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SOME ASPECTS OF G_H-RINGS

BY

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Abstract. In this paper we introduce and study a class of hyperstructure called G_H-ring. We show that every commutative group admits a G_H-ring structure. Here we obtain necessary and sufficient conditions for a G_H-ring to be a division G_H-ring (and a strong division G_H-ring). The notion of ideals in a G_H-ring is also introduced and studied here. We show that every G_H-ring with an identity set (i.e., an i-set, in short) always contains a maximal ideal. This is obtained that the maximality of an ideal I of a G_H-ring R with condition $(\mathcal{R})$ (i.e., a multiplicative hyperring with absorbing zero) having an i-set, is a necessary and sufficient condition for the quotient G_H-ring R/I of R to be a G_H-field. We establish an isomorphism theorem on G_H-rings in analogy to the first isomorphism theorem on rings. We construct, over any ring R, a G_H-ring structure R_A, induced by each $A \in P^*(R)$ with $|A| \geq 2$. We study such G_H-ring R_A over a ring R, in accordance with the nature of the set A chosen.

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1. Introduction

Krasner's hyperring [7], introduced and studied by KRASNER is a hypercompositional structure $(S, +, \cdot)$ where $(S, +)$ is a canonical hypergroup, $(S, \cdot)$ is a semigroup [4] in which the zero element is absorbing and the operation $\cdot$ is a two-sided distributive one over the hypercomposition $+$. MARTY initiated the study of hypergroups in 1934. A hypergroup [5] (or multigroup) $(G, \circ)$ is a semihypergroup (i.e., a hypercompositional structure with single associative hyperoperation) such that for all $a \in G, G \circ a = a \circ G = G$. An H_V-semigroup [17] is a hypercompositional structure $(G, \star)$ with a single weak associative hyperoperation $\star$ (in the sense that $a\star(b\star c)\cap(a\star b)\star c \neq \phi$, $\forall a, b, c \in G$). A canonical hypergroup [5] $(G, \circ)$ is a commutative $(a \circ b =$

$b \circ a$, $\forall a, b \in G$) hypergroup with a scalar identity e (in the sense that $e \circ a = \{a\}$, $\forall a \in G$), of which every element has a unique inverse $a' \in G$ (in the sense that $e \in a \circ a'$) and is also invertible (in the sense that $b \in a \circ x \Rightarrow x \in a' \circ b$, $\forall a, b, x \in G$). Krasner's hyperring is a generalization of superring [10] (in which both the addition and multiplication are hypercompositions) introduced by MITTAS in 1973. In [3] CHAOPRAKNOI and KEMPRASIT introduce in 2005 the notion of semihyperring $(S, +, \cdot)$ where $(S, +)$ is a semihypergroup, $(S, \cdot)$ is a semigroup and the operation $\cdot$ is both left and right distributive accross the hyperopreration $+$. Interchanging the modes of the operations involved in the hyperstructure semihyperring, we define in [16] another class of hyperstructure called hypersemiring which is an (additive) commutative semigroup $(S, +)$ endowed with a hyperoperation $\odot : S \times S \to P(S)$ such that for all $x, y, z \in S$, (i) $x \odot (y \odot z) = (x \odot y) \odot z$ and (ii) $(x + y) \odot z = x \odot z + y \odot z, x \odot (y + z) = x \odot y + x \odot z$ (where for any $A, B \in P(S), A \oplus B = \{a + b : a \in A, b \in B\}$). In contrary to Krasner's hyperring, another kind of hyper-ring is introduced in [16]. This hyper-ring is a hypersemiring $(S, +, \odot)$ where $(S, +)$ is a commutative group whose identity element 0_S is absorbing in the hypersemiring $(S, +, \odot)$ (in the sense that $0_S \in 0_S \odot x = x \odot 0_S$, $\forall x \in S$). G_H-ring $(R, +, \odot)$ is a generalized hyper-ring [16] which being assumed to have absorbing zero 0_R (in the sense, $0_R \odot x = x \odot 0_R = \{0_R\}$, instead of $0_R \in 0_R \odot x = x \odot 0_R$) is defined by replacing "set equality" type distributive axioms of hyper-ring by "one-sided containment ($\subseteq$)" type distributive axioms $(x + y) \odot z \subseteq x \odot z + y \odot z$, and $x \odot (y + z) \subseteq x \odot y + x \odot z$, $\forall x, y, z \in R$.

In [15], ROTA initiates the study of multiplicative hyperring which is an additive commutative group $(R, +)$ endowed with a hyperoperation $\cdot$ such that

(i) $(R, \cdot)$ is semihypergroup;
(ii) $(x + y) \cdot z \subseteq x \cdot z + y \cdot z$, and $x \cdot (y + z) \subseteq x \cdot y + x \cdot z$, $\forall x, y, z \in R$ and
(iii) $(-x) \cdot y = x \cdot (-y) = -(x \cdot y)$, $\forall x, y, z \in R$.

The G_H-rings and the multiplicative hyperrings form two distinct classes of hyperstructures, none of which being contained in other (see examples 2.3 (a), (b), (c)).

