi, \rho)] \cap \mathcal{K}$ $F(t, u)$ is relatively compact. Indeed, $\beta(F(t, u)) \leq \omega(t, \beta(\{u\}) = \omega(t, 0) = 0$, as claimed.

(ii) If X is separable (sufficiently $\mathcal{K}$ separable) and F is β-compact then there exists a negligible set $N \subseteq I$ such that for all $(\tau, \xi) \in [(I \setminus N) \times X] \cap \mathcal{K}$, $F(\tau, \xi)$ is relatively compact.

Remark 5.3. If F has compact values and is almost u.s.c., and $\mathcal{K}$ is locally compact, then F is β-compact.

Theorem 5.1. *Let $\mathcal{K}$ be locally closed from the left and let $F : \mathcal{K} \rightsquigarrow X$ be convex valued and β-compact. If (H_1), (H_2) and (H_4) are satisfied, then a necessary and sufficient condition in order that $\mathcal{K}$ be viable with respect to F is (H_6).*

Theorem 5.2. *Let $\mathcal{K}$ be locally compact from the left and let $F : \mathcal{K} \rightsquigarrow X$ be convex and weakly compact valued. If (H_1), (H_2) and (H_4) are satisfied, then a necessary and sufficient condition in order that $\mathcal{K}$ be viable with respect to F is (H_6).*

From Theorems 5.1, 5.2 and Brezis-Browder Ordering Principle, i.e. Theorem 2.1.1, p. 30 in Cârjă, Necula, Vrabie [3], we easily deduce the two global viability results stated below.

Theorem 5.3. *Let $\mathcal{K}$ be closed from the left and let $F : \mathcal{K} \rightsquigarrow X$ be convex valued and β-compact. If (H_1), (H_2) and (H_4) are satisfied, then a necessary and sufficient condition in order that $\mathcal{K}$ be globally viable with respect to F is (H_6).*

Theorem 5.4. *Let $\mathcal{K}$ be compact from the left and let $F : \mathcal{K} \rightsquigarrow X$ be convex and weakly compact valued. If (H_1), (H_2) and (H_4) are satisfied, then a necessary and sufficient condition in order that $\mathcal{K}$ be globally viable with respect to F is (H_6).*

The next lemma, essentially inspired from Cârjă, Monteiro-Marques [1], is the main step through the proof of both Theorems 5.1 and 5.2.

Lemma 5.1. *Let I be a nonempty and bounded interval and $K : I \rightsquigarrow X$ a multi-function with locally closed from the left graph, $\mathcal{K}$, let $(\tau, \xi) \in \mathcal{K}$ and let $F : \mathcal{K} \rightsquigarrow X$ be a nonempty and closed valued multi-function satisfying (H_1), (H_2) and (H_6). Let $Z \subseteq I$ be a negligible set including the negligible set in (H_6).*

Let $\rho > 0$, $T > \tau$ and $\ell \in L^1(I, \mathbb{R})$ be from the construction of $E_{\tau,\xi,\ell,h}$ and such that $([\tau, T] \times D(\xi, \rho)) \cap \mathcal{K}$ is closed from the left, and

$$T - \tau + \int_\tau^T \ell(s)\, ds \le \rho. \tag{5.1}$$

Then, for each $\varepsilon \in (0, 1)$ and each open set $\mathcal{O} \subseteq I$, with $Z \subseteq \mathcal{O}$, there exist a family $\mathcal{P}_T = \{[t_m, s_m); m \in \Gamma\}$, of disjoint intervals, with Γ finite or at most countable, and $f, r, v \in L^1(\tau, T; X)$ and $u \in W^{1,1}([\tau, T]; X)$ such that:

(*i*) $\cup[t_m, s_m) = [\tau, T)$ *and* $s_m - t_m \le \varepsilon$, *for all* $m \in \Gamma$;

(*ii*) *if* $t_m \in \mathcal{O}$, *then* $[t_m, s_m) \subseteq \mathcal{O}$;

