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# RIEMANNIAN METRICS ON THE TANGENT BUNDLE OF A FINSLER SUBMANIFOLD

BY

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**Abstract.** Let $\mathbb{F}^n = (M, F)$ be a Finsler submanifold of a Finsler manifold $\mathbb{F}^{n+p} = (\widetilde{M}, \widetilde{F})$. Then the induced non-linear connection $HTM^\circ$ and the canonical non-linear connection $GTM^\circ$ define two Riemannian metrics $G$ and $G^*$ on $TM^\circ = TM \setminus \{0\}$, both of Sasaki-Finsler type. On the other hand, the Sasaki-Finsler metric $\widetilde{G}$ on $T\widetilde{M}^\circ = T\widetilde{M} \setminus \{0\}$ induces a Riemannian metric $G_{\text{ind}}$ on $TM^\circ$. We prove that $\mathbb{F}^n$ is totally geodesic immersed in $\mathbb{F}^{n+p}$ if and only if $G = G^* = G_{\text{ind}}$ on $TM^\circ$.

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**Key words:** Finsler submanifold, Sasaki metric, totally geodesic immersion.

## 1. Preliminaries

Let $\mathbb{F}^{n+p} = (\widetilde{M}, \widetilde{F})$ be an $(n+p)$-dimensional Finsler manifold, where $\widetilde{F}$ is the fundamental function of $\mathbb{F}^{n+p}$ that is of class $C^\infty$ on the slit tangent bundle $T\widetilde{M}^\circ = T\widetilde{M} \setminus \{0\}$ (see Bao-Chern-Shen [1], Bejancu-Farran [3], Matsumoto [4], Rund [5], for basic results on Finsler geometry). Denote by $(x^i, y^i)$, $i = \{1, ..., n+p\}$ the local coordinates on $T\widetilde{M}$, where $(x^i)$ are the local coordinates of a point $x \in \widetilde{M}$ and $(y^i)$ are the coordinates of a vector $y \in T_x\widetilde{M}$. The vertical vector bundle $VT\widetilde{M}^\circ$ on $T\widetilde{M}^\circ$ is endowed with a Riemannian metric whose local components are given by

$$\widetilde{g}_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 \widetilde{F}^2}{\partial y^i \partial y^j}.$$

Denote by $\widetilde{g}^{ij}(x, y)$ the entries of the inverse of the $(n+p) \times (n+p)$-matrix $[\widetilde{g}_{ij}(x, y)]$. Then the geodesics of $\mathbb{F}^{n+p}$ are given by the solutions of the system of differential equations

$$\frac{d^2x^i}{ds^2} + \widetilde{G}^i\left(x^j(s), \frac{dx^j}{ds}\right) = 0,$$

where $\widetilde{G}^i$ are given by

$$\widetilde{G}^i(x,y) = \frac{1}{4}\,\widetilde{g}^{ih}\left\{\frac{\partial^2 \widetilde{F}^2}{\partial y^h \partial x^j}\,y^j - \frac{\partial \widetilde{F}^2}{\partial x^h}\right\}.$$

We note that $VT\widetilde{M}^\circ$ is an integrable distribution on $T\widetilde{M}^\circ$ which is locally spanned by $\{\frac{\partial}{\partial y^i}\}$, $i \in \{1,...,n+p\}$. Also, on $T\widetilde{M}^\circ$ there exists a distribution $GT\widetilde{M}^\circ$ (in general, non-integrable) which is given by the non-holonomic frame field:

$$\frac{\delta^*}{\delta^* x^i} = \frac{\partial}{\partial x^i} - \widetilde{G}^j_i\,\frac{\partial}{\partial y^j}\,, \quad i \in \{1,...,n+p\},$$

where $\widetilde{G}^j_i$ are given by

$$\widetilde{G}^j_i = \frac{\partial \widetilde{G}^j}{\partial y^i}.$$

It is important to note that we have

$$TT\widetilde{M}^\circ = GT\widetilde{M}^\circ \oplus VT\widetilde{M}^\circ,$$

that is, $GT\widetilde{M}^\circ$ and $VT\widetilde{M}^\circ$ are complementary distributions on $T\widetilde{M}^\circ$. We call $GT\widetilde{M}^\circ$ the *canonical non-linear connection* on $T\widetilde{M}^\circ$. The above decomposition of $TT\widetilde{M}^\circ$ enables us to define the *Sasaki-Finsler metric* $\widetilde{G}$ on $T\widetilde{M}^\circ$ given locally as follows (cf. Bao-Chern-Shen [1], p. 48, Bejancu-Farran [3], p. 35, Matsumoto [4], p. 136)

$$(1)\quad \widetilde{G}\left(\frac{\delta^*}{\delta^* x^j}\,, \frac{\delta^*}{\delta^* x^i}\right) = \widetilde{G}\left(\frac{\partial}{\partial y^j}\,, \frac{\partial}{\partial y^i}\right) = \widetilde{g}_{ij}(x,y), \quad \widetilde{G}\left(\frac{\delta^*}{\delta^* x^j}\,, \frac{\partial}{\partial y^i}\right) = 0.$$