Γ-ring is the motivating algebraic structure to conceptualise the idea of G_H-ring. The notion of Γ-ring was introduced by NOBUSAWA in [11] and subsequently, was studied and developed by BARNES, BOOTH, KYUNO, LUH, RAVISANKAR and many others (see [1], [2], [8], [9], [14]). BARNES in [1] defined the Γ-ring by restricting some axioms of Nobusawa's Γ-ring. In Barnes sense, an additive commutative group M is called a Γ-*ring* if for another additive commutative group Γ, there exists a mapping $M\times\Gamma\times M \to M$ described by $(x,\alpha,y)\mapsto x\alpha y$ such that

(i) $x\alpha(y+z)=x\alpha y+x\alpha z$;
(ii) $(x+y)\alpha z=x\alpha z+y\alpha z$;
(iii) $x(\alpha+\beta)y=x\alpha y+a\beta y$;
(iv) $x\alpha(y\beta z)=(x\alpha y)\beta z$.

for any $x,y,z\in M$ and $\alpha,\beta\in\Gamma$.

The identity element 0_M of the group M is absorbing in the Γ-ring M in the sense that, for any $x,y\in M$ and for any $\alpha\in\Gamma$, $x\alpha 0_M=0_M\alpha x=x0_\Gamma y=0_M$. For any Γ-ring M, we define a hyperoperation $\odot$ on M as follows: $a\odot b=\{a\alpha b:\alpha\in\Gamma\}$ for any $a,b\in M$. Then, $(M,+,\odot)$ is a G_H-ring which we call as the associated G_H-ring of the Γ-ring M and we denote it by M_Γ.

Corresponding to every subset $A\in P^*(R)=P(R)\setminus\{0_R\}$ ($|A|\geq 2$) of a ring $(R,+,\cdot)$, there exists a G_H-ring $(R_A,+,\odot)$ where $R_A=R$ and for any $x,y\in R_A, x\odot y=\{x\cdot a\cdot y:a\in A\}$. This G_H-ring is called the G_H-ring over the ring R induced by A, which we study in section 4.

2. Properties of a G_H-ring

Let $(M_{2\times3}(\mathbb{R}),+)$ be the additive group of 2×3 real matrices and A (with $|A|\geq 2$) be any non-empty subset of the set $M_{3\times2}(\mathbb{R})$ of 3×2 real matrices. For any $a,b\in M_{2\times3}(\mathbb{R})$, let $a\odot b=\{a\cdot\alpha\cdot b:\alpha\in A\}$, where $\cdot$ is the usual multiplication of matrices. Then, $(M_{2\times3}(\mathbb{R}),\odot)$ is a semihypergroup. Also, this is to be observed that, $p\in a\odot(b+c)\Rightarrow p=a\cdot\alpha\cdot(b+c)$ (for some $\alpha\in A$) $=a\cdot\alpha\cdot b+a\cdot\alpha\cdot c\in a\odot b+a\odot c$. Thus, $a\odot(b+c)\subseteq a\odot b+a\odot c$. Similarly, $(b+c)\odot a\subseteq b\odot a+c\odot a$. Moreover, corresponding to $A=\left\{\begin{bmatrix}1&0\\0&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\\0&0\end{bmatrix}\right\}$, we see that, for $a=\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}$, $b=c=\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}$, $a\odot(b+c)=\left\{\begin{bmatrix}0&2&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}\right\}$; while, $a\odot b=a\odot c=\left\{\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}\right\}$ and so, $a\odot b+a\odot c=\left\{\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&2&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}\right\}$, whence, $a\odot(b+c)\neq a\odot b+a\odot c$. Similarly, $(b+c)\odot a\neq b\odot a+c\odot a$, for $b=c=\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}$ and $a=\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}$. Thus, here the hyperoperation $\odot$ is not in general exactly distributive over the operation $+$, but is "inclusively" distributive in the sense that, $a \odot (b + c) \subseteq a \odot b + a \odot c$ and $(b + c) \odot a \subseteq b \odot a + c \odot a$. This evokes the idea to perceive formally and study generally a new class of hyperstructure (called the G_H-ring) defined as follows.

Definition 2.1. A non-empty set R endowed with a binary operation $+$ and a hyperoperation $\odot$ is called a *generalized hyperring* or simply a *G_H-ring* if
(i) $(R, +)$ is a commutative group (with identity 0_R),
(ii) $(R, \odot)$ is a semihypergroup,
(iii) $(x+y) \odot z \subseteq x \odot z + y \odot z$ and $x \odot (y+z) \subseteq x \odot y + x \odot z$, $\forall x, y, z \in R$,
(iv) $0_R \odot x = x \odot 0_R = \{0_R\}$, $\forall x \in R$ (absorbing property of 0_R).

A G_H-ring $(R, +, \odot)$ is called commutative if $x \odot y = y \odot x$, $\forall x, y \in R$. In case of a ring $(R, +, \cdot)$, the identity 0_R of the group $(R, +)$ is inherently absorbing within the structure of the ring. The insertion of the axiom (iv) in the definition of G_H-ring in addition to first three axioms, is simply due to make the absorbing property of its zero element inherent within its hypercompositional structure. In fact, the following two examples show that the fourth axiom is not in general a consequence of the first three in the definition of G_H-ring.