(*iii*) $u(t_m) \in D(\xi, \rho) \cap K(t_m)$, *for all* $m \in \Gamma$, $u(T) \in D(\xi, \rho) \cap K(T)$;

(*iv*) $v \in W^{1,1}([t_m, s_m); X)$, *for all* $m \in \Gamma$; $(t, v(t)) \in ([\tau, T) \times S(\xi, \rho)) \cap \mathcal{K}$, *for all* $t \in [\tau, T)$ *and* $\| v(t) - u(t_m) \| \le \varepsilon$, *for all* $t \in [t_m, s_m)$

(*v*) $f(s) \in F(s, v(s))$ *for* $s \in [t_m, s_m)$ *if* $t_m \notin \mathcal{O}$, $f(s) = v'(s)$ *a.e.* $s \in [t_m, s_m)$ *if* $t_m \in \mathcal{O}$ *and* $\|f(s)\| \le \ell(s)$ *a.e.* $s \in [\tau, T]$

(*vi*) $\|r(s)\| \le \varepsilon$ *for* $s \in [\tau, T]$;

(*vii*) $u(t) = \xi + \int_\tau^t f(s)\, ds + \int_\tau^t r(s)\, ds$, *for all* $t \in [\tau, T]$.

Proof. Let ε be arbitrary but fixed in $(0, 1)$ and let $\mathcal{O} \subseteq R$ be an open subset with $Z \subseteq \mathcal{O}$. We will show that there exist $\delta = \delta(\varepsilon, \mathcal{O}) \in (\tau, T)$ and $\mathcal{P}_\delta, f, r, u$ such that (i)$\sim$(vi) hold true with δ instead of T. We distinguish between the following different cases.

Case 1. If $\tau \in \mathcal{O}$, we take $\Gamma = \{1\}$, $t_1 = \tau$, $s_1 = \delta$ with $\delta \in (\tau, T)$ small enough in order to $[\tau, \delta] \subseteq \mathcal{O}$ and there exists a simple solution issuing from (τ, ξ), defined on $[\tau, \delta]$. Further, let us fix such a simple solution, v, diminish δ such that $\|v(t) - \xi\| < \min\{\varepsilon, \rho\}$ for all $s \in [\tau, \delta]$ and let us define $\mathcal{P}_\delta = \{[\tau, \delta)\}$, $f(s) = v'(s)$ a.e for $s \in [\tau, \delta]$, $r = 0$ and $u(t) = v(t)$ for all $t \in [\tau, \delta]$.

Case 2. If $\tau \notin \mathcal{O}$ then $\tau \notin Z$ which implies that there exist $h_n \downarrow 0$, $v_n \in W^{1,1}([\tau, \tau + h_n]; X)$ such that $(t, v_n(t)) \in ([\tau, \tau + h_n] \times D(\xi, \rho)) \cap \mathcal{K}$ and $f_n \in L^1(\tau, \tau + h_n; X)$, $f_n \in F(s, v_n(s))$ for $s \in [\tau, \tau + h_n]$ and $p_n \in X$ with $p_n \to 0$, such that

$$\xi + \int_\tau^{\tau + h_n} f_n(s)\, ds + h_n p_n \in K(\tau + h_n) \cap D(\xi, \rho)$$

for all $n \geq 1, n \in \mathbb{N}$.

Let $n_0 \in N$ and $\delta = \tau + h_{n_0}$ be such that $\delta \in (\tau, T)$, $h_{n_0} < \varepsilon$ and $\|p_{n_0}\| < \varepsilon$ and $\int_\tau^\delta \ell(s)\, ds < \varepsilon$

We define $\mathcal{P}_\delta = \{[\tau, \delta)\}$, $f(s) = f_{n_0}(s)$, $r(s) = p_{n_0}$, $v(s) = v_{n_0}(s)$ for $s \in [\tau, \delta]$ and let $u : [\tau, \delta] \to X$ be given by (vii). One may easily see that (i)$\sim$(vii) are satisfied. Let

$$\mathcal{U} = \{(\mathcal{P}_\delta, f, r, v, u);\ \delta \in (\tau, T],\ \text{(i)}\sim\text{(vii) hold true with } \delta \text{ instead of } T\}.$$