Next, let $M$ be an $n$-dimensional submanifold of $\widetilde{M}$ that is locally given by the equations

$$x^i = x^i(u^1,...,u^n),\ i \in \{1,...,n+p\}.$$

Throughout the paper we use the following ranges for indices: $i, j, k, ... \in \{1, ..., n+p\}$, $\alpha, \beta, \gamma, ... \in \{1, ..., n\}$, $a, b, c, ... \in \{n+1, ..., n+p\}$. Consider $(u^\alpha, v^\alpha)$ as local coordinates on $TM^\circ = TM \setminus \{0\}$ and define the function $F(u^\alpha, v^\alpha) = \widetilde{F}(x^i(u), y^i(u, v))$, where we put

$$y^i(u, v) = B^i_\alpha v^\alpha, \ B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}.$$

Then $\mathbb{F}^n = (M, F)$ is a Finsler submanifold of $\mathbb{F}^{n+p}$.

As above, we have the decomposition

$$TTM^\circ = GTM^\circ \oplus VTM^\circ, \tag{2}$$

where $VTM^\circ$ and $GTM^\circ$ are the vertical vector bundle and the canonical non-linear connection on $TM^\circ$, locally spanned by $\{\frac{\partial}{\partial v^\alpha}\}$ and

$$\frac{\delta^*}{\delta^* u^\alpha} = \frac{\partial}{\partial u^\alpha} - G^\beta_\alpha \frac{\partial}{\partial v^\beta},$$

respectively. Thus the Sasaki-Finsler metric $G^*$ on $TM^\circ$ defined by the decomposition (1.2) is locally given by

$$G^*\left(\frac{\delta^*}{\delta^* u^\beta}, \frac{\delta^*}{\delta^* u^\alpha}\right) = G^*\left(\frac{\partial}{\partial v^\beta}, \frac{\partial}{\partial v^\alpha}\right) = g_{\alpha\beta}(u, v), G^*\left(\frac{\delta^*}{\delta^* u^\beta}, \frac{\partial}{\partial v^\alpha}\right) = 0, \tag{3}$$

where we put $g_{\alpha\beta}(u, v) = \frac{1}{2}\frac{\partial^2 F^2}{\partial v^\alpha \partial v^\beta}$. By direct calculations it follows that

$$g_{\alpha\beta} = \widetilde{g}_{ij} B^i_\alpha B^j_\beta. \tag{4}$$

On the other hand, $\widetilde{G}$ induces a Riemannian metric $G_{\text{ind}}$ on $TM^\circ$ as follows

$$G_{\text{ind}}(X, Y) = \widetilde{G}(X, Y), \quad \forall X, Y \in \Gamma(TTM^\circ). \tag{5}$$

Here, and in the sequel, we denote by $\Gamma(TTM^\circ)$ the $\mathcal{F}(TM^\circ)$-module of vector fields on $TM^\circ$, where $\mathcal{F}(TM^\circ)$ is the algebra of smooth functions on $TM^\circ$. The same notation will be used for the module of sections of any vector bundle.

Now, we denote by $VTM^{\circ\perp}$ the orthogonal complementary vector subbundle to $VTM^\circ$ in $V\widetilde{TM}_{|TM^\circ}$ with respect to $\widetilde{G}$ and call it the *Finsler normal bundle* of the immersion of $\mathbb{F}^n$ in $\mathbb{F}^{n+p}$ (cf. BEJANCU [2], p. 47). Then we consider a local field of orthonormal frames $\{B_a = B^i_a \, \partial/\partial y^i\}$,

$a \in \{n+1, ..., n+p\}$ in $VTM^{\circ\perp}$, and taking into account that $\{\partial/\partial v^\alpha = B^i_\alpha\, \partial/\partial y^i\}$ is a local field of frames in $VTM^\circ$, we deduce that

$$\text{(6)} \qquad \text{(a) } \widetilde{g}_{ij} B^i_\alpha B^j_a = 0, \quad \text{(b) } \widetilde{g}_{ij} B^i_a B^j_b = \delta_{ab}.$$

Denote by $[\widetilde{B}^\alpha_i \ \widetilde{B}^a_i]$ the inverse of the transition matrix $[B^i_\alpha \ B^i_a]$ from the natural field of frames $\{\frac{\partial}{\partial y^i}\}$ on $V\widetilde{TM}^\circ_{|TM^\circ}$ to the field of frames $\{\frac{\partial}{\partial v^\alpha}, B_a\}$.

