ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
MATEMATICĂ, Tomul LV, 2009, f.2

A PSEUDO PROJECTIVE CURVATURE TENSOR ON A LORENTZIAN PARA-SASAKIAN MANIFOLD

BY

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Abstract. In this paper, we consider pseudo projectively flat, pseudo projectively conservative and $\phi-$pseudo projectively flat Lorentzian para-Sasakian manifold. It has also been proved that an Einstein Lorentzian para-Sasakian manifold satisfying the relation $R(X,Y).\tilde{P}=0$, where $\tilde{P}$ is pseudo projective curvature tensor, then it is locally isometric with a unit sphere.

Mathematics Subject Classification 2000: 53B05, 53B15, 53C25.
Key words: LP-Sasakian manifold, pseudo projective curvature tensor, $\eta-$ Einstein manifold, pseudo projectively conservative.

1. Preliminaries. An n-dimensional differentiable manifold M^n is Lorentzian para-Sasakian (LP-Sasakian) manifold, if it admits a $(1,1)$-tensor field ϕ, vector field ξ, 1-form η and a Lorentzian metric g, which satisfies

$$\phi^2(X) = X + \eta(X)\xi \tag{1.1}$$

$$\eta(\xi) = -1 \tag{1.2}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y) \tag{1.3}$$

$$g(X,\xi) = \eta(X) \tag{1.4}$$

$$(D_X\phi)(Y) = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi \tag{1.5}$$

and

$$D_X\xi = \phi X, \tag{1.6}$$

for arbitrary vector fields X and Y; where D_X denotes covariant differentiation with respect to g, ([4], [5]).

In an LP-Sasakian manifold M^n with structure (ϕ, ξ, η, g) it is easily seen that

$$(a) \quad \phi\xi = 0 \quad (b) \quad \eta(\phi X) = 0 \quad (c) \quad \text{rank } \phi = (n-1). \tag{1.7}$$

Let us put

$$F(X, Y) = g(\phi X, Y), \tag{1.8}$$

then the tensor field F is symmetric (0, 2)-tensor field. Thus, we have

$$F(X, Y) = F(Y, X) \tag{1.9}$$

$$F(X, Y) = (D_X\eta)(Y) \tag{1.10}$$

and

$$(D_X\eta)(Y) - (D_Y\eta)(X) = 0. \tag{1.11}$$

An LP-Sasakian manifold M^n is said to be Einstein manifold if its Ricci tensor S is of the form

$$S(X, Y) = kg(X, Y) \tag{1.12}$$

where $k = (n-1)$.

An LP-Sasakian manifold M^n is said to be $\eta-$Einstein manifold if its Ricci tensor S is of the form

$$S(X, Y) = \alpha g(X, Y) + \beta\eta(X)\eta(Y). \tag{1.13}$$

for any vector fields X and Y, where α, β are functions on M^n.

Let M^n be an n-dimensional LP-Sasakian manifold with structure (ϕ, ξ, η, g). Then we have ([5], [6]).

$$g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y) \tag{1.14}$$

(1.15)(a) $$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)(X)$$

(1.15)(b) $$R(\xi,X)\xi = X + \eta(X)\xi$$

(1.15)(c) $$R(X,Y)\xi = \eta(Y)X - \eta(X)Y$$

(1.16) $$S(X,\xi) = (n-1)\eta(X)$$

and

(1.17) $$S(\phi X,\phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y)$$

for any vector fields X, Y, Z where $R(X,Y)Z$ is the Riemannian curvature tensor of type $(1,3)$, S is the Ricci tensor of type $(0,2)$. Q is $(1,1)$ Ricci tensor and r is the scalar curvature $g(QX,Y) = S(X,Y)$ for all X, Y. Recently one of the author defined Pseudo projective curvature tensor $\tilde{P}$ on a Riemannian manifold $(M^n, g)(n > 2)$ of type $(1,3)$ as follows (Prasad [9]).

(1.18) $$\begin{aligned}\tilde{P}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y]\\ &-\frac{r}{n}\Big[\frac{a}{n-1} + b\Big][g(Y,Z)X - g(X,Z)Y]\end{aligned}$$

where a and b are constants such that $a, b \neq 0$.

