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WEAK-COMPATIBILITY IN MENGER SPACE

BY

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Abstract. The object of this paper is to establish a unique common fixed point theorem for six self mappings using the concept of weak-compatibility in Menger space which is an alternate result of SINGH and JAIN [10]. As an application of this theorem two common fixed point theorems for sequence of self mappings have been established, which significantly generalize the results of DEDEIC and SARAPA [2] and of SEHGAL and BHARUCHA-REID [9] and many others. All the results of this paper are new.

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1. Introduction. There has been a number of generalizations of metric space. One such generalization is Menger space initiated by MENGER [5]. It is a probabilistic generalization in which we assign to any two points x and y, a distribution function $F_{x,y}$. SCHWEIZER and SKLAR [8] studied this concept and gave some fundamental results on this space. It is observed by many authors that contraction condition in metric space may be exactly translated into PM space endowed with the min norm. SEHGAL and BHARUCHA-REID [9] obtained a generalization of Banach contraction principle in a complete Menger space, which is a milestone in developing fixed point theory in Menger space. SESSA [7] initiated the tradition of improving commutativity conditions in fixed-point theorems by introducing the notion of weakly commuting maps in metric spaces. JUNGCK [3] soon enlarged this concept to compatible maps. Recently, JUNGCK and ROHADES [4] termed a pair of self-maps to be coincidentally commuting or equivalently weak-compatible if they commute at their coincidence points.

This concept is most general among all the commutativity concepts in this field as every pair of R-weakly commuting maps is compatible and each pair of compatible maps is weak-compatible but the reverse is not true always.

In this paper a fixed point theorem for six self maps has been established through weak-compatibility, which turns out to be an alternate result of Singh and Jain [10]. Apart from this, two theorems on unique common fixed point of a sequence of maps have been established, which generalize the result of [2], [9] and [11].

2. Preliminaries.

Through out this paper we use all symbols and basic definitions of Mishra [6].

Definition 1. *A mapping $F : R \to R^+$ is called a distribution if it is non-decreasing left continuous with* $\inf\{F(t) : t \in R\} = 0$ *and* $\sup\{F(t) : t \in R\} = 1$. *We shall denote by L the set of all distribution functions while H will always denote the specific distribution function defined by*

$$H(t) = \begin{cases} 0, & t \le 0, \\ 1, & t > 0. \end{cases}$$

Definition 2. *A Probabilistic metric space (PM-space) is an ordered pair (X, F), where X is an abstract set of elements and $F : X \times X \to L$, defined by $(p, q) \longmapsto F_{p,q}$, where L is the set of all distribution functions i.e. $L = \{F_{p,q} | p, q \in X\}$, if the functions $F_{p,q}$ satisfy:*

(a) $F_{p,q}(x) = 1$ for all $x > 0$ if and only if $p = q$;
(b) $F_{p,q}(0) = 0$;
(c) $F_{p,q} = F_{q,p}$;
(d) If $F_{p,q}(x) = 1$ and $F_{q,r}(y) = 1$ then $F_{p,r}(x + y) = 1$.

Definition 3. *A mapping $t : [0, 1] \times [0, 1] \to [0, 1]$ is called a t-norm if*

(a) $t(a, 1) = a; t(0, 0) = 0$;
(b) $t(a, b) = t(b, a)$;
(c) $t(c, d) \ge t(a, b)$ for $c \ge a, d \ge b$;
(d) $t(t(a, b), c) = t(a, t(b, c))$ for all $a, b, c, d \in [0, 1]$.

Definition 4. *A Menger space is a triplet (X, F, t) where (X, F) is PM-space and t is a t-norm such that $\forall p, q, r \in X$ and $\forall x, y \ge 0$*

$$F_{p,r}(x + y) \ge t(F_{p,q}(x), F_{q,r}(y)).$$

Schweizer and Sklar [7] proved that if (X, F, t) is a Menger space with $\sup_{0<x<1} t(x, x) = 1$, then (X, F, t) is a Hausdorff topological space in the topology induced by the family of (ϵ, λ)-neighborhoods $\{U_p(\epsilon, \lambda) : p \in X, \epsilon > 0, \lambda > 0\}$, where $U_p(\epsilon, \lambda) = \{x \in X : F_{x.p}(\epsilon) > 1 - \lambda\}$.

Definition 5. *Let (X, F, t) be a Menger space with $\sup_{0<x<1} t(x, x) = 1$. A sequence $\{p_n\}$ in X is said to converge to a point p in X (written as $p_n \to p$) if for every $\epsilon > 0$ and $\lambda > 0, \exists$ an integer $M(\epsilon, \lambda)$ such that $F_{p_n,p}(\epsilon) > 1-\lambda, \forall n \geq M(\epsilon, \lambda)$. Further, the sequence is said to be a Cauchy sequence in X, if for each $\epsilon > 0$ and $\lambda > 0, \exists$ an integer $M(\epsilon, \lambda)$ such that $F_{p_n,p_m}(\epsilon) > 1 - \lambda, \forall n, m \geq M(\epsilon, \lambda)$. A Menger space (X, F, t) is said to be complete if every Cauchy sequence in it converges to a point of it.*

A complete metric space can be treated as a complete Menger space in the following way.