Example 2.2. (a) On the additive commutative group of real numbers $(\mathbb{R}, +)$, the hyperoperation $\odot$ defined as $x \odot y = \{r \in R : 0 < r < m\}$, $\forall (x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ where, m =max$\{|x|, |y|\}$), and $0 \odot 0 = \{0\}$, is commutative and satisfies first three axioms in the definition of G_H-ring, whereas $0 \notin 0 \odot x$, $\forall x \in \mathbb{R} \setminus \{0\}$.

(b) On the additive commutative group of integers $(\mathbb{Z}, +)$, the hyperoperation $\odot$ defined as $x \odot y = \{0, x, y\}$ is commutative and satisfies first three axioms in the definition 2.1, but it does not satisfy axiom (iv) as we see that, $0 \in 0 \odot x = x \odot 0 \neq \{0\}$, $\forall x \in \mathbb{R} \setminus \{0\}$. So, $(\mathbb{Z}, +, \odot)$ is not a G_H-ring. However, every commutative group admits a G_H-ring structures and it is evident from the following construction of a G_H-ring in the example 2.3 (a).

Example 2.3. (a) For any commutative group $(G, +)$, the hyperstructure $(G, +, \odot)$ is a commutative G_H-ring when $x \odot y = \{0_G, x, y\}$, $\forall x, y \in G \setminus \{0_G\}$, and $x \odot 0_G = 0_G \odot x = \{0_G\}$, $\forall x \in G$. Clearly, this G_H-ring is not a multiplicative hyperring.

(b) For any commutative group $(G,+)$, the hyperstructure $(G,+,\circ)$ is a commutative multiplicative hyperring when $x \circ y = \{0_G, x, -x, y, -y\}$, $\forall x, y \in G$. Clearly, this multiplicative hyperring is not a G_H-ring.

(c) To deal with complementary multiplicative hyperrings, PROCESI and ROTA construct in [13] a multiplicative hyperring $(A,+,\circ)$ which is not a G_H-ring, where $A = \mathbb{Z}_6$ and $\forall \overline{a}, \overline{b} \in A, \overline{a} \circ \overline{b} = \overline{ab} + I, I = \{\overline{0}, \overline{2}, \overline{4}\}$.

In case of multiplicative hyperrings in the examples 2.3 (b), (c), though 0 is not absorbing, it is seen that $0 \in a \circ 0 \cap 0 \circ a$ for every element a. PROCESI and ROTA show that there are some multiplicative hyperrings R in which even $0 \notin a \circ 0$ or $0 \notin 0 \circ a$ for some $a \in R$. In fact, let R be a unitary domain, S be one of its unitary subrings and $T = S \setminus \{0\}$. On R, $\circ$ is a hyperoperation defined by $a \circ b = aT + bT$ for all $a, b \in R$. Then, $(R,+,\circ)$ is a multiplicative hyperring in which $0 \notin aT$, for all $a \in R \setminus \{0\}$. In [12], OLSON and WARD obtain that *"In any multiplicative hyperring* $(A,+,\cdot)$*, if there are elements* $a, b \in A$ *such that* $|a \cdot b| = 1$*, then* $0 \cdot 0 = \{0\}$*"*. Adopting the same arguments applied to prove the aforesaid result, we establish the following.

Proposition 2.4. *A Multiplicative hyperring* $(A,+,\cdot)$ *is a* G_H*-ring if and only if for each* $a \in A$ *there exists* $(b,c) \in A^2$ *such that* $|a \cdot b| = |c \cdot a| = 1$.

Proof. If the multiplicative hyperring $(A,+,\cdot)$ is a G_H-ring, then for each $a \in A$ we have $(0,0) \in A^2$ such that $|a \cdot 0| = |0 \cdot a| = 1$.

Conversely let $(A,+,\cdot)$ be a multiplicative hyperring such that for each $a \in A$ there exists $(b,c) \in A^2$ such that $|a \cdot b| = |c \cdot a| = 1$. Then, $a \cdot 0 = a \cdot (b - b) \subseteq a \cdot b - a \cdot b = \{0\}$ (since, $|a \cdot b| = 1$) $\Rightarrow a \cdot 0 = \{0\}$. Similarly, $|c \cdot a| = 1 \Rightarrow 0 \cdot a = \{0\}$. Thus, $(A,+,\cdot)$ is a G_H-ring. □

Throughout this section wherever will be mentioned, $(R,+,\odot)$ will denote a G_H-ring.

Proposition 2.5. *If* x, y *in* $(R,+,\odot)$ *and for any* $A \in P^*(R), -A = \{-a : a \in A\}$*, then,* (i) $(-x) \odot y \cap (-(x \odot y)) \neq \phi$ *and* (ii) $x \odot (-y) \cap (-(x \odot y)) \neq \phi$.

Proof. (i) For any x, y in $(R,+,\odot)$, $0_R \in 0_R \odot y = (x + (-x)) \odot y \subseteq x \odot y + (-x) \odot y \Rightarrow \exists a \in x \odot y$ such that $-a \in (-x) \odot y \cap (-(x \odot y))$.

(ii) can be proved similarly.