If $a = 1$ and $b = -\frac{1}{(n-1)}$ then (1.18) takes the form

$$\tilde{P}(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)}[S(Y,Z)X - S(X,Z)Y] = P(X,Y)Z.$$

where P is the projective curvature tensor ([7]). Hence the projective curvature P is a particular case of the tensor P. For the reason P is called pseudo projective curvature tensor.

2. An Einstein LP-Sasakian manifold satisfying $\widetilde{P}(X,Y)Z = 0$. In this section we assume that $\tilde{P}(X,Y)Z = 0$. Then from (1.18), we get

(2.1) $$\begin{aligned}&a'R(X,Y,Z,W) = -b[S(Y,Z)g(X,W) - S(X,Z)g(Y,W)]\\ &+\frac{r}{n}\Big[\frac{a}{n-1} + b\Big][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]\end{aligned}$$

where $'R(X, Y, Z, W) = g(R(X, Y)Z, W)$. Putting $X = W = \xi$ in (2.1), we get

$$a'R(\xi, Y, Z, \xi) = -b[-S(Y, Z) - S(\xi, Z)\eta(Y)] + \frac{r}{n}\Big[\frac{a}{n-1} + b\Big][-g(Y, Z) - \eta(Y)\eta(Z)]. \tag{2.2}$$

In view of (1.12), (1.14) and (2.2), we get

$$[a + (n-1)b][r - n(n-1)]g(\phi Y, \phi Z) = 0. \tag{2.3}$$

Thus we see that $g(\phi Y, \phi Z) \neq 0$.

Hence from (2.3), we get $r = n(n-1)$, provided $a+(n-1)b \neq 0$. Hence, we can state the following theorem:

Theorem 1. *The scalar curvature r of a pseudo projectively flat LP-Sasakian manifold M^n is constant, given by $r = n(n-1)$, provided $a+(n-1)b \neq 0$.*

Contracting (1.18) with respect to X, we get

$$\begin{aligned}(C_1^1\tilde{P})(Y, Z) &= aS(Y, Z) + b(n-1)S(Y, Z) \\ &\quad -\frac{r}{n}\Big[\frac{a}{n-1} + b\Big](n-1)g(Y, Z) \\ &= [a + (n-1)b]\Big[S(Y, Z) - \frac{r}{n}\, g(Y, Z)\Big]\end{aligned} \tag{2.4}$$

where contraction of $\tilde{P}(X, Y)Z$ with respect to X is defined by $(C_1^1\tilde{P})(Y, Z)$.

Let us assume that in an LP-Sasakian manifold

$$(C_1^1\tilde{P})(Y, Z) = 0. \tag{2.5}$$

From (2.4) and (2.5), we get $[a + (n-1)b][S(Y, Z) - \frac{r}{n}\, g(Y, Z)] = 0$.

If $a + (n-1)b \neq 0$, then from (2.6), we get $S(Y, Z) = \frac{r}{n}\, g(Y, Z)$, which shows that M^n is an Einstein manifold.

Putting $Z = \xi$ in (2.6), we get $[a+(n-1)b]\,[r-n(n-1)]\eta(Y) = 0$. Thus we see that $\eta(Y) \neq 0$, hence, we get $r = n(n-1)$, provided $a+(n-1)b \neq 0$.

Hence, we can state the following theorem:

Theorem 2. *If in an LP-Sasakian manifold the relation $(C_1^1\tilde{P})(Y, Z) = 0$ hold, then M^n is an Einstein manifold with scalar curvature $r = n(n-1)$, provided $a + (n-1)b \neq 0$.*

3. Einstein LP-Sasakian manifold satisfying (div$\widetilde{P}$)$(X,Y)Z=0$.

Definition. A manifold $(M^n, g)(n > 2)$ is called pseudo projectively conservative if div $\widetilde{P} = 0$, ([3]).