Proposition 6. *If (X, d) is a metric space then the metric d induces a mapping $X \times X \to L$, defined by $F_{p,q}(x) = H(x - d(p, q)), \forall p, q \in X$ and $x \in R$. Further, if $t : [0, 1] \times [0, 1] \to [0, 1]$ is defined by $t(a, b) = \min\{a, b\}$, then (X, F, t) is a Menger space. It is complete if (X, d) is complete. The space (X, F, t) so obtained is called induced Menger space.*

Proposition 7. *In a Menger space (X, F, t) if $t(x, x) \geq x, \forall x \in [0, 1]$ then $t(a, b) = \min\{a, b\}, \forall a, b \in [0, 1]$. In the following T_M will denote the minimum t-norm.*

Definition 8 ([10, 12]). *Self mappings A and S of a Menger space (X, F, t) are said to be weak compatible if they commute at their coincidence points i.e. $Ax = Sx$ for some $x \in X$, then $ASx = SAx$.*

Definition 9. *Self mappings A and S of a Menger space (X, F, t) are called compatible if $F_{ASx_n,SAx_n}(\epsilon) \to 1, \forall \epsilon > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Sx_n \to u$, for some $u \in X$, as $n \to \infty$.*

In the following example self mappings A and S are weak-compatible but they are not compatible.

Example 10. Let (X, d) be a metric space where $X = [0, 1]$ and (X, F, t) be the induced Menger space with $F_{p,q}(\epsilon) = H(\epsilon - d(p, q)), \forall p, q \in$

X and $\epsilon > 0$. Let I be the identity map on X. Define self maps S and T as follows:

$$A(x) = \begin{cases} 2 - x, & x \in [0,1) \\ 2, & x \in [1,2] \end{cases} \qquad S(x) = \begin{cases} x, & x \in [0,1) \\ 2, & x \in [1,2] \end{cases}$$

and $x_n = 1/2 - 1/n$. Now, $F_{Ax_n,1}(\epsilon) = H(\epsilon - \frac{1}{n})$. Hence $\lim_{n\to\infty} F_{Ax_n,1}(\epsilon) = 1$. Thus, $Ax_n \to 1$. Similarly $Sx_n \to 1$ as $n \to \infty$. Again $F_{ASx_n,SAx_n}(\epsilon) = H(\epsilon-(1-1/n)) \lim_{n\to\infty} F_{ASx_n,SAx_n}(\epsilon) = H(\epsilon-1) \neq 1, \forall \epsilon > 0$. Hence (A, S) is not compatible. Also set of coincident points of A and S is $[1, 2]$. Now for any $x \in [1, 2], Ax = Sx = 2$ and $AS(x) = A(2) = 2 = S(2) = SA(x)$.

Thus mappings A and S are weak-compatible but they are not compatible. From above example, it is obvious that the concept of weak-compatibility in Menger space is more general than one of compatibility.

Let $\Phi = \{\phi : R^+ \to R^+ : \phi$ is upper semi continuous with $\phi(x) < x$ for each $x > 0$ and $\phi(0) = 0\}$, where R^+ is the set of non-negative real numbers.

Lemma 11 ([1]). *Let $\phi \in \Phi$, then there exists a strictly increasing continuous function $\psi : R^+ \to R^+$ such that $\phi(u) \leq \psi(u) < u$ for each $u > 0, \lim_{u\to\infty} \psi(u) = \infty$ and $\psi(u) > 0$ for each $u > 0$.*

Remark 12 ([1]). In the above case the function ψ is invertible. If for each $u > 0$, we denote $\psi^0(u) = u$ and $\psi^{-n}(u) = \psi(\psi^{-n+1}(u))$ for each $n \in N$, then $\lim_{n\to\infty} \psi^{-n}(u) = \infty$.

Proposition 13. *Let $\{x_n\}$ be a Cauchy sequence in a Menger space (X, F, t) with continuous t-norm t. If the subsequence $\{x_{2n}\}$ converges to x in X, then $\{x_n\}$ also converges to x.*

Proof. As $\{x_{2n}\}$ converges to x, we have

$$F_{x_n,x}(\epsilon) \geq t(F_{x_n,x_{2n}}(\frac{\epsilon}{2}), F_{x_{2n},x}(\frac{\epsilon}{2})).$$

Taking limit as $n \to \infty$ we get $\lim_{n\to\infty} F_{x_n,x}(\epsilon) \geq t(1, 1)$, which gives $\lim_{n\to\infty} F_{x_n,x}(\epsilon) = 1, \forall \epsilon > 0$ and the result follows. □

3. Main results

Theorem 14. *Let A, B, S, T, L and M are self-maps on a Menger space (X, F, t) with continuous t-norm t satisfying:*

(1) $L(X) \subseteq ST(X), M(X) \subseteq AB(X)$;

(2) $AB = BA, ST = TS, LB = BL, MT = TM$;

(3) *One of $ST(X), M(X), AB(X)$ or $L(X)$ is complete;*

(4) *The pairs (L, AB) and (M, ST) are weak-compatible;*

(5) *There exists $\phi \in \Phi$ such that $F_{Lp,Mq}(\phi(x)) \geq \min\{F_{ABp,Lp}(x), F_{STq,Mq}(x), F_{STq,Lp}(\beta x), F_{ABp,Mq}((2-\beta)x), F_{ABp,STq}(x)\}$, for all $p, q \in X$, $\beta \in (0, 2)$ and $x > 0$. Then A, B, S, T, L and M have a unique common fixed point in X.*

Note: In view of Lemma 11 and Remark 12 the function ϕ can be taken to be increasing and invertible and we have $\lim_{n\to\infty} \phi^{-n}(u) = \infty$, for $u > 0$.