As we already have shown, $\mathcal{U} \neq \emptyset$. On $\mathcal{U}$ we define a partial order by:

$$(\mathcal{P}_{\delta_1}, f_1, r_1, v_1, u_1) \preceq (\mathcal{P}_{\delta_2}, f_2, r_2, v_2, u_2),$$

if

$$\begin{cases} \delta_1 \leq \delta_2, \quad \mathcal{P}_{\delta_1} \subseteq \mathcal{P}_{\delta_2}, \\ f_1(s) = f_2(s), r_1(s) = r_2(s), v_1(s) = v_2(s) \text{ a.e. for } s \in [\tau, \delta_1] \\ u_1(s) = u_2(s), \text{for all } s \in [\tau, \delta_1]. \end{cases}$$

We will prove that each nondecreasing sequence in $\mathcal{U}$ is bounded from above. Let $(\mathcal{P}_{\delta_j}, f_j, r_j, v_j, u_j)_{j \geq 1}$ be a nondecreasing sequence in $\mathcal{U}$ and let $\delta = \sup_{j \geq 1} \delta_j$. If there exists $j_0 \in N$ such that $\delta_{j_0} = \delta$, then $(\mathcal{P}_{\delta_{j_0}}, f_{j_0}, r_{j_0}, v_{j_0}, u_{j_0})$ is an upper bound for the sequence. So, let us assume that $\delta_j < \delta$, for all $j \geq 1$. Obviously, $\delta \in (\tau, T]$. We define $\mathcal{P}_\delta = \cup_{j \geq 1} \mathcal{P}_{\delta_j}$, $f(s) = f_j(s)$,$r(s) = r_j(s)$ and $v(s) = v_j(s)$ for all j and all $s \in [\tau, \delta_j]$. Clearly, $f, r, v \in L^1(\tau, T; X)$. Next, we define $u : [\tau, \delta] \to X$ by

$$u(t) = \xi + \int_\tau^t f(s)\, ds + \int_\tau^t r(s)\, ds,$$

for all $t \in [\tau, \delta]$. We have $u \in W^{1,1}([\tau, \delta]; X)$ and $u(s) = u_j(s)$, for all $j \geq 1$ and all $s \in [\tau, \delta_j]$. Since $u(\delta) = \lim_{t\uparrow\delta} u(t) = \lim_{j\to\infty} u(\delta_j) = \lim_{j\to\infty} u_j(\delta_j)$, and $u_j(\delta_j) \in D(\xi, \rho) \cap K(\delta_j)$ which is closed from the left, we deduce that that $u(\delta) \in D(\xi, \rho) \cap K(\delta)$. The rest of conditions in lemma being obviously satisfied, it follows that $(\mathcal{P}_\delta, f, r, v, u)$ is an upper bound for the sequence. Thus, the partially ordered set $(\mathcal{U}, \preceq)$ and the function $\mathcal{N} : (\mathcal{U}, \preceq) \to R$, defined by $\mathcal{N}(\mathcal{P}_\delta, f, r, v, u) = \delta$, for each $(\mathcal{P}_\delta, f, r, v, u) \in \mathcal{U}$, satisfy the hypotheses of the Brezis-Browder Ordering Principle, i.e. Theorem 2.1.1, p. 30 in Cârjă, Necula, Vrabie [3]. Accordingly, there exists an $\mathcal{N}$-maximal element in $\mathcal{U}$. This means that there exists $(\mathcal{P}_{\delta^*}, f^*, r^*, v^*, u^*) \in \mathcal{U}$ such that, whenever $(\mathcal{P}_{\delta^*}, f^*, r^*, v^*, u^*) \preceq (\mathcal{P}_{\overline{\delta}}, \overline{f}, \overline{r}, \overline{v}, \overline{u})$, we necessarily have $\mathcal{N}(\mathcal{P}_{\delta^*}, f^*, r^*, v^*, u^*) = \mathcal{N}(\mathcal{P}_{\overline{\delta}}, \overline{f}, \overline{r}, \overline{v}, \overline{u})$. We will show that $\delta^* = T$. To this aim, let us assume by contradiction that $\delta^* < T$. We distinguish between two cases.