In this section we assume that

$$\text{div}\tilde{P} = 0, \tag{3.1}$$

where div denotes divergence. Now differentiating (1.18) covariantly, we get

$$\begin{aligned}(D_U\tilde{P})(X,Y)Z &= a(D_U R)(X,Y)Z \\ &+b[(D_U S)(Y,Z)X - (D_U S)(X,Z)Y] \\ &-\frac{1}{n}\Big[\frac{a}{n-1}+b\Big][D_U r][g(Y,Z)X - g(X,Z)Y].\end{aligned} \tag{3.2}$$

Contraction of (3.2) gives

$$\begin{aligned}(\text{div}\,\tilde{P})(X,Y)Z &= a(\text{div}R)(X,Y)Z \\ &+b[(D_X S)(Y,Z) - (D_Y S)(X,Z)] \\ &-\frac{1}{n}\Big[\frac{a}{n-1}+b\Big][g(Y,Z)dr(X) - g(X,Z)dr(Y)].\end{aligned} \tag{3.3}$$

But from [2], we have

$$(\text{div}R)(X,Y)Z = (D_X S)(Y,Z) - (D_Y S)(X,Z). \tag{3.4}$$

If LP-Sasakian manifold is an Einstein manifold, then from (1.12) and (3.4), we get

$$(\text{div}R)(X,Y)Z = (D_X S)(Y,Z) - (D_Y S)(X,Z) = 0. \tag{3.5}$$

From (3.3) and (3.5), we get

$$(\text{div}\tilde{P})(X,Y)Z = -\frac{1}{n}\Big[\frac{a}{n-1}+b\Big][g(Y,Z)dr(X) - g(X,Z)dr(Y)]. \tag{3.6}$$

From (3.1) and (3.6), we get $g(Y,Z)dr(X) - g(X,Z)dr(Y) = 0$, provided $a + (n-1)b \neq 0$ which shows that r is constant. Again if r is constant then from (3.6), we get $(\text{div}\tilde{P})(X,Y)Z = 0$.

Hence, we can state the following theorem:

Theorem 3. *An Einstein LP-Sasakian manifold $(M^n, g)(n > 2)$ is pseudo projectively conservative if and only if the scalar curvature is constant, provided $a + (n-1)b \neq 0$.*

4. $\phi-$ Pseudo projectively flat LP-Sasakian manifold

Definition. A differentiable manifold $(M^n, g)(n > 2)$ satisfying the condition ([1])

$$\phi^2 \tilde{P}(\phi X, \phi Y)\phi Z = 0, \tag{4.1}$$

is called $\phi-$pseudo projectively flat LP-Saskian manifold.

Suppose that $(M^n, g)(n > 2)$, is a $\phi-$pseudo projectively flat LP-Sasakian manifold. It is easy to see that $\phi^2 \tilde{P}(\phi X, \phi Y)\phi Z = 0$, holds if and only if $g(\tilde{P}(\phi X, \phi Y)\phi Z, \phi W) = 0$, for any vector fields X, Y, Z, W. By the use of (1.18), $\phi-$ pseudo projectively flat means,

$$\begin{aligned} a'R(\phi X, \phi Y, \phi Z, \phi W) &= -b[S(\phi Y, \phi Z)g(\phi X, \phi W) \\ &\quad -S(\phi X, \phi Z)g(\phi Y, \phi W)] \\ &\quad +\frac{r}{n}\left[\frac{a}{n-1} + b\right][g(\phi Y, \phi Z)g(\phi X, \phi W) \\ &\quad -g(\phi X, \phi Z)g(\phi Y, \phi W)] \end{aligned} \tag{4.2}$$

where ${}'R(X, Y, Z, W) = g(R(X, Y)Z, W)$.

Let $\{e_1, e_2, ..., e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in M^n by using the fact that $\{\phi e_1, \phi e_2, ..., \phi e_{n-1}, \xi\}$ is also a local orthonormal basis, if we put $X = W = e_i$ in (4.2) and sum up with respect to i, then we have

$$\begin{aligned} &\sum_{i=1}^{n-1} a'R(\phi e_i, \phi Y, \phi Z, \phi e_i) = -b\sum_{i=1}^{n-1}[S(\phi Y, \phi Z)g(\phi e_i, \phi e_i) \\ &-S(\phi e_i, \phi Z)g(\phi Y, \phi e_i)] \\ &+\frac{r}{n}\left[\frac{a}{n-1} + b\right][g(\phi Y, \phi Z)g(\phi e_i, \phi e_i) - g(\phi e_i, \phi Z)g(\phi Y, \phi e_i)]. \end{aligned} \tag{4.3}$$