Proof. Let $x_0 \in X$. From (1) there exists $x_1, x_2 \in X$ such that $Lx_0 = STx_1 = y_0$ and $Mx_1 = ABx_2 = y_1$. Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $Lx_{2n} = STx_{2n+1} = y_{2n}$ and $Mx_{2n+1} = ABx_{2n+2} = y_{2n+1}, n = 0, 1, 2, \ldots$. First of all we show that $\{y_n\}$ is a Cauchy sequence in X. Putting $p = x_{2n}, q = x_{2n+1}$ for $x > 0$ and $\beta = 1 - \alpha$ with $\alpha \in (0, 1)$ in (5) we get

$$F_{Lx_{2n},Mx_{2n+1}}(\phi(x)) = F_{y_{2n},y_{2n+1}}(\phi(x))$$

$$\geq \min \begin{Bmatrix} F_{ABx_{2n},Lx_{2n}}(x), F_{STx_{2n+1},Mx_{2n+1}}(x), F_{STx_{2n+1},Lx_{2n}}((1-\alpha)x), \\ F_{ABx_{2n},Mx_{2n+1}}((1+\alpha)x), F_{ABx_{2n},STx_{2n+1}}(x) \end{Bmatrix}$$

$$= \min \left\{F_{y_{2n-1},y_{2n}}(x), F_{y_{2n},y_{2n+1}}(x), 1, F_{y_{2n-1},y_{2n+1}}((1+\alpha)x), F_{y_{2n-1},y_{2n}}(x)\right\}$$

$$= \min \left\{F_{y_{2n-1},y_{2n}}(x), F_{y_{2n},y_{2n+1}}(x), F_{y_{2n-1},y_{2n}}(x), F_{y_{2n},y_{2n+1}}(\alpha x)\right\}$$

$$\geq \min\{F_{y_{2n-1},y_{2n}}(x), F_{y_{2n},y_{2n+1}}(x), F_{y_{2n},y_{2n+1}}((\alpha)x)\}.$$

Letting $\alpha \to 1$ we get

$$F_{y_{2n},y_{2n+1}}(\phi(x)) \geq \min\{F_{y_{2n-1},y_{2n}}(x), F_{y_{2n},y_{2n+1}}(x), F_{y_{2n},y_{2n+1}}(x)\}$$

$$= \min\{F_{y_{2n-1},y_{2n}}(x), F_{y_{2n},y_{2n+1}}(x)\}.$$

Hence $F_{y_{2n},y_{2n+1}}(\phi(x)) \geq \min\{F_{y_{2n-1},y_{2n}}(x), F_{y_{2n},y_{2n+1}}(x)\}$. Similarly, we can have $F_{y_{2n+1},y_{2n+2}}(\phi(x)) \geq \min\{F_{y_{2n},y_{2n+1}}(x), F_{y_{2n+1},y_{2n+2}}(x)\}$. Therefore, for all n even or odd we have

$$F_{y_n,y_{n+1}}(\phi(x)) \geq \min\{F_{y_{n-1},y_n}(x), F_{y_n,y_{n+1}(x)}\}.$$

Consequently, $F_{y_n,y_{n+1}}(x) \geq \min\{F_{y_{n-1},y_n}(\phi^{-1}(x)), F_{y_n,y_{n+1}}(\phi^{-1}(x))\}$. By repeated application of above inequality we get

$$F_{y_n,y_{n+1}}(x) \geq \min\{F_{y_{n-1},y_n}(\phi^{-1}(x)), F_{y_n,y_{n+1}}(\phi^{-i}(x))\}.$$

Since $F_{y_n,y_{n+1}}(\phi^{-i}(x)) \to 1$ as $i \to \infty$ it follows $F_{y_n,y_{n+1}}(x) \geq F_{y_{n-1},y_n}(\phi^{-1}(x))$, $\forall n$ and $x > 0$. Therefore, $F_{y_n,y_{n+1}}(x) \geq F_{y_{n-1},y_n}(\phi^{-1}(x)) \geq F_{y_{n-2},y_{n-1}}(\phi^{-2}(x))$ $\geq \ldots \geq F_{y_0,y_1}(\phi^{-n}(x))$ Taking limit as $n \to \infty$, we have $\lim_{n\to\infty} F_{y_n,y_{n+1}}(x)$ $= 1$. Now, $F_{y_n,y_{n+2}}(x) \geq \min(F_{y_n,y_{n+1}}(x/2), F_{y_{n+1},y_{n+2}}(x/2))$. Taking limit as $n \to \infty$ we have $\lim_{n\to\infty} F_{y_n,y_{n+2}}(x) = 1$. By using the method of induction we can show that $\lim_{n\to\infty} F_{y_n,y_{n+p}}(x) = 1, \forall p > 0$. Thus $\{y_n\}$ is a Cauchy sequence in X.

Case I. *ST(X) is complete.* In this case $\{y_{2n}\} = \{STx_{2n+1}\}$ is a Cauchy sequence in $ST(X)$, which is complete. Thus $\{y_{2n+1}\}$ converges to some $z \in ST(X)$. By Proposition 13 we have

(1) $$\{Mx_{2n+1}\} \to z \quad \text{and} \quad \{STx_{2n+1}\} \to z$$

(2) $$\{Lx_{2n}\} \to z \quad \text{and} \quad \{ABx_{2n}\} \to z.$$

As $z \in ST(X)$ there exists $u \in X$ such that $z = STu$.

Step I. Putting $p = x_{2n}, q = u$ with $\beta = 1$ in (5) we get

$$F_{Lx_{2n},Mu}(\phi(x)) \geq \min\left\{\begin{matrix} F_{ABx_{2n},Lx_{2n}}(x), F_{STu,Mu}(x), F_{STu,Lx_{2n}}(x), \\ F_{ABx_{2n},Mu}(x), F_{ABx_{2n},STu}(x) \end{matrix}\right\}.$$

Taking limit as $n \to \infty$ we get

$$F_{z,Mu}(\phi(x)) \geq \min\{F_{z,z}(x), F_{z,Mu}(x), F_{z,z}(x), F_{z,Mu}(x), F_{z,z}(x)\}$$

i.e. $F_{z,Mu}(\phi(x)) \geq F_{z,Mu}(x), \forall x > 0$, which gives $Mu = z$. Hence $STu = Mu = z$. As (M, ST) is weak-compatible so we have $Mz = STz$.