Let M be a Γ-ring and F be the free abelian group generated by the set of all ordered pairs in $\Gamma \times M$. Then, $A = \{\sum_{i=1}^{n} m_i(\gamma_i, x_i) : \sum_{i=1}^{n} m_i x \gamma_i x_i =$

$0_M, \forall x \in M\}$ is a group. Let the factor group F/A be denoted by R and the coset $A+(\gamma, x)$ by $[\gamma, x]$. then, every element of R can be expressed as a finite sum $\sum_{i=1}^{n}[\gamma_i, x_i]$ and thus R forms a ring if the multiplication is defined by $\sum_{i=1}^{n}[\alpha_i, x_i]\sum_{i=1}^{m}[\beta_j, y_j] = \sum_{i=1}^{n}\sum_{j=1}^{m}[\alpha_i, x_i\beta_j y_j] = \sum_{i,j}[\alpha_i, x_i\beta_j y_j]$ (in short). This ring R is called the right operator ring [2] of the Γ-ring M. Analogously, the left operator ring L can also be defined. If there exists an element $\sum_i[\delta_i, e_i] \in R$ such that $\sum_i x\delta_i e_i = x$, $\forall x \in M$, then the Γ-ring M is said to have the right identity $\sum_i[\delta_i, e_i]$, [14]. Similarly, if there exists an element $\sum_i[f_i, \gamma_i] \in L$ such that $\sum_i f_i\gamma_i x = x$, $\forall x \in M$, then M is said to have the left identity $\sum_i[f_i, \gamma_i]$, [12].

Let M be a Γ-ring with right identity $\sum_i[\delta_i, e_i]$. Then, the non-empty finite subset $A = \{e_1, e_2, \ldots, e_n\}$ of M has the property that, at least one $e_i \neq 0_M$ and for any x of the associated G_H-ring M_Γ of the Γ-ring$M, x \in \sum_i x \odot e_i$. Similarly, if M has the left identity $\sum_i[f_i, \gamma_i]$ then the set $B = \{f_1, f_2, \ldots, f_n\}$ has the property that, at least one $f_i \neq 0_M$ and for any x of the associated G_H-ring M_Γ of the Γ-ring $M, \{x \in \sum_i f_i \odot x\}$. The existence of such properties of sets in the associated G_H-ring of a Γ-ring motivates us to fomally conceptualise these properties over the sets of any G_H-ring. □

Definition 2.6. A non-empty finite subset $\mathcal{E}_l$ (resp. $\mathcal{E}_r$) $= \{e_1, e_2, \ldots, e_n\}$ of a G_H-ring $(R, +, \odot)$ is called a left (resp. right) identity set (in short i-set) of R if

(i) $e_i \neq 0_R$ for at least one $i = 1, 2, \ldots, n$, and

(ii) for any $a \in R$, $a \in \sum_{i=1}^{n} e_i \odot a$ (resp. $a \in \sum_{i=1}^{n} a \odot e_i$).

A non-empty finite subset $\mathcal{E}$ of a G_H-ring $(R, +, \odot)$ is called an i-set of R if it is both a left i-set and a right i-set of R. An element e of a G_H-ring $(R, +, \odot)$ is called a (left, right) hyperidentity of R if the set $\{e\}$ is an (resp. left, right) i-set of R.

Example 2.7. (a) Every non-empty finite subset ($\neq \{0_G\}$) of the G_H-ring as stated in the example 2.3(a) is an i-set of the same.

(b) Let $R = \{a\sqrt{2} + b\sqrt{3} : a, b \in \mathbb{Q}\}$ and $A = \{\sqrt{2}, \sqrt{3}\}$. Then, with respect to usual addition $+$ of reals, $(R, +)$ is a commutative group, with identity 0. On R, $\odot$ is a hyperoperation defined by

$$\begin{aligned}(a\sqrt{2} + b\sqrt{3}) \odot (c\sqrt{2} + d\sqrt{3}) &= \{(a\sqrt{2} + b\sqrt{3}) \cdot t \cdot (c\sqrt{2} + d\sqrt{3}) : t \in A\} \\ &= \{p\sqrt{2} + 2q\sqrt{3}, 3q\sqrt{2} + p\sqrt{3}\},\end{aligned}$$

where $p = 2ac + 3bd$ and $q = bc + ad$. Then, $(R, +, \odot)$ is a commutative G_H-ring, in which $\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{3}}$ are two hyperidentities. This is also to be noted that $\{\sqrt{2}+\sqrt{3}, -\sqrt{2}-\sqrt{3}\}$ is an i-set of $(R, +, \odot)$. In fact, for any $x \in R$, we see that,

$$\begin{aligned} x &= x \cdot (\sqrt{3}-\sqrt{2}) \cdot (\sqrt{2}+\sqrt{3}) = x \cdot \sqrt{3} \cdot (\sqrt{2}+\sqrt{3}) + x \cdot \sqrt{2} \cdot (-\sqrt{2}-\sqrt{3}) \\ &\in x \odot (\sqrt{2}+\sqrt{3}) + x \odot (-\sqrt{2}-\sqrt{3}). \end{aligned}$$

(c) On a non-trivial commutative group $(G, +)$, a hyperoperation $\odot$ is defined as $x \odot y = G \setminus \{0_G\} = G^*$, $\forall x, y \in G^*$ and $0_G \odot x = x \odot 0_G = \{0_G\}$, $\forall x \in G$. Then, $(G, +, \odot)$ is a commutative G_H-ring in which every non-empty finite subset is an i-set and thus in particular, every non-zero element of G_H-ring $(G, +, \odot)$ is its hyperidentity.