Next, we consider the *induced non-linear connection* $HTM^\circ$ on $TM^\circ$, which is defined as the complementary orthogonal distribution to $VTM^\circ$ in $TTM^\circ$ with respect to $G_{\text{ind}}$. Thus, apart from (2), $TTM^\circ$ admits the decomposition

$$\text{(7)} \qquad TTM^\circ = HTM^\circ \oplus VTM^\circ.$$

We should note that both (2) and (7) are orthogonal decompositions, but with respect to different metrics $G^*$ and $G_{\text{ind}}$, respectively. Locally, $HTM^\circ$ is given by the field of frames $\left\{\frac{\delta}{\delta u^\alpha}\right\}$, $\alpha \in \{1, ..., n\}$, expressed as follows (cf. BEJANCU-FARRAN [3], pp. 74, 75)

$$\text{(8)} \qquad \frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} - N^\beta_\alpha \frac{\partial}{\partial v^\beta}, N^\beta_\alpha = \widetilde{B}^\beta_i \left(B^i_{\alpha 0} + B^j_\alpha \widetilde{G}^i_j\right),$$

or

$$\text{(9)} \qquad \frac{\delta}{\delta u^\alpha} = B^i_\alpha \frac{\delta^*}{\delta^* x^i} + H^a_\alpha B_a,\ H^a_\alpha = \widetilde{B}^a_i \left(B^i_{\alpha 0} + B^j_\alpha \widetilde{G}^i_j\right),$$

where we put

$$B^i_{\alpha 0} = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}\, v^\beta.$$

Finally, by using the decomposition (7) we can define another Sasaki-Finsler metric $G$ on $TM^\circ$ given by

$$\text{(10)} \qquad G\left(\frac{\delta}{\delta u^\beta}, \frac{\delta}{\delta u^\alpha}\right) = G\left(\frac{\partial}{\partial v^\beta}, \frac{\partial}{\partial v^\alpha}\right) = g_{\alpha\beta},\ G\left(\frac{\delta}{\delta u^\beta}, \frac{\partial}{\partial v^\alpha}\right) = 0.$$

Summing up, we constructed on $TM^\circ$ three Riemannian metrics: $G^*$, $G_{\text{ind}}$ and $G$ given by (3), (5) and (10), respectively. Thus, it is natural to ask the following question. What class of Finsler submanifolds is characterized by the condition $G^* = G_{\text{ind}} = G$? In the second part of this paper we give the answer to this question.

## 2. The main result

Let $\mathbb{F}^n = (M, F)$ be a Finsler submanifold of $\mathbb{F}^{n+p} = (\widetilde{M}, \widetilde{F})$. Then we say that $\mathbb{F}^n$ is *totally geodesic* immersed in $\mathbb{F}^{n+p}$ if any geodesic of $\mathbb{F}^n$ is a geodesic of $\mathbb{F}^{n+p}$. The main purpose of this section is to prove the following theorem.

**Theorem 2.1.** *Let $\mathbb{F}^n = (M, F)$ be a Finsler submanifold of $\mathbb{F}^{n+p} = (\widetilde{M}, \widetilde{F})$. Then the following assertions are equivalent:*

*(i)* *$\mathbb{F}^n$ is totally geodesic immersed in $\mathbb{F}^{n+p}$.*

*(ii)* *$G_{\text{ind}} = G^*$ on $TM^\circ$.*

*(iii)* *$G_{\text{ind}} = G$ on $TM^\circ$.*

As a direct consequence of this result we state the following.

**Corollary 2.1.** *$\mathbb{F}^n$ is a totally geodesic Finsler submanifold of $\mathbb{F}^{n+p}$ if and only if $G^* = G_{\text{ind}} = G$ on $TM^\circ$.*

First, we prove the following lemma.