Step II. Putting $p = x_{2n}, q = z$ with $\beta = 1$ in (5) we get

$$F_{Lx_{2n},Mz}(\phi(x)) \geq \min\left\{\begin{matrix} F_{ABx_{2n},Lx_{2n}}(x), F_{STz,Mz}(x), F_{STz,Lx_{2n}}(x), \\ F_{ABx_{2n},Mz}(x), F_{ABx_{2n},STz}(x) \end{matrix}\right\}.$$

Letting $n \to \infty$ and using $STz = Mz$ we get

$$F_{z,Mz}(\phi(x)) \geq \min\left\{F_{z,z}(x), F_{Mz,Mz}(x), F_{Mz,z}(x), F_{z,Mz}(x), F_{z,Mz}(x)\right\}$$

i.e. $F_{z,Mz}(\phi(x)) \geq F_{z,Mz}(x)$ which gives $z = Mz$.

Step III. Putting $p = x_{2n,}q = Tz$ with $\beta = 1$ in (5) we get

$$F_{Lx_{2n},MTz}(\phi(x)) \geq \min\begin{Bmatrix}F_{ABx_{2n},Lx_{2n}}(x), F_{STTz,MTz}(x), F_{STTz,Lx_{2n}}(x),\\ F_{ABx_{2n},MTz}(x), F_{ABx_{2n},STTz}(x)\end{Bmatrix}$$

As $MT = TM$ and $ST = TS$ we have $MTz = TMz = Tz$ and $ST(Tz) = T(STz) = Tz$. Letting $n \to \infty$ we get

$$F_{z,Tz}(\phi(x)) \geq \min\left\{F_{z,z}(x), F_{Tz,Tz}(x), F_{Tz,z}(x), F_{z,Tz}(x), F_{z,Tz}(x)\right\}$$

i.e. $F_{z,Tz}(\phi(x)) \geq F_{z,Tz}(x)$ which gives $z = Tz$. Now $STz = Tz = z$ implies $Sz = z$. Hence $Sz = Tz = Mz = z$.

Step IV. As $M(X) \subseteq AB(X)$ there exists $v \in X$ such that $z = Mz = ABv$. Putting $p = v, q = x_{2n+1}$ with $\beta = 1$ in (5) we get

$$\begin{aligned}&F_{Lv,Mx_{2n+1}}(\phi(x))\\ &\geq \min\begin{Bmatrix}F_{ABv,Lv}(x), F_{STx_{2n+1},MTx_{2n+1}}(x), F_{STx_{2n+1},Lv}(x),\\ F_{ABv,MTx_{2n+1}}(x), F_{ABv}, STx_{2n+1}(x)\end{Bmatrix}.\end{aligned}$$

Letting $n \to \infty$ we get,

$$F_{Lv,z}(\phi(x)) \geq \min\left\{F_{z,Lv}(x), F_{z,z}(x), F_{z,Lv}(x), F_{z,z}(x), F_{z,z}(x)\right\}$$

i.e. $F_{z,Lv}(\phi(x)) \geq F_{z,Lv}(x)$ which gives $z = Lv$. Therefore, $ABz = Lz$.

Step V. Putting $p = z, q = x_{2n+1}$ with $\beta = 1$ in (5) we get

$$\begin{aligned}&F_{Lz,Mx_{2n+1}}(\phi(x))\\ &\geq \min\begin{Bmatrix}F_{ABz,Lz}(x), F_{STx_{2n+1},MTx_{2n+1}}(x), F_{STx_{2n+1},Lz}(x),\\ F_{ABz,MTx_{2n+1}}(x), F_{ABz}, STx_{2n+1}(x)\end{Bmatrix}.\end{aligned}$$

Letting $n \to \infty$ we get,

$$F_{Lz,z}(\phi(x)) \geq \min\left\{F_{z,Lz}(x), F_{z,z}(x), F_{z,Lz}(x), F_{z,z}(x), F_{z,z}(x)\right\}$$

i.e. $F_{z,Lz}(\phi(x)) \geq F_{z,Lz}(x)$ which gives $z = Lz$. Therefore, $ABz = Lz = z$.

Step VI. Putting $p = Bz, q = x_{2n+1}$ with $\beta = 1$ in (5) we get

$$F_{LBz,Mx_{2n+1}}(\phi(x)) \geq \min\begin{Bmatrix} F_{ABBz,LBz}(x), F_{STx_{2n+1},MT_{2n+1}}(x), F_{STx_{2n+1},LBz}(x), \\ F_{ABBz,MTx_{2n+1}}(x), F_{ABBz}, STx_{2n+1}(x) \end{Bmatrix}.$$

As $BL = LB, AB = BA$, so we have $L(Bz) = B(Lz) = Bz$ and $AB(Bz) = B(ABz) = Bz$. Letting $n \to \infty$ we get

$$F_{Bz,z}(\phi(x)) \geq \min\{F_{Bz,Bz}(x), F_{z,Bz}(x), F_{Bz,z}(x), F_{Bz,z}(x), F_{z,z}(x)\}$$

i.e. $F_{z,Bz}(\phi(x)) \geq F_{z,Bz}(x)$. It gives $Bz = z$ and $ABz = z$ implies $Az = z$. Therefore $Az = Bz = Lz = z$. Combining the results from different steps we have $Az = Bz = Lz = Mz = Tz = Sz = z$. Hence the six self-maps have a common fixed point in this case. Case when $L(X)$ is complete follows from above case as $L(X) \subseteq ST(X)$.

Case II. *AB(X) is complete.* This case follows by symmetry. As $M(X) \subseteq AB(X)$, therefore the result also holds when $M(X)$ is complete.