On an LP-Sasakian manifold, we have ([8]).

$$\sum_{i=1}^{n-1} {}'R(\phi e_i, \phi Y, \phi Z, \phi e_i) = S(\phi Y, \phi Z) + g(\phi Y, \phi Z) \tag{4.4}$$

$$\sum_{i=1}^{n-1} S(\phi e_i, \phi e_i) = r + n - 1 \tag{4.5}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi Z) S(\phi Y, \phi e_i) = S(\phi Y, \phi Z) \tag{4.6}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = n + 1 \tag{4.7}$$

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi Z) g(\phi Y, \phi e_i) = g(\phi Y, \phi Z). \tag{4.8}$$

So by virtue of (4.4)-(4.8), the equation (4.3) takes the form,

$$(a + bn) S(\phi Y, \phi Z) = \left[r \left\{ \frac{a}{n-1} + b \right\} - a \right] g(\phi Y, \phi Z). \tag{4.9}$$

By making the use of (1.3) and (1.17) in (4.9), we get

$$\begin{aligned} (a + bn) S(Y, Z) &= \left[r \left\{ \tfrac{a}{n-1} + b \right\} - a \right] g(Y, Z) + \left[r \left\{ \tfrac{a}{n-1} + b \right\} \right. \\ &\quad \left. - n(a + bn) + bn \right] \eta(Y) \eta(Z) \end{aligned} \tag{4.10}$$

which shows that, M^n is an $\eta-$Einstein manifold. Contracting (4.10), we get

$$b[r - n(n-1)] = 0. \tag{4.11}$$

If $b \neq 0$, then from (4.11), we have $r = n(n-1)$.

Hence we can state the following theorem:

Theorem 4. *Let M^n be an n-dimensional ($n > 2$) $\phi-$pseudo projectively flat LP-Sasakian manifold, then M^n is an $\eta-$Einstein manifold with the scalar curvature $r = n(n-1)$, provided $b \neq 0$.*

5. An Einstein LP-Sasakian manifold satisfying $R(X, Y).\widetilde{P} = 0$.

In this section we assume that

$$R(X, Y).\tilde{P}(U, V) W = 0. \tag{5.1}$$

Let the Riemannian manifold M^n be an Einstein manifold, then (1.18) gives

$$\begin{aligned}\tilde{P}(X,Y)Z &= aR(X,Y)Z + bk[g(Y,Z)X - g(X,Z)Y]\\ &-\frac{r}{n}\Big[\frac{a}{n-1}+b\Big][g(Y,Z)X - g(X,Z)Y]\end{aligned} \tag{5.2}$$

Now (5.2) can be written as

$$\begin{aligned}&\tilde{P}(X,Y,Z,W) = a'R(X,Y,Z,W)\\ &+\Big[bk - \frac{r}{n}\Big\{\frac{a}{n-1}+b\Big\}\Big][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)],\end{aligned} \tag{5.3}$$

where $'\tilde{P}(X,Y,Z,W)=g(\tilde{P}(X,Y)Z,W)$ and $'R(X,Y,Z,W)=g(R(X,Y)Z, W)$.

Using (1.14) in (5.3), we get

$$\eta(\tilde{P}(X,Y)Z) = \Big[a+bk-\frac{r}{n}\Big\{\frac{a}{n-1}+b\Big\}\Big][g(Y,Z)\eta(X)-g(X,Z)\eta(Y)] \tag{5.4}$$

Taking $X=\xi$ in (5.4), we get

$$\eta(\tilde{P}(\xi,Y)Z) = \Big[a+bk-\frac{r}{n}\Big\{\frac{a}{n-1}+b\Big\}\Big][-g(Y,Z)-\eta(Y)\eta(Z)] \tag{5.5}$$

$$\eta(\tilde{P}(X,Y)\xi) = 0. \tag{5.6}$$

Now,

$$\begin{aligned}&R(X,Y).\tilde{P}(U,V)W = R(X,Y)\tilde{P}(U,V)W - \tilde{P}(R(X,Y)U,V)W\\ &-\tilde{P}(U,R(X,Y)V)W - \tilde{P}(U,V)R(X,Y)W.\end{aligned}$$