(d) The commutative (additive) group of integers $(\mathbb{Z}, +)$ endowed with the hyperoperation $x \odot y = \{xy, -xy\}$, $\forall x, y \in \mathbb{Z}$ is a commutative G_H-ring in which a non-empty finite subset $\mathcal{E} = \{e_1, e_2, \ldots, e_n\}$ is an i-set if and only if 1 or -1 is an element of $\sum_{i=1}^n A_i$, where $A_i = \{e_i, -e_i\}$. So, G_H-ring $(\mathbb{Z}, +, \odot)$ has exactly two hyperidentities which are 1 and -1. In fact, for any $x \in \mathbb{Z}, x \odot e_i = \{xf_i : f_i \in A_i\}$ and also $1 = \sum_{i=1}^n f_i \in \sum_{i=1}^n A_i \Leftrightarrow -1 = \sum_{i=1}^n (-f_i) \in \sum_{i=1}^n A_i$. Thus here, the subset $\mathcal{E} = \{e_1, e_2, \ldots, e_n\}$ of $\mathbb{Z}$ is an i-set $\Leftrightarrow x \in \sum_{i=1}^n x \odot e_i$ (for any $x \in \mathbb{Z}$) $\Leftrightarrow x = \sum_{i=1}^n xf_i$ (for some $f_i \in A_i$) $= x\sum_{i=1}^n f_i \Leftrightarrow 1 = \sum_{i=1}^n f_i \in \sum_{i=1}^n A_i$ (taking in $(\Rightarrow)$ implication, only a particular $x \neq 0$ in $\mathbb{Z}$).

(e) The additive group of 2×2 real matrices $(M_2(\mathbb{R}), +)$ endowed with hyperoperation $\odot$ defined as

$$\begin{bmatrix} x & y \\ z & t \end{bmatrix} \odot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \left\{ \begin{bmatrix} ay+cx & by+dx \\ at+cz & bt+dz \end{bmatrix}, \begin{bmatrix} ay-cx & by-dx \\ at-cz & bt-dz \end{bmatrix} \right\}$$

is a non-commutative G_H-ring with only two hyperidentities $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

(f) On the (additive) commutative group of integers $(\mathbb{Z}, +)$, $\odot$ is a hyperoperation defined by $x \odot y = \{2xy, 3xy\}$ for all $x, y \in \mathbb{Z}$. Then, $(\mathbb{Z}, +, \odot)$ is a commutative G_H-ring without having any hyperidentity. For any $x \in \mathbb{Z}, x = 5 \cdot 2 \cdot x + (-3) \cdot 3 \cdot x \in 5 \odot x + (-3) \odot x$ and hence, $(\mathbb{Z}, +, \odot)$ is a commutative G_H-ring with an i-set $\mathcal{E} = \{5, -3\}$.

(g) On the (additive) commutative group of integers $(\mathbb{Z},+)$, $\odot$ is a hyperoperation defined by $x \odot y = \{6xy, 9xy\}$ for all $x, y \in \mathbb{Z}$. Then, $(\mathbb{Z},+,\odot)$ is a commutative G_H-ring without having any i-set (and thus having no hyperidentity), since, the equation $6x + 9y = 1$ does not have any integral solution.

Definition 2.8. A non-zero element a of a G_H-ring $(R,+,\odot)$ with an i-set $\mathcal{E} = \{e_1, e_2, \dots, e_n\}$ is said to be left (right) invertible with respect to $\mathcal{E}$ if for each $i = 1, 2, \dots, n, (n \in \mathbb{N})$, there exist $r_{ij} \in R$ such that $e_i \in \sum_{j=1}^{m} r_{ij} \odot a$ (resp. $e_i \in \sum_{j=1}^{m} a \odot r_{ij}$). A non-zero element a of a G_H-ring with an i-set $\mathcal{E}$ is said to be invertible or a unit with respect to $\mathcal{E}$ if it is both left and right invertible with respect to $\mathcal{E}$.

A non-zero element a of a G_H-ring $(R,+,\odot)$ with a hyperidentity e is said to be left (right) hyperinvertible with respect to e if there exist $r \in R$ such that $e \in r \odot a$ (resp. $e \in a \odot r$). A non-zero element a of a G_H-ring with a hyperidentity e is said to be hyperinvertible or a hyperunit with respect to e if it is both left and right invertible with respect to e.

Example 2.9. (a) The G_H-ring in the example 2.7(f), having no hyperidentity does not contain any hyperunit. But it has two units 1 and -1 with respect to the i-set $\mathcal{E} = \{5, -3\}$.

(b) Every non-zero element of the G_H-ring in the example 2.7(b) is a hyperunit with respect to the hyperidentity $\frac{1}{\sqrt{2}}$. In fact, for any $x \in R$, we see that $\frac{1}{\sqrt{2}} = x \cdot \sqrt{2} \cdot r \in x \odot r$ where $r = \frac{1}{2x} \in R$.