Uniqueness. Let u be another common fixed point of A, B, L, M, S and T, then $Au = Bu = Lu = Su = Tu = Mu = u$. Putting $p = z, q = u$ in (5) with $\beta = 1$ we get

$$F_{Lz,Mu}(\phi(x)) \geq \min\{F_{ABz,Lz}(x), F_{STu,Mu}(x), F_{STu,Lz}(x), F_{ABz,Mu}(x), F_{ABz,STu}(x)\}$$

i.e $F_{z,u}(\phi(x)) \geq F_{z,u}$. It gives $z = u$. Therefore, z is a unique common fixed point of A, B, L, M, S and T. □

If we define $\phi(x) = kx$ for some $k \in (0, 1)$ in Theorem 14 we get

Corollary 15. *Let A, B, S, T, L and M are self mappings on a Menger space (X, F, t), with continuous t-norm T_M satisfying* (1), (2), (3), (4) *and* (6)*, there exists $k \in (0, 1)$ such that $F_{Lp,Mq}(kx) \geq \min\{F_{ABp,Lp}(x), F_{STq,Mq}(x), F_{STq,Lp}(\beta x), F_{ABp,Mq}((2-\beta)x), F_{ABp,STq}(x)\}$, for all $p, q \in X$, $\beta \in (0, 2)$ and $x > 0$. Then A, B, S, T, L and M have a unique common fixed point in X.*

In [10] Singh and Jain have proved the following result:

Theorem 16 ([10]). *Let A, B, S, T, L and M are self-maps on a complete Menger space (X, F, t) with $t(a, a) \geq a$, for all $a \in [0, 1]$, satisfying:*

(1) $L(X) \subseteq ST(X), M(X) \subseteq AB(X)$.

(2) $AB = BA, ST = TS, LB = BL, MT = TM$.

(3) (L, AB) *is compatible and* (M, ST) *is weak-compatible.*

(4) *Either* AB *or* L *is continuous.*

(5) *There exists* $k \in (0, 1)$ *such that* $F_{Lp,Mq}(kx) \geq \min\{F_{ABp,Lp}(x), F_{STq,Mq}(x), F_{STq,Lp}(\beta x), F_{ABp,Mq}(2-\beta)x, F_{ABp,STq}(x)\}$, *for all* $p, q \in X, \beta \in (0, 2)$ *and* $x > 0$. *Then* A, B, S, T, L *and* M *have a unique common fixed point in* X.

Remark 17. In view of proposition 7, $t(a, b) = \min\{a, b\}$. Thus corollary 15 is an alternate result of the above quoted result of [10] reducing the compatibility of the pair (L, AB) to its weak-compatibility and dropping the condition (4) of continuity in a Menger space with continuous t-norm.

If we take $B = T = I$, the identity map and $\phi(x) = kx$, $k \in (0, 1)$ in Theorem 14 we get,

Corollary 18. *Let* A, S, L *and* M *are self-maps on a Menger space* (X, F, t), *with continuous t-norm* t *satisfying*

(1) $L(X) \subseteq S(X), M(X) \subseteq A(X)$;

(2) *One of* $S(X), M(X), A(X)$ *or* $L(X)$ *is complete;*

(3) *The pairs* (L, A) *and* (M, S) *are weak-compatible;*

(4) *There exists* $k \in (0, 1)$ *such that,* $F_{Lp,Mq}(kx) \geq \min\{F_{Ap,Lp}(x), F_{Sq,Mq}(x), F_{Sq,Lp}(\beta x), F_{Ap,Mq}(2-\beta)x, F_{Ap,Sq}(x)\}$ *for all* $p, q \in X, \beta \in (0, 2)$ *and* $x > 0$. *Then* A, S, L *and* M *have the unique common fixed point in* X.

Example 19. (Theorem 14 and Corollary 18) Let (X, d) be a metric space where $X = [0, 1]$ and (X, F, t) be the induced Menger space with $F_{p,q}(\epsilon) = H(\epsilon - d(p, q)), \forall p, q \in X$ and $\epsilon > 0$. Define self maps L, M, A and S as follows:

$$L(x) = M(x) = \begin{cases} 0, & x \in [0, \frac{3}{4}] \\ 1 - x, & \text{otherwise} \end{cases}$$

$$A(x) = \begin{cases} 0, & x \in [0, \frac{2}{3}] \\ 1 - x, & \text{otherwise} \end{cases}$$

$$S(x) = \begin{cases} 0, & x \in [0, \frac{1}{2}] \\ 1 - x, & \text{otherwise.} \end{cases}$$

Then $L(X) = M(X) = [0, \frac{1}{4}), A(X) = [0, \frac{1}{3}]$ and $S(X) = [0, \frac{1}{2})$. Hence the containment condition (1) is satisfied. Also, the pairs (L, A) and (M, S) are weak-compatible and $A(X)$ is complete. Further for $k = \frac{1}{2}$ the condition

(4) is satisfied. Thus all the conditions of corollary 18 are satisfied and 0 is the unique common fixed point of the mappings L, M, A and S.

Remark 20. This problem can not be solved by the quoted result of [10] as all the mappings are discontinuous in the above example.

If we take $A = I$, the identity map in Corollary 18 we get

Corollary 21. *Let S, L and M are self-maps on a complete Menger space (X, F, t), with continuous t-norm t satisfying*

(1) $L(X) \subseteq S(X)$;

(2) *The pair (M, S) is weak-compatible;*

(3) *There exists $k \in (0,1)$ such that $F_{Lp,Mq}(kx) \geq \min\{F_{p,Lp}(x),$ $F_{Sq,Mq}(x)$, $F_{Sq,Lp}(\beta x)$, $F_{p,Mq}(2-\beta)x$, $F_{p,Sq}(x)\}$, for all $p, q \in X, \beta \in (0,2)$ and $x > 0$. Then S, L and M have the unique common fixed point in X.*

If we take $S = A = I$, the identity map on X and writing $L = T_i$ and $M = T_j$ in Corollary 18, we have

Corollary 22. *Let T_i and T_j be self-maps on a complete Menger space (X, F, t),with continuous t-norm t and there exist $k \in (0,1)$ such that, $F_{T_ip,T_jq}(kx) \geq \min\{F_{p,T_ip}(x), F_{q,T_jq}(x), F_{q,T_ip}(\beta x), F_{p,T_jq}(2-\beta)x, F_{p,q}(x)\}$ for all $p, q \in X, \beta \in (0,2)$ and $x > 0$.*

Then T_i and T_j have a unique common fixed point in X.