**Lemma 2.1.** *Let $\mathbb{F}^n = (M, F)$ be a Finsler submanifold of $\mathbb{F}^{n+p} = (\widetilde{M}, \widetilde{F})$. Then we have:*

$$\begin{aligned} &\text{(a)} \quad G_{\text{ind}}\left(\frac{\delta}{\delta u^\beta}, \frac{\delta}{\delta u^\alpha}\right) = g_{\alpha\beta} + \sum_{a=n+1}^{n+p} H^a_\alpha H^a_\beta, \\ (11) \qquad &\text{(b)} \quad G_{\text{ind}}\left(\frac{\delta}{\delta u^\beta}, \frac{\partial}{\partial v^\alpha}\right) = 0, \\ &\text{(c)} \quad G_{\text{ind}}\left(\frac{\partial}{\partial v^\beta}, \frac{\partial}{\partial v^\alpha}\right) = g_{\alpha\beta}. \end{aligned}$$

**Proof.** By direct calculations using (5), (9), (1) and (4) we obtain

$$\begin{aligned} G_{\text{ind}}\left(\frac{\delta}{\delta u^\beta}, \frac{\delta}{\delta u^\alpha}\right) &= \widetilde{G}\left(B^j_\beta \frac{\delta^*}{\delta^* x^j} + H^b_\beta B_b, B^i_\alpha \frac{\delta^*}{\delta^* x^i} + H^a_\alpha B_a\right) \\ &= \widetilde{g}_{ij} B^i_\alpha B^j_\beta + H^a_\alpha H^b_\beta \delta_{ab} = g_{\alpha\beta} + \sum_{a=n+1}^{n+p} H^a_\alpha H^a_\beta, \end{aligned}$$

which proves (11a). Then, (11b) follows from the definition of the non-linear connection $HTM^\circ$. Finally, by using (5), (1) and (4) we deduce that

$$G_{\text{ind}}\left(\frac{\partial}{\partial v^\beta}, \frac{\partial}{\partial v^\alpha}\right) = \widetilde{G}\left(B^j_\beta \frac{\partial}{\partial y^j}, B^i_\alpha \frac{\partial}{\partial y^i}\right) = \widetilde{g}_{ij} B^i_\alpha B^j_\beta = g_{\alpha\beta},$$

which completes the proof of the lemma. □

Next, we recall the following characterization of totally geodesic Finsler submanifolds.

**Theorem 2.2** (Bejancu-Farran [3], p. 134)**.** $\mathbb{F}^n=(M,F)$ *is a totally geodesic Finsler submanifold of* $\mathbb{F}^{n+p}=(\widetilde{M},\widetilde{F})$ *if and only if we have*

$$H^a_\alpha = 0, \quad \forall \alpha \in \{1, ..., n\},\ a \in \{n+1, ..., n+p\}. \tag{12}$$

Now, we can state the following.

**Theorem 2.3.** $\mathbb{F}^n$ *is a totally geodesic Finsler submanifold of* $\mathbb{F}^{n+p}$ *if and only if* $G_{\text{ind}} = G$ *on* $TM^\circ$.

**Proof.** By comparing (11) with (10) we deduce that $G_{\text{ind}} = G$ on $TM^\circ$ if and only if

$$\sum_{a=n+1}^{n+p} H^a_\alpha H^a_\beta = 0, \quad \forall \alpha, \beta \in \{1, ..., n\},$$

which is equivalent to (12).

In order to find another characterization of totally geodesic Finsler submanifolds we present the relationship between the induced non-linear connection $HTM^\circ$ and the canonical non-linear connection $GTM^\circ$ on $TM^\circ$. First we consider the Cartan tensor field of type $(0,3)$ on $\mathbb{F}^{n+p}$, whose local components are given by

$$\widetilde{g}_{ijk} = \frac{1}{4}\,\frac{\partial^3 \widetilde{F}^2}{\partial y^i \partial y^j \partial y^k}.$$

Then we define on $TM^\circ$ $p$ Finsler tensor fields of type $(1,1)$ whose local components are given by

$$g_a{}^\beta{}_\alpha = g^{\beta\gamma} \widetilde{g}_{ijk} B^i_a B^j_\gamma B^k_\alpha, \quad a \in \{n+1, ..., n+p\}, \alpha, \beta \in \{1, ..., n\}.$$