Case 1. If $\delta^* \in \mathcal{O}$, we take $\overline{\delta} \in (\delta^*, T)$ such that $[\delta^*, \overline{\delta}] \subseteq \mathcal{O}$ and we define

$$\overline{f}(s) = \begin{cases} f^*(s) \text{ for } s \in [\tau, \delta^*] \\ v'(s) \text{ a.e for } s \in (\delta^*, \overline{\delta}] \end{cases}, \overline{r}(s) = \begin{cases} r^*(s) \text{ for } s \in [\tau, \delta^*] \\ 0 \text{ for } s \in (\delta^*, \overline{\delta}] \end{cases},$$

$$\overline{v}(s) = \begin{cases} v^*(s) \text{ for } s \in [\tau, \delta^*) \\ v(s) \text{ a.e for } s \in [\delta^*, \overline{\delta}] \end{cases},$$

where $v : [\delta^*, \overline{\delta}] \to X$ is a simple solution issuing from $(\delta^*, u(\delta^*))$ and defined on $[\delta^*, \overline{\delta}]$. We can diminish $\overline{\delta}$ such that $\|v(t) - u^*(\delta^*)\| < \varepsilon$ and $\|v(t) - \xi\| < \rho$ for all $t \in [\delta^*, \overline{\delta}]$. Let

$$\overline{u}(s) = \begin{cases} u^*(s), \text{ for } s \in [\tau, \delta^*] \\ v(s), \text{ for } s \in (\delta^*, \overline{\delta}] \end{cases} \quad \text{and} \quad \mathcal{P}_{\overline{\delta}} = \mathcal{P}_{\delta^*} \cup \{[\delta^*, \overline{\delta})\}.$$

It follows that $(\mathcal{P}_{\overline{\delta}}, \overline{f}, \overline{r}, \overline{v}, \overline{u}) \in \mathcal{U}$, $(\mathcal{P}_{\delta^*}, f^*, r^*, v^*, u^*) \preceq (\mathcal{P}_{\overline{\delta}}, \overline{f}, \overline{r}, \overline{v}, \overline{u})$, but $\delta^* < \overline{\delta}$ which contradicts the maximality of $(\mathcal{P}_{\delta^*}, f^*, r^*, v^*, u^*)$.

Case 2. If $\delta^* \notin \mathcal{O}$ then $\delta^* \notin Z$ which implies that there exist $h_n \downarrow 0$, $v_n \in W^{1,1}([\delta^*, \delta^* + h_n]; X)$ such that $(t, v_n(t)) \in \mathcal{K}$ and $f_n \in L^1(\delta^*, \delta^* + h_n; X)$, $f_n(s) \in F(s, v_n(s))$ for $s \in [\delta^*, \delta^* + h_n]$ and $p_n \in X$ with $p_n \to 0$, such that

$$u^*(\delta^*) + \int_{\delta^*}^{\delta^* + h_n} f_n(s)\, ds + h_n p_n \in K(\delta^* + h_n),$$

for all $n \geq 1, n \in \mathbb{N}$.

We may suppose that $v_n(t) \in S(\xi,\rho)$ for all $n \geq 1$ and $t \in [\delta^*, \delta^* + h_n]$. Indeed, since $u^*(\delta^*) \in S(\xi,\rho)$, for every $v \in ACS_h(\delta^*, u^*(\delta^*), \ell^*)$ we have

$$\|v(t)-\xi\| \leq \|v(t)-v(\delta^*)\| + \|v(\delta^*)-\xi\| \leq \int_{\delta^*}^{h} \ell^*(s)ds + \|u^*(\delta^*)-\xi\| < \rho$$

for h small enough and for all $t \in [\delta^*, \delta^* + h]$.