(c) Every non-singular matrix in the G_H-ring stated in example 2.7 (e) is a hyperunit with respect to each of the hyperidentities $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Proposition 2.10. *Let $\mathcal{E} = \{e_1, e_2, \dots, e_n\}$ and $\mathcal{E}' = \{e'_1, e'_2, \dots, e'_m\}$ be two i-sets of a G_H-ring $(R,+,\odot)$ and $a \in R$ be a unit with respect to $\mathcal{E}$. Then, a is also a unit with respect to $\mathcal{E}'$.*

Proof. Since a is a unit in G_H-ring $(R,+,\odot)$, with respect to the i-set $\mathcal{E} = \{e_1, e_2, \dots, e_n\}$, so for each $i = 1, 2, \dots, n$, there exist $r_{ik}, s_{il} \in R$ such that $e_i \in (\sum_{k=1}^{p} r_{ik} \odot a) \cap (\sum_{l=1}^{q} a \odot s_{il})$. Again, for each $j = 1, 2, \dots, m$, $e'_j \in \sum_{i=1}^{n} e_i \odot e'_j \cap \sum_{i=1}^{n} e'_j \odot e_i$. Now, $e'_j \in \sum_{i=1}^{n} e_i \odot e'_j \subseteq \sum_{i=1}^{n} (\sum_{l=1}^{q} a \odot s_{il}) \odot e'_j \subseteq \sum_{i=1}^{n} (\sum_{l=1}^{q} (a \odot s_{il}) \odot e'_j) = \sum_{i=1}^{n} (\sum_{l=1}^{q} a \odot (s_{il} \odot e'_j)) \Rightarrow e'_j \in \sum_{i=1}^{n} (\sum_{l=1}^{q} a \odot t_{ilj})$ for some $t_{ilj} \in R$. Similarly, $e_i \in \sum_{k=1}^{p} r_{ik} \odot a$ and

$e'_j \in \sum_{i=1}^{n} e'_j \odot e_i$ imply that $e'_j \in \sum_{i=1}^{n}(\sum_{k=1}^{p} u_{ikj} \odot a)$ for some $u_{ikj} \in R$. Hence, a is a unit with respect to $\mathcal{E}'$. □

Definition 2.11. A non-zero element x of $(R, +, \odot)$ is called a left(resp. right) zero divisor in R if $\exists y \in R^*$ (resp. $z \in R^*$) such that $0_R \in x \odot y$ (resp. $0_R \in z \odot x$). A both left and right zero devisor in a G_H-ring R is called a zero divisor of R.

Example 2.12. Every element of the G_H-ring in the example 2.3(a) is a zero divisor, while the G_H-rings in examples 2.7. (b), (c), (d), (f), (g) have no zero divisor.

In a ring, a left (resp. right) invertible element [6] can never be a left (resp. right) divisor of zero. This not true in general for a G_H-ring. In fact, on the commutative group of integers $(\mathbb{Z}, +)$, if we define a hypercomposition $\odot$ by stating that, $x \odot y = \{0, xy\}$, $\forall x, y \in \mathbb{Z}$, then $(\mathbb{Z}, +, \odot)$ is a commutative G_H-ring with a hyperidentity 1. Every non-zero element of $(\mathbb{Z}, +, \odot)$ is a zero divisor and $1, -1 \in \mathbb{Z}$ are , in particular, hyperunits of $(\mathbb{Z}, +, \odot)$. To get a parity with the ring theory in this regard, we perceive the notion of strong hyperinvertibility of an element of a G_H-ring.

Definition 2.13. A left (resp. right) hyperinvertible element x of a G_H-ring $(R, +, \odot)$ (with respect to a hyperidentity e) is said to be strongly left (resp. right) hyperinvertible in R if it is not a left (resp. right) zero divisor in R. An element that is both strongly left and strongly right hyperinvertible with respect to e is said to be strongly hyperinvertible or to be a strong hyperunit with respect to e.

Example 2.14. The hyperunits in the G_H-rings stated in examples 2.7 (b), (c), (d), (f) are all strong hyperunits.

Proposition 2.15. *Let e be a hyperidentity of $(R, +, \odot)$ and $\star_r$ (resp. $\star_l$) be a hyperoperation on the set $V_r(R)$ (resp. $V_l(R)$) of all right (resp. left) hyperinvertible elements in R (with respect to e) defined by $a \star_r b = (a \odot b) \cap V_r(R)$, $\forall a, b \in V_r(R)$ (resp. $a \star_l b = (a \odot b) \cap V_l(R)$, $\forall a, b \in V_l(R)$). Then, $a \star_r (b \star_r c) \subseteq (a \star_r b) \star_r c$, $\forall a, b, c \in V_r(R)$ (resp. $(a \star_l b) \star_l c) \subseteq a \star_l (b \star_l c)$, $\forall a, b, c \in V_l(R)$) i.e., $(V_r(R), \star_r)$ (resp. $(V_l(R), \star_l)$) is an H_V-semigroup.*

Proof. For any $a, b \in V_r(R)$, $\exists x, y \in R$ such that $e \in a \odot x$ and $e \in b \odot y$. Then $a \in a \odot e \subseteq a \odot (b \odot y) = (a \odot b) \odot y \Rightarrow e \in a \odot x \subseteq ((a \odot b) \odot y) \odot x =$

$(a \odot b) \odot (y \odot x) \Rightarrow e \in c \odot u$ for some $c \in a \odot b$ and $u \in y \odot x$. Now since $e \neq 0_R, c \neq 0_R$. Hence, $c \in V_r(R)$ and thus $c \in (a \odot b) \cap V_r(R) = a \star_r b$ i.e., $a \star_r b \in P^*(V_r(R))$, $\forall a, b \in V_r(R)$.