In [2] Dedeic and Sarapa established the following theorem:

Theorem 23 ([2]). *Let $\{T_n\}(n \in N)$ be a sequence of self mappings of a complete Menger space (X, F, t) with continuous t norm t, where $t(a,b) = Min\{a,b\}$, for $a, b \in [0,1]$ such that, for some $r \in N$ and some $k \in (0,1)$ we have $F_{T^r_{n+1}(p),T^r_n(q)}(k\varepsilon) \geq F_{p,q}(\epsilon)$ and $F_{T^r_n(p),T_m(q)}(k\varepsilon) \geq F_{p,q}(\epsilon)$, for all $n, m \in N$, for all $p, q \in X$ and for every $\epsilon > 0$. Then the sequence $\{T_n\}$ has a unique common fixed point in X. The following theorem is a more complete result with an additional self map S and a more general contraction.*

Theorem 24. *Let $\{A_n\}$ be a sequence of self-maps and S be a self-map on a complete Menger space (X, F, t) with continuous t-norm such that for some positive integers m_n*

(1) $A_nS = SA_n, \forall n$;

(2) $A_n^{m_n}(X) \subseteq S(X), \forall n$;

(3) *there exists* $k_i \in (0,1)$ *such that, for all* $p, q \in X$ *and* $0 < \beta < 2$ *and* $x > 0$ *and for all* i,

$$F_{A_i^{m_i}p, A_{i+1}^{m_{i+1}}q}(k_i(x)) \geq \min \begin{Bmatrix} F_{A_i^{m_i}p,p}(x), F_{A_{i+1}^{m_{i+1}}q,Sq}(x), F_{p,S_q}(x), \\ F_{A_i^{m_i}p,Sq}(\beta x), F_{A_{i+1}^{m_{i+1}}q,p}((2-\beta)x) \end{Bmatrix},$$

for all $p, q \in X, \beta \in (0,2)$ *and* $x > 0$. *Then the sequence* $\{A_n\}$ *and* S *have a unique common fixed point in* X.

Proof. From (1) it follows that $A_n^{m_n}S = SA_n^{m_n}$, for all n and thus the pair $(A_n^{m_n}, S)$ is weak-compatible for all n. Thus from Corollary 21 in view of (3) each consecutive pair $(A_i^{m_i}, A_{i+1}^{m_{i+1}})$ and S have a unique common fixed point in X. Let u_1 be the common fixed point of the pair $(A_1^{m_1}, A_2^{m_2})$ and S i.e. $A_1^{m_1}(u_1) = A_2^{m_2}(u_1) = Su_1 = u_1$ and u_2 be the common fixed point of the pair $(A_2^{m_2}, A_3^{m_3})$ and S i.e. $A_2^{m_2}(u_2) = A_3^{m_3}(u_2) = Su_2 = u_2$. It follows that $A_1^{m_1}(A_1u_1) = A_1u_1$ and $S(A_1u_1) = A_1Su_1 = A_1u_1$ and $S(A_2u_1) = A_2Su_1 = A_2u_1$. Taking $p = A_1u_1$ and $q = u_1$ in (3) with $i = 1$ and $\alpha = 1$ we have, $F_{A_1^{m_1}A_1u_1, A_2^{m_{i+1}}u_1}(k_1x) \geq \min\{F_{A^{m_1}A_1u_1, A_1u_1}(x), F_{A_2^{m_2}u_1,Su_1}(x), F_{A_1u_1,Su_1}(x), F_{A_1^{m_1}A_1u_1,Su_1}(x), F_{A_2^{m_2}u_1,A_1u_1}(x)\}$ i.e.

$$F_{A_1u_1,u_1}(k_1(x)) \geq \min\{1, 1, F_{A_1u_1,Su_1}(x), F_{A_1u_1,Su_1}(x), F_{u_1,A_1u_1}(x)\}$$

which gives $F_{A_1u_1,u_1}(k_1(x)) \geq F_{A_1u_1,u_1}(x), \forall x > 0$, which gives $A_1u_1 = u_1$. Similarly, by taking $p = u_1$ and $q = A_2u_1$ in (3) with $i = 1$ and $\alpha = 1$ we get $A_2u_1 = u_1$. Hence u_1 is a common fixed point of the pair (A_1, A_2) and S. If v is another fixed point of the pair (A_1, A_2) and S then $A_1^{m_1}v = A_2^{m_2}v = v = Sv$. Hence v becomes a common fixed point of $(A_1^{m_1}, A_2^{m_2})$ and S, which is unique. Hence, $u_1 = v$. Thus u_1 is the unique common fixed of the maps A_1, A_2 and S. Similarly,u_2 the common fixed of the pair $(A_2^{m_2}, A_3^{m_3})$ and S becomes the unique common fixed point of the maps A_2, A_3 and S. Again by putting $p = u_1$ and $q = u_2$ in (3) with $i = 1$ and $\alpha = 1$, we have

$$F_{A_1^{m_1}u_1, A_2^{m_2}u_2}(k_2x)$$

$$\geq \min \begin{Bmatrix} F_{A_1^{m_1}u_1,u_1}(x), F_{A_2^{m_2}u_2,Su_2}(x), F_{u_1,Su_2}(x), \\ F_{A_2^{m_1}p,Su_2}(x), F_{A_2^{m_2}q,u_1}(x) \end{Bmatrix}$$

i.e. $F_{A_1^{m_1}u_1, A_2^{m_2}u_2}(k_2x) \geq \min\ F_{u_1,u_2}(x), \forall x > 0$. It gives $u_1 = u_2$. Thus the unique common fixed point of the maps A_1, A_2 and S is the same as that of A_2, A_3 and S and so on. Hence self maps of the sequence $\{An\}$ and S have a unique common fixed point. □

Remark 25. If in Theorem 24 we take $m_i = r$, for all i, $S = I$ and restrict the right hand side of the contractive condition to the third factor, we obtain the quoted result of [2]. It is to be noted that Theorem 24 here drops the assumption $F_{T_n^r(p),T_m^r(q)}(k\varepsilon) \geq F_{p,q}(\epsilon)$ of [2] totally.