Let us fix n_0 in order that $\overline{\delta} = \delta^* + h_{n_0}$ satisfy $\overline{\delta} \in (\delta^*, T)$, $h_{n_0} < \varepsilon$, $\|p_{n_0}\| < \varepsilon$ and $\int_{\delta^*}^{\delta^*+h_{n_0}} \ell^*(s)\,ds < \varepsilon$. We define $\mathcal{P}_{\overline{\delta}} = \mathcal{P}_{\delta^*} \cup \{[\delta^*, \overline{\delta})\}$, and

$$\overline{f}(s) = \begin{cases} f^*(s), s \in [\tau, \delta^*] \\ f_{n_0}, s \in (\delta^*, \overline{\delta}] \end{cases}, \overline{r}(s) = \begin{cases} r^*(s), s \in [\tau, \delta^*] \\ p_{n_0}, s \in (\delta^*, \overline{\delta}] \end{cases},$$

$$\overline{v}(s) = \begin{cases} v^*(s) \text{ for } s \in [\tau, \delta^*) \\ v_{n_0}(s) \text{ for } s \in [\delta^*, \overline{\delta}] \end{cases},$$

$$\overline{u}(s) = \begin{cases} u^*(s), s \in [\tau, \delta^*] \\ u(\delta^*) + \int_{\delta^*}^{t} f_{n_0}(s)\,ds + \int_{\delta^*}^{t} p_{n_0}\,ds, \text{for } s \in (\delta^*, \overline{\delta}]. \end{cases}$$

Clearly $(\mathcal{P}_{\overline{\delta}}, \overline{f}, \overline{r}, \overline{v}, \overline{u}) \in \mathcal{U}$, $(\mathcal{P}_{\delta^*}, f^*, r^*, v^*, u^*) \preceq (\mathcal{P}_{\overline{\delta}}, \overline{f}, \overline{r}, \overline{v}, \overline{u})$, but $\delta^* < \overline{\delta}$ which contradicts the maximality of $(\mathcal{P}_{\delta^*}, f^*, r^*, v^*, u^*)$. Hence $\delta^* = T$, and $\mathcal{P}_{\delta^*}$, f^*, r^*, v^* and u^* satisfy all the conditions (i)$\sim$(vi). The proof is complete. □

Definition 5.4. Let $\varepsilon > 0$, Z and $\mathcal{O}$ be as in Lemma 5.1. A quintuple $(\mathcal{P}_T, f, r, v, u)$ satisfying (i)$\sim$(vii) in Lemma 5.1, is called an $(\varepsilon, \mathcal{O})$-*approximate solution of* (1.1).

6. Proof of Theorem 5.1

Proof. Since the necessity follows from Theorem 3.1, we will confine ourselves only to the proof of the sufficiency.

Let $Z \subseteq \mathbb{R}$ be a negligible set including the negligible sets appearing in (H_6) and Definition 5.3. Let $\varepsilon_n \in (0,1)$, with $\varepsilon_n \downarrow 0$, let $(\mathcal{O}_n)_{n\geq 1} \subseteq \mathbb{R}$ be a sequence of open sets, and let ℓ the function in Lemma 5.1. We notice that we may assume with no loss of generality that the sequence $(\mathcal{O}_n)_{n\geq 1}$ is so chosen to satisfy:

(a) $Z \subseteq \mathcal{O}_n$ for each $n \in \mathbb{N}$, $n \geq 1$;

(b) $\mathcal{O}_{n+1} \subseteq \mathcal{O}_n$ and $\lambda([\tau, T] \cap \mathcal{O}_n) \leq \varepsilon_n$ for each $n \in \mathbb{N}$, $n \geq 1$;

(c) $F_{|[(I\setminus\mathcal{O}_n)\times D(\xi,\rho)]\cap\mathcal{K}}$ is strongly-weakly u.s.c., for each $n \in \mathbb{N}$, $n \geq 1$;