Now, for any $a, b, c \in V_r(R), p \in a \star_r (b \star_r c) = a \star_r ((b \odot c) \cap V_r(R)) \Rightarrow p \in a \odot s$ for some $s \in (b \odot c) \cap V_r(R) \Rightarrow p \in (a \odot (b \odot c)) \cap V_r(R) = ((a \odot b) \odot c) \cap V_r(R) \Rightarrow p \in (t \odot c) \cap V_r(R)$ for some $t \in a \odot b$. Again, $p \in V_r(R) \Rightarrow \exists q \in R^*$ such that $e \in p \odot q \Rightarrow e \in (t \odot c) \odot q = t \odot (c \odot q)$ (since, $p \in t \odot c$) $\Rightarrow e \in t \odot v$ for some $v \in c \odot q \subseteq R \Rightarrow t \in V_r(R) \Rightarrow t \in (a \odot b) \cap V_r(R) = a \star_r b$. Thus, we have that $p \in (t \odot c) \cap V_r(R) = t \star_r c$ for some $t \in a \star_r b \Rightarrow p \in (a \star_r b) \star_r c$; whence, $a \star_r (b \star_r c) \subseteq (a \star_r b) \star_r c$.

Similarly, it can be proved that $a \star_l b \in P^*(V_l(R))$ and $(a \star_l b) \star_l c \subseteq a \star_l (b \star_l c)$, $\forall a, b, c \in V_l(R)$. □

Proposition 2.16. *Let $(R, +, \odot)$ be commutative and have a hyperidentity e. Then, $(V, \star)$ is a commutative hypergroup where V is the set of all hyperunits (with respect to e) and $a \star b = (a \odot b) \cap V$, $\forall a, b \in V$.*

Proof. Since $(R, +, \odot)$ is commutative, so $V_r(R) = V_l(R) = V$ and hence, on $V, \star_r = \star_l = \star$. Thus from proposition 2.15,$(V, \star)$ is semihypergroup. Now, for each $a \in V$, there exists $x \in V$ (since, $(R, +, \odot)$ is commutative) such that $e \in a \odot x$ and so, $e \in a \star x$ (since $e \in V$). Let $b \in V$ be arbitrary. Then, $b \in e \star b \subseteq (a \star x) \star b = a \star (x \star b) \Rightarrow b \in a \star r$ for some $r \in x \star b \Rightarrow b \in a \star V$ (note that $r \in x \star b \Rightarrow r \in V$). So, $V \subseteq a \star V$ whence, $a \star V = V$ for all $a \in V$. Hence, $(V, \star)$ is a commutative hypergroup. □

Definition 2.17. A G_H-ring $(R, +, \odot)$ with a hyperidentity e is called a division G_H-ring (resp. strong division G_H-ring) if every non-zero element of R is a hyperunit (resp. strong hyperunit) in R with respect to e.

Proposition 2.18. *A G_H-ring $(R, +, \odot)$ with a hyperidentity is a division G_H-ring if and only if $(R \setminus \{0_R\}, \star)$ is a hypergroup, where $a \star b = (a \odot b) \setminus \{0_R\}$, $\forall a, b \in R \setminus \{0_R\}$.*

Proof. Let e be a hyperidentity of $(R, +, \odot)$ and $(S, \star)$ be a hypergroup where $S = R \setminus \{0_R\}$ and $a \star b = a \odot b \setminus \{0_R\}$ for all $a, b \in S$. Then $a \star S = S = S \star a$ for any $a \in S$. So, $e \in a \star x \cap y \star a$ i.e., $e \in a \odot x \cap y \odot a$ for some $x, y \in S$, Thus every non-zero element of R is a hyperunit (with recpect to e). i.e., $(R, +, \odot)$ is a division G_H-ring.

Conversely let $(R,+,\odot)$ be a division G_H-ring with a hyperidentity e. Then, $V_r(R)=V_l(R)=R\setminus\{0_R\}$. So, $a\star b=(a\odot b)\cap(R\setminus\{0_R\})$, $\forall a,b\in R\setminus\{0_R\}$ and hence by proposition 2.14, $(R\setminus\{0_R\},\star)$ is a semihypergroup. Now, let $b\in R\setminus\{0_R\}$ be arbitrary. Then for each $a\in R\setminus\{0_R\}, e\in a\star c\cap d\star a$ for some $c,d\in R\setminus\{0_R\}$ (since, $e\in R\setminus\{0_R\}$). Hence, $b\in e\star b\subseteq a\star(c\star b)\Rightarrow b\in a\star t$, for some $t\in c\star b\subseteq R\setminus\{0_R\}\Rightarrow b\in a\star(R\setminus\{0_R\})$; so, $(R\setminus\{0_R\})\subseteq a\star(R\setminus\{0_R\})$, whence, $(R\setminus\{0_R\})=a\star(R\setminus\{0_R\})$. Similarly, $(R\setminus\{0_R\})\star a=(R\setminus\{0_R\})$. Hence,$(R\setminus\{0_R\},\star)$ is a hypergroup. □

Corollary 2.19. *A G_H-ring $(R,+,\odot)$ with a hyperidentity is a strong division G_H-ring if and only if $(R\setminus\{0_R\},\odot)$ is a hypergroup.*

Proof. It follows from definition 2.17 and proposition 2.18. □

Definition 2.20. A commutative G_H-ring $(R,+,\odot)$ with an i-set $\mathcal{E}$ is said to be a G_H-field if every element of $(R\setminus\{0_R\},\odot)$ is a unit with respect to $\mathcal{E}$. A commutative G_H-ring $(R,+,\odot)$ with a hyperidentity e is said to be a strong G_H-field if every element of $(R\setminus\{0_R\},\odot)$ is a strong hyperunit with respect to e.