Example 26. Let (X, F, t) be the Menger space as in Example 19. Define the sequence of self mappings $\{A_n\}$ and S as follows

$$A_n(x) = \begin{cases} 0, & x \in [0, \frac{1}{n}] \\ \frac{1}{n+1}, & \text{otherwise } \forall n \end{cases}, \qquad S(x) = \begin{cases} \frac{1}{n} & x \in (\frac{1}{n+1}, \frac{1}{n}] \\ 0, & \text{otherwise} \end{cases}.$$

Then $S(X) = \{0, \frac{1}{2}, \frac{1}{3}, \ldots\}$ and $A_n(X) = \{0, \frac{1}{n+1}\}, A_n^2(X) = \{0\}, \forall n$. Thus $A_n^2(X) \subseteq S(X), \forall n$. Also, $A_nS = SA_n, , \forall n$. On taking $m_n = 2, \forall n$ the contractive condition (3) is satisfied for $k_i = \frac{1}{2}$ for all i. Thus all the conditions of theorem 24 are satisfied and 0 is the unique common fixed point of all the maps of the sequence $\{A_n\}$ and S.

Remark 27. In the above example we have an additional map S. Hence it is not solvable by the quoted result of [2].

On the lines of above theorem the following result can be proved:

Corollary 28. *Let $\{A_n\}$ be a sequence of self-maps and S be a self-map on a complete Menger space (X, F, t), with continuous t-norm such that*
(1) $A_n(X) \subseteq S(x), \forall n$;
(2) *The pairs of mappings (A_n, S) are weak-compatible;*
(3) *there exists $k_i \in (0, 1)$ for each i, such that for all $p, q \in X$ and $0 < \alpha < 2$ and $x > 0$,*

$$F_{A_ip,A_{i+1}q}(kx) \geq \min \begin{Bmatrix} F_{p,A_ip}(x), & F_{Sq,A_{i+1}q}(x), F_{p,Sq}(x), \\ F_{A_ip,Sq}(\alpha x), & F_{A_{i+1}q,p}((2-\alpha)x) \end{Bmatrix}$$

Then the sequence $\{A_n\}$ and S have a unique common fixed point in X.

Example 29. Let (X, F, t) be the Menger space as in Example 19. Define the sequence of self mappings $\{A_n\}$ and S as follows

$$A_n(x) = \begin{cases} 0, & x \in [0, \frac{1}{n+2}] \\ \frac{1}{n+3}, & \textit{otherwise}, \end{cases} \forall n \quad S(x) = \begin{cases} x & x \in [0, \frac{1}{3}] \\ \frac{1}{3}, & \textit{otherwise} \end{cases}$$

Then $S(X) = [0, \frac{1}{3}]$ and $A_n(X) = \{0, \frac{1}{n+2}\}$. Thus $A_n(X) \subseteq S(X), \forall n$. Also, the pair (A_n, S) are weak compatible for all n. Also choosing $k_n = \frac{1}{n+2}$, for all n the contraction (3) is satisfied by the maps A_n and A_{n+1}. Thus all the conditions of Corollary 28 are satisfied and 0 is the unique common fixed point of all the maps of the sequence $\{A_n\}$ and S.

The study of fixed point in theory of PM-space was started by Sehgal and Bharucha-Reid in [9]. The following definition and theorem appeared in their paper.

Definition 30 ([9]). *A mapping f of a PM-space (X, F) into itself is a contraction if there exist $0 < k < 1$ such that for each x and y in X, $F_{fx,fy}(kt) \geq F_{x,y}(t), \forall t > 0$.*

Theorem 31 ([9]). *Let (X, F, t) be a complete Menger space where $t(a, b) = Min\{a, b\}$. If f is any contraction, then there exists a unique $p \in X$ such that $f(p) = p$. Moreover, $\lim_{n\to\infty} f^n(q) = p$ for each $q \in X$. As an application of Corollary 28 the following theorem is a more complete result for sequence of maps $\{T_n\}$ as with a larger contraction and a continuous t-norm.*

Theorem 32. *Let $\{T_n\}$ be sequence of self-maps on complete Menger space (X, F, t), with continuous t-norm. If there exists $k \in (0, 1)$ such that for any pair (T_i, T_j) and for all $p, q \in X, \beta \in (0, 2)$ and $x > 0$, $F_{T_ip,T_jq}(kx) \geq \min\{F_{p,T_ip}(x), F_{q,T_jq}(x), F_{q,T_ip}(\beta x), F_{p,T_jq}(2 - \beta)x, F_{p,q}(x)\}$. Then for each fixed x_0 in X the sequence $x_n = T_n x_{n-1}$ is convergent and its limit u is the common fixed point of the sequence $\{T_n\}$.*