Let $\rho > 0$ and $T > \tau$ be as in Lemma 5.1, and such that ρ satisfies Definition 5.3 and let $n \in \mathbb{N}$, $n \geq 1$ be arbitrary but fixed. Let $((\mathcal{P}_T^n, f_n, r_n, v_n, u_n))_n$ be a sequence of $(\varepsilon_n, \mathcal{O}_n)$-approximate solutions of (1.1), sequence whose existence is ensured, again by Lemma 5.1. If $\mathcal{P}_T^n = \{[t_m^n, s_m^n);\ m \in \Gamma_n\}$ with Γ_n finite or at most countable, we denote by $a_n : [\tau, T) \to [\tau, T)$ the step function, defined by $a_n(s) = t_m^n$ for each $s \in [t_m^n, s_m^n)$. Clearly

$$\lim_n a_n(s) = s \tag{6.1}$$

uniformly for $s \in [\tau, T)$.

We will show that, on a subsequence at least, $(u_n)_n$ is uniformly convergent on $[\tau, T]$ to some function u.

We analyze first the case when X is separable. From (vi) in Lemma 5.1, it follows that, for each $k \geq 1$ and $t \in [\tau, T]$, we have

$$\beta\left(\left\{\int_\tau^t r_n(s)\, ds;\ n \geq k\right\}\right) = 0. \tag{6.2}$$

Next, let us observe that

$$\|f_n(t)\| \leq \ell(t) \tag{6.3}$$

for each $n \geq 1$ and a.e for $t \in [\tau, T]$.

Now, from (vi), (vii) (6.3) and (6.1), we conclude that

$$\lim_n \|u_n(a_n(s)) - u_n(s)\| = 0 \text{ and } \lim_n \|u_n(a_n(s)) - v_n(s)\| = 0$$

uniformly for $s \in [\tau, T)$. So we have $\lim_n \|v_n(s) - u_n(s)\| = 0$ uniformly for $s \in [\tau, T)$. Then

$$\beta(\{v_n(s) - u_n(s);\ n \geq k\}) = 0, \tag{6.4}$$

for each $k \geq 1$ and $s \in [\tau, T)$. Let us observe that, in view of (6.2), (6.3), (6.4), (b), (vi) in Lemma 5.1 and Lemma 2.1, we have

$$\beta(\{u_n(t);\ n \geq k\})$$

$$\leq \beta\left(\left\{\int_\tau^t f_n(s)\, ds;\ n \geq k\right\}\right) + \beta\left(\left\{\int_\tau^t r_n(s)\, ds;\ n \geq k\right\}\right)$$

$$\leq \int_{[\tau,t]\setminus\mathcal{O}_k} \beta(\{f_n(s);\ n\geq k\})\,ds + \int_{\mathcal{O}_k} \beta(\{f_n(s);\ n\geq k\})\,ds$$

$$\leq \int_{[\tau,t]\setminus\mathcal{O}_k} \omega(s,\beta(\{v_n(s);\ n\geq k\})\,ds + \int_{\mathcal{O}_k} \ell(s)\,ds$$

$$\leq \int_{[\tau,t]\setminus\mathcal{O}_k} \omega(s,\beta(\{u_n(s);\ n\geq k\} + \{v_n(s)-u_n(s);\ n\geq k\}))\,ds + \int_{\mathcal{O}_k} \ell(s)\,ds$$

$$\leq \int_{[\tau,t]\setminus\mathcal{O}_k} \omega(s,\beta(\{u_n(s);\ n\geq k\}) + \beta(\{v_n(s)-u_n(s);\ n\geq k\}))\,ds + \int_{\mathcal{O}_k} \ell(s)\,ds$$

$$\leq \int_{\tau}^{t} \omega(s,\beta(\{u_n(s);\ n\geq k\}) + \int_{\mathcal{O}_k} \ell(s)\,ds.$$