Example 2.21. Each of the G_H-rings in the examples 2.7(b), (c) is a strong G_H-field.

Remark 2.22. A G_H-ring is a strong G_H-field if and only if it is a commutative strong division G_H-ring.

3. Ideals in G_H-rings

Let I be a non-empty subset of a G_H-ring $(R,+,\odot)$ such that $a+b\in I$ and $a\odot b\subseteq I$ for all $a,b\in I$. Then I is called a sub-G_H-ring of $(R,+,\odot)$ if $(I,+,\odot)$ is itself a G_H-ring. A sub-G_H-ring I of $(R,+,\odot)$ is called a left (resp. right) ideal of $(R,+,\odot)$ if $x\in R, a\in I\Rightarrow x\odot a\subseteq I$ (resp. $x\in R, a\in I\Rightarrow a\odot x\subseteq I$) ; I is called an ideal of $(R,+,\odot)$ if it is both a left and a right ideal of $(R,+,\odot)$.

For a Γ-ring M, a subgroup A of the additive group M is called a *left* (*right*) *ideal* of Γ-ring M if for any $x\in A, \alpha\in\Gamma, y\in M$, $y\alpha x\in A$ (resp. $x\alpha y\in A$). A subgroup A of the additive group M is called an *ideal* [8] of the Γ-ring M if it is both a left and a right ideal of M.

Remark 3.1. (a) A non-empty subset I of a Γ-ring M is an ideal of M if and only if I is an ideal of the associated G_H-ring M_Γ of M.

(b) For any ideals I and J of $(R, +, \odot), 0_R \in I$; both $I \cap J$ and $I + J = \{i + j : i \in I, j \in J\}$ are ideals of $(R, +, \odot)$.

(c) An ideal I of a G_H-ring $(R, +, \odot)$ with an i-set $\mathcal{E}$ is proper if and only if $\mathcal{E} \nsubseteq I$.

(d) A division G_H-ring (resp. a strong division G_H-ring) and thus a G_H-field (resp. a strong G_H-field) do not contain any proper ideal.

In case of a G_H-ring $(R, +, \odot)$,we have seen in proposition 2.4 that $(-x) \odot y \cap (-(x \odot y)) \neq \phi$ and $x \odot (-y) \cap (-(x \odot y)) \neq \phi$ for any x, y in R, where for any $A \in P^*(R), -A = \{-a : a \in A\}$. Unlike a ring, the equality of the set-expressions $(-x) \odot y$, $x \odot (-y)$ and $-(x \odot y)$ does not hold in general, on a G_H-ring $(R, +, \odot)$ for any $x, y \in R$. In fact, for any two non-zero elements x and y of the G_H-ring $(G, +, \odot)$, as stated in the example 2.3(a), we see that $(-x) \odot y \neq -(x \odot y)$ and $x \odot (-y) \neq -(x \odot y)$, while $(-x) \odot y = x \odot (-y)$ if and only if $x = y$.

Definition 3.2. A G_H-ring $(R, +, \odot)$ is said to satisfy the condition $(\mathcal{R})$ if the set equality $(-x) \odot y = x \odot (-y) = -(x \odot y)$ (called the condition$(\mathcal{R})$) holds true for any two elements x and y of R.

This is clear from the definition that the class of G_H-rings with condition $(\mathcal{R})$ is precisely that of multiplicative hyperrings with absorbing zero. Following are examples of some G_H-rings with condition$(\mathcal{R})$.

Example 3.3. (a) The associated G_H-ring M_Γ of a Γ-ring$M$ is a G_H-ring with condition$(\mathcal{R})$.

(b) Let $(G, +)$ be any (additive) commutative group and let, $a \otimes b = \{0_G, a, b, -a, -b\}$ for any $a, b \in G \setminus 0_G$ and $a \otimes 0_G = 0_G \otimes a = \{0_G\}$ for any $a \in G$. Then, $(G, +, \otimes)$ is a commutative G_H-ring with condition $(\mathcal{R})$.

(c) The G_H-ring in the example 2.7(b) is a G_H-ring with condition $(\mathcal{R})$.

(d) Olson and Ward consider in [12] a non-zero ring $(R, +, \cdot)$ and define on the additive group $(R, +)$ a hyperoperation $\star$ by stating that $a \star b = \{na \cdot b : n \in \mathbb{N}\}$, $\forall a, b \in R$. Then, $(R, +, \star)$ is a G_H-ring with condition $(\mathcal{R})$.

Definition 3.4. Let A be a subset of a G_H-ring $(R, +, \odot)$. Then, the (left, right) ideal of the G_H-ring R generated by A is the smallest (resp. left, right) ideal of the G_H-ring R containing A which is denoted by (resp.

$(A)_l, (A)_r)$ (A). The principal (left, right) ideal of the G_H-ring R generated by an element