Finally, we consider the *deformation Finsler tensor field* $D$ with respect to the pair $(HTM^\circ, GTM^\circ)$, whose local components are given by

$$D_\alpha^\beta = g_a{}^\beta{}_\alpha H_\gamma^a v^\gamma. \tag{13}$$

Then it is proved in Bejancu-Farran [3], p. 112, that the local fields of frames $\left\{\frac{\delta^*}{\delta^* u^\alpha}\right\}$ and $\left\{\frac{\delta}{\delta u^\alpha}\right\}$ in $GTM^\circ$ and $HTM^\circ$ are related by

$$\frac{\delta}{\delta u^\alpha} = \frac{\delta^*}{\delta^* u^\alpha} + D_\alpha^\beta \frac{\partial}{\partial v^\beta}\,. \tag{14}$$

□

**Lemma 2.2.** *Let* $\mathbb{F}^n = (M, F)$ *be a Finsler submanifold of* $\mathbb{F}^{n+p} = (\widetilde{M}, \widetilde{F})$. *Then we have:*

$$\begin{aligned} &\text{(a)}\quad G_{\text{ind}}\left(\frac{\delta^*}{\delta^* u^\beta}, \frac{\delta^*}{\delta^* u^\alpha}\right) = g_{\alpha\beta} + \sum_{a=n+1}^{n+p} H_\alpha^a H_\beta^a + g_{\gamma\mu} D_\alpha^\gamma D_\beta^\mu, \\ &\text{(b)}\quad G_{\text{ind}}\left(\frac{\delta^*}{\delta^* u^\beta}, \frac{\partial}{\partial v^\alpha}\right) = -g_{\alpha\gamma} D_\beta^\gamma. \end{aligned} \tag{15}$$

**Proof.** First by using (14) and (11) we obtain

$$\begin{aligned} G_{\text{ind}}\left(\frac{\delta^*}{\delta^* u^\beta}, \frac{\delta^*}{\delta^* u^\alpha}\right) &= G_{\text{ind}}\left(\frac{\delta}{\delta u^\beta} - D_\beta^\mu \frac{\partial}{\partial v^\mu}, \frac{\delta}{\delta u^\alpha} - D_\alpha^\gamma \frac{\partial}{\partial v^\gamma}\right) \\ &= g_{\alpha\beta} + \sum_{a=n+1}^{n+p} H_\alpha^a H_\beta^a + g_{\gamma\mu} D_\alpha^\gamma D_\beta^\mu, \end{aligned}$$

and (15a) is proved. In a similar way we obtain (15b). □

**Theorem 2.4.** $\mathbb{F}^n$ *is a totally geodesic Finsler submanifold of* $\mathbb{F}^{n+p}$ *if and only if* $G_{\text{ind}} = G^*$.

**Proof.** Suppose $\mathbb{F}^n$ is totally geodesic in $\mathbb{F}^{n+p}$. Then by (12) and (13) we have $H_\alpha^a = 0$ and $D_\alpha^\beta = 0$, $\forall \alpha,\ \beta \in \{1, ..., n\}$, $a \in \{n+1, ..., n+p\}$. Using these in (15) and comparing with (3) we deduce that $G_{\text{ind}} = G^*$. Conversely, if $G_{\text{ind}} = G^*$, from (15) and (3) we deduce that

$$g_{\alpha\gamma} D_\beta^\gamma = 0 \ \text{ and } \sum_{a=n+1}^{n+p} H_\alpha^a H_\beta^a + g_{\gamma\mu} D_\alpha^\gamma D_\beta^\mu = 0,\ \forall \alpha, \beta \in \{1, ..., n\},$$

which imply (12). Hence $\mathbb{F}^n$ is totally geodesic. □

By Theorems 2.3 and 2.4 we have a proof of Theorem 2.1.

In particular, let us consider $\mathbb{F}^{n+p}=(\widetilde{M}, \widetilde{F})$ a Riemannian manifold, that is $\widetilde{g}_{ijk} = 0$, $\forall i, j, k \in \{1, ..., n+p\}$. Then by (13) we deduce that $D_\alpha^\beta = 0$ for all $\alpha, \beta \in \{1, ..., n\}$, and therefore by (14) we obtain $HTM^\circ = GTM^\circ$. As a conclusion we have $G = G^*$. Therefore, in the particular case of Riemannian submanifolds we obtain the following result.

**Theorem 2.5.** *Let $(M, g)$ be a submanifold of a Riemannian manifold $(\widetilde{M}, \widetilde{g})$. Then $(M, g)$ is totally geodesic if and only if the Riemannian metric induced on $TM$ by the Sasaki metric of $T\widetilde{M}$ coincides with the Sasaki metric induced on $TM$ by $g$.*

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