Proof. First we shall prove that the sequence $\{x_n\}$ defined by $x_n = T_n x_{n-1}$ is a Cauchy sequence. For the pair of maps T_n and T_{n+1}, taking $p = x_{n-1}$ and $q = x_n$ with $\beta = 1 - \alpha, \alpha \in (0, 1)$ in the given contractive condition we have

$$F_{T_n x_{n-1}, T_{n+1} x_n}(kx)$$

$$\geq \min \begin{Bmatrix} F_{x_{n-1},T_n x_{n-1}}(x), F_{x_n,T x_{n+1}}(x), F_{x_n,T_n x_{n-1}}((1-\alpha)x), \\ F_{x_{n-1},T_{n+1}x_n}((1+\alpha)x), F_{x_{n-1},x_n}(x) \end{Bmatrix}$$

i.e.

$$F_{x_n,x_{n+1}}(kx) \geq \min \begin{Bmatrix} F_{x_{n-1},x_n}(x), F_{x_n,x_{n+1}}(x), F_{x_n,x_n}((1-\alpha)x), \\ F_{x_{n-1},x_{n+1}}((1+\alpha)x), F_{x_{n-1},x_n}(x) \end{Bmatrix}$$

which gives

$$F_{x_n,x_{n+1}}(kx) \geq \min\left\{F_{x_{n-1},x_n}(x), F_{x_n,x_{n+1}}(x), F_{x_n,x_{n+1}}(\alpha x), F_{x_{n-1},x_n}(x)\right\}.$$

Letting $\alpha \to 1$ we get,

$$F_{x_n,x_{n+1}}(kx) \geq \min\left\{F_{x_{n-1},x_n}(x), F_{x_n,x_{n+1}}(x)\right\}.$$

Consequently,

$$F_{x_n,x_{n+1}}(x) \geq \min\left\{F_{x_{n-1},x_n}(k^{-1}x), F_{x_n,x_{n+1}}(k^{-1}x)\right\}.$$

By repeated application of above inequality we get,

$$F_{x_n,x_{n+1}}(x) \geq \min\left\{F_{x_{n-1},x_n}(k^{-1}x), F_{x_n,x_{n+1}}(k^{-m}x)\right\}.$$

Since $F_{x_n,x_{n+1}}(k^{-m}x) \to 1$ as $m \to \infty$ it follows that $F_{x_n,x_{n+1}}(x) \geq F_{x_{n-1},x_n}(x)$, $\forall n \in N$ and $\forall x > 0$. Therefore, as in last half of step I of theorem 14, $\{x_n\}$ is a Cauchy sequence in X, which is complete. Hence $\{y_n\} \to u \in X$. Now, for any pair of maps (T_n, T_m) using the contractive condition with $p = x_{n-1}, q = u$ and $\beta = 1$ we have, $F_{x_n,T_m u}(kx) \geq \min\{F_{x_{n-1},x_n}(x), F_{u,T_m u}(x), F_{u,x_n}(x), F_{x_{n-1},T_m u}(x), F_{x_{n-1},u}(x)\}$ i.e. $F_{T_n x_{n-1},T_m u}(kx) \geq \min\{F_{x_{n-1},T_n x_{n-1}}(x), F_{u,T_m u}(x), F_{u,T_n x_{n-1}}(x), F_{x_{n-1},T_m u}(x), F_{x_{n-1},u}(x)\}$. Taking limit as $n \to \infty$ we get, $F_{u,T_m u}(kx) \geq \min\{F_{u,u}(x), F_{u,T_m u}(x), F_{u,u}(x), F_{u,T_m u}(x), F_{u,u}(x)\}$ which gives $F_{u,T_m u}(kx) \geq, F_{u,T_m u}(x)$ for all $x > 0$. Hence $u = Tm_u$, for all m. Therefore, u is the common fixed point of the sequence of maps $\{T_n\}$. The uniqueness follows from corollary 22. □

Corollary 33. *Let f be a self-map of a complete Menger space (X, F, t), with continuous t-norm t. If there exists $k \in (0, 1)$ such that $F_{fp,fq}(kx) \geq \min\{F_{p,q}(x), F_{fp,p}(x), F_{q,fq}(x)\}$ for all $p, q \in X$ and $x > 0$. Then f has a unique fixed point u in X and $\lim_{n\to\infty} f_n(p) = u$, for all p in X.*

Proof. The proof follows from theorem 32 by taking $T_n = f$ for all n and restricting the contractive condition of it to first three factors only. □

Remark 34. Taking t-norm $t = T_M$ and restricting the contraction to the first factor the quoted result of [9] follows.

Theorem 35 ([11]). *Let A be self mapping of a complete Menger space (X, F, Min). Suppose for some $k \in (0, 1)$, $F_{Ap,Aq}(kt) \geq b_0 F_{p,q}(t) + c_0 F_{Ap,p}(t), \forall p, q \in X$ and $\forall t > 0$, where b_0, c_0 are some non-negative real numbers such that $b_0 + c_0 = 1$. Then A has unique fixed point u in X. Moreover, $\lim_{n\to\infty} A_n(v) = u$, for all v in X. The following corollary is a more complete result generalizing above result of* [11].

Corollary 36. *Let f be a self-map of a complete Menger space (X, F, t), with continuous t-norm t. If there exist $k \in (0,1), a, b$ and $c \in [0,1]$ with $a+b+c = 1$ such that, $F_{fp,fq}(kx) \geq aF_{p,q}(x) + bF_{fp,p}(x) + cF_{fq,q}(x), \forall p, q \in X$ and $\forall x > 0$. Then f has unique fixed point u in X. Moreover, $\lim_{n\to\infty} f_n(p) = u$, for all p in X.*

Proof. Let $M(x) = \min\{F_{p,q}(x), F_{fp,p}(x), F_{fq,q}(x)\}$. Hence from the given contractive condition, we have $F_{fp,fq}(kx) \geq aM(x)+bM(x)+cM(x) = (a+b+c)M(x) = M(x)$ and the result follows from Corollary 33.

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