Since $\beta(\{u_n(t);\ n\geq k\}) = \beta(\{u_n(t);\ n\geq 1\}) := x(t)$, the inequality above rewrites as $x(t)\leq \gamma_k + \int_\tau^t \omega(s,x(s))\,ds$ for each $t\in[\tau,T]$, where $\gamma_k = \int_{\mathcal{O}_k} \ell(s)\,ds$. Passing to the limit for $k\to\infty$ in the inequality above and taking into account that ω is a Carathéodory uniqueness function, it follows that $x(t)=0$ for each $t\in[\tau,T]$. Thus $\beta(\{u_n(t);\ n\geq 1\})=0$ which means that $\{u_n(t);\ \ n\geq 1\}$ is relatively compact. In view of (6.3), it follows that $\{u_n;\ n\in\mathbb{N},\ n\geq 1\}$ is equicontinuous on $[\tau,T]$. By virtue of Arzelà-Ascoli's Theorem 1.3.6, p. 9, in Cârjă, Necula, Vrabie [3], we conclude that $(u_n)_n$ has at least one uniformly convergent subsequence to some function u. But $\lim_n v_n(t)=u(t)$, uniformly for $t\in[\tau,T)$, and hence, for each $k\geq 1$, the set

$$C_k = \overline{\{(t,v_n(t));\ n\geq k,\ t\in[\tau,T)\setminus\mathcal{O}_k\}}$$

is compact. Further, since F is β-compact, it follows that, for each $k\geq 1$, each $n\geq k$ and each $t\in[\tau,T)\setminus\mathcal{O}_k$, $F(t,v_n(t))$ is compact. See Remark 5.2.

By (c), the observation above and Lemma 2.6.1, p. 47, in Cârjă, Necula, Vrabie [3], it follows that, for each $k\geq 1$,

$$B_k := \overline{\mathrm{conv}}\left(\bigcup_{n\geq k}\ \bigcup_{t\in[\tau,T)\setminus\mathcal{O}_k} F(t,v_n(t))\right)$$

is weakly compact. Let us observe that $\|f_n(s)\|\leq \ell(s)$ a.e. for $s\in[\tau,T]$ and $f_n(s)\in B_k$ for each $k\geq 1$, each $n\geq k$ and a.e. for $s\in[\tau,T]\setminus\mathcal{O}_k$. Since $\ell\in L^1(\tau,T;\mathbb{R})$, B_k is weakly compact, and $\lim_k \lambda(\mathcal{O}_k)=0$, by Diestel's Theorem 1.3.8, p. 10, in Cârjă, Necula, Vrabie [3], it follows that, on a subsequence at least, $\lim_k f_n = f$ weakly in $L^1(\tau, T; X)$. As $\lim_n v_n(t) = u(t)$ uniformly for $t \in [\tau, T]$, and, by Lemma 5.1, for each $k \geq 1$, each $n \geq k$ and a.e. for $s \in [\tau, T] \setminus \mathcal{O}_k$, we have $f_n(s) \in F(s, v_n(s))$, from Theorem 3.1.2, p. 88, in Vrabie [7], we conclude that $f(s) \in F(s, u(s))$ for each $k \geq 1$ and a.e. for $s \in [\tau, T] \setminus \mathcal{O}_k$. Since $\lim_k \lambda(\mathcal{O}_k) = 0$, we conclude that $f(s) \in F(s, u(s))$ a.e. for $s \in [\tau, T]$.

Next, again by Lemma 5.1, u_n and f_n satisfy

$$u_n(t) = \xi + \int_\tau^t f_n(s)\, ds + \int_\tau^t r_n(s)\, ds \tag{6.5}$$

for all $n \geq 1$ and $t \in [\tau, T]$.

Finally, passing to the limit both sides in (6.5), for $n \to \infty$, we get

$$u(t) = \xi + \int_\tau^t f(s)\, ds,$$

for each $t \in [\tau, T]$. Thus u is solution of (1.1), and this completes the proof in the case when X is separable.