In view of (5.1), we get

$$\begin{aligned}&R(X,Y)\tilde{P}(U,V)W - \tilde{P}(R(X,Y)U,V)W - \tilde{P}(U,R(X,Y)V)W\\ &-\tilde{P}(U,V)R(X,Y)W = 0.\end{aligned} \tag{5.7}$$

Therefore,

$$\begin{aligned}&g(R(\xi,Y)\tilde{P}(U,V)W,\xi) - g(\tilde{P}(R(\xi,Y)U,V)W,\xi)\\ &-g(\tilde{P}(U,R(\xi,Y)V)W,\xi) - g(\tilde{P}(U,V)R(\xi,Y)W,\xi) = 0.\end{aligned}$$

From this it follows that

$$(5.8)\qquad \begin{aligned} &-{}'\tilde{P}(U,V,W,Y) - \eta(Y)\eta(\tilde{P}(U,V)W) \\ &+\eta(U)\eta(\tilde{P}(Y,V)W) + \eta(V)\eta(\tilde{P}(U,Y)W) \\ &+\eta(W)\eta(\tilde{P}(U,V)Y) - g(Y,U)\eta(\tilde{P}(\xi,V)W) \\ &-g(Y,V)\eta(\tilde{P}(U,\xi)W) - g(Y,W)\eta(\tilde{P}(U,V)\xi) = 0. \end{aligned}$$

Putting $Y = U$ in (5.8), we get

$$(5.9)\qquad \begin{aligned} &-{}'\tilde{P}(U,V,W,U) - \eta(U)\eta(\tilde{P}(U,V)W) \\ &+\eta(U)\eta(\tilde{P}(U,V)W) + \eta(V)\eta(\tilde{P}(U,U)W) \\ &+\eta(W)\eta(\tilde{P}(U,V)U) - g(U,U)\eta(\tilde{P}(\xi,V)W) \\ &-g(U,V)\eta(\tilde{P}(U,\xi)W) - g(U,W)\eta(\tilde{P}(U,V)\xi) = 0. \end{aligned}$$

Let $\{e_i\}$, i = 1, 2, 3, ..., n be an orthonormal basis of the tangent space at any point. Then the sum for $1 \leq i \leq n$ of the relation (5.9), for $U = e_i$, gives

$$(5.10)\qquad \begin{aligned} &\eta(\tilde{P}(\xi,V)W) = \frac{1}{n-1}\Big[- aS(V,W) \\ &-\Big\{bk - \frac{r}{n}\Big(\frac{a}{n-1} + b\Big)\Big\}(n-1)g(V,W) \\ &-\Big\{a + bk - \frac{r}{n}\Big(\frac{a}{n-1} + b\Big)\Big\}(n-1)\eta(V)\eta(W)\Big]. \end{aligned}$$

Using (5.4) and (5.10), it follows from (5.8) that

$$(5.11)\qquad \begin{aligned} {}'\tilde{P}(U,V,W,Y) &= \Big[bk - \frac{r}{n}\Big\{\frac{a}{n-1} + b\Big\}\Big][g(V,W)g(Y,U) - g(U,W)g(V,Y)] \\ &+\frac{a}{n-1}[S(V,W)g(Y,U) - S(U,W)g(V,Y)]. \end{aligned}$$

Using (1.12) in (5.11), we get

$$(5.12)\qquad \begin{aligned} {}'\tilde{P}(U,V,W,Y) &= \Big[a + bk - \frac{r}{n}\Big\{\frac{a}{n-1} + b\Big\}\Big][g(V,W)g(Y,U) \\ &-g(U,W)g(V,Y)] \end{aligned}$$

From (5.3) and (5.12), we get

$$a'R(U,V,W,Y) = a[g(V,W)g(Y,U) - g(U,W)g(V,Y)],$$

which gives $'R(U,V,W,Y) = g(V,W)g(Y,U) - g(U,W)g(V,Y)$, provided $a \neq 0$ where $'R(U,V,W,Y) = g(R(U,V)W,Y)$.

Hence we can state the following theorem:

Theorem 5. *If in an Einstein LP-Sasakian manifold, the relation* $R(X,Y).\tilde{P} = 0$ *holds, then it is locally isometric with a unit sphere, provided* $a \neq 0$.

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Received: 23.XI.2007

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