If X is not separable, we have to observe that there exists a separable and closed subspace $Y \subseteq X$ such that the families: $\{f_n;\ n \geq 1\}$, $\{u_n;\ n \geq 1\}$, $\{v_n;\ n \geq 1\}$ and $\{r_n;\ n \geq 1\}$ are Y-valued. Then, to complete the proof, it suffices to follows the very same arguments as before and to make use of (iv) in Remark 2.1. □

7. Proof of Theorem 5.2

Proof. Let $Z \subseteq \mathbb{R}$ be a negligible set including the negligible set N appearing in (H_6). Let $\varepsilon_n \in (0,1)$, with $\varepsilon_n \downarrow 0$, let $(\mathcal{O}_n)_{n \geq 1} \subseteq \mathbb{R}$ be a sequence of open sets, and let ℓ as in the proof of Theorem 5.1. Clearly, we may assume with no loss of generality that the sequence $(\mathcal{O}_n)_{n \geq 1}$ is so chosen to satisfy (a), (b), (c) in the proof of Theorem 5.1.

Throughout, we will use the notations in the proof of Theorem 5.1. So, let $((\mathcal{P}_T^n, f_n, r_n, v_n, u_n))_n$ be a sequence of $(\varepsilon_n, \mathcal{O}_n)$-approximate solutions of (1.1), sequence whose existence is ensured by Lemma 5.1. First, let us observe that, on a subsequence at least, $(u_n)_n$ is uniformly convergent on $[\tau, T]$ to some function u. Indeed, since $\mathcal{K}$ is locally compact from the left, it follows that the set $\{v_n(t); n \geq 1\}$ is relatively compact. Moreover, recalling that $\lim_n \|v_n(s) - u_n(s)\| = 0$ for $s \in [\tau, T)$, it follows that $\{u_n(t);\ n \geq 1\}$ is relatively compact for all $t \in [\tau, T)$. In addition, by (6.3) and (vii) in

Lemma 5.1, we deduce that $\{u_n;\ n \geq 1\}$ is equicontinuous on $[\tau, T]$. By Arzelà-Ascoli Theorem 1.3.6, p. 9, in Cârjă, Necula, Vrabie [3], we conclude that, on a subsequence at least, $(u_n)_n$ is uniformly convergent on $[\tau, T]$ to some function u.

Observing that $\lim_n v_n(t) = u(t)$ uniformly for $t \in [\tau, T)$, and using the fact that $\mathcal{K}$ is locally closed from the left, we conclude that $(t, u(t)) \in \mathcal{K}$ for each $t \in [\tau, T]$.

Since, by hypothesis, F has weakly compact values and, by (c), for each $k \geq 1$, $F_{|[([\tau,T]\setminus\mathcal{O}_k)\times D(\xi,\rho)]\cap\mathcal{K}}$ is strongly-weakly u.s.c., from Lemma 2.6.1, p. 47, in Cârjă, Necula, Vrabie [3], it follows that the set B_k, defined as in the proof of Theorem 5.1, is weakly compact.

From now on, the proof is identical to that one of Theorem 5.1. $\square$

REFERENCES

1. Cârjă, O.; Monteiro Marques, M.D.P. – *Viability for nonautonomous semilinear differential equations,* J. Differential Equations, 166 (2000), 328–346.
2. Cârjă, O.; Necula, M.; Vrabie, I.I. – *Necessary and sufficient conditions for viability for semilinear differential inclusions,* Trans. Amer. Math. Soc., 361 (2009), 343-390.
3. Cârjă, O.; Necula, M.; Vrabie, I.I. – *Viability, Invariance and Applications*, North-Holland Mathematics Studies, 207, Elsevier, 2007.
4. Mönch, H. – *Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces,* Nonlinear Anal. T.M.A., 4 (1980), 985–999.
5. Necula, M. – *Viability of variable domains for differential equations governed by Carathéodory perturbations of nonlinear m-accretive operators,* An. Ştiinţ. Univ. Al. I. Cuza Iaşi Secţ. I a Mat., 48 (2002), 41–60.
6. Necula, M.; Popescu, M.; Vrabie, I.I. – *Viability for differential inclusions on graphs*, Set-Valued Anal., 16 (2008), 961-981.
7. Vrabie, I.I. – *Compactness Methods for Nonlinear Evolutions*, Second Edition, Pitman Monographs and Surveys in Pure and Applied Mathematics, 75, Longman 1995.
8. Vrabie, I.I. – *C_0-Semigroups and Applications*, North-Holland Publishing Co. Amsterdam, 2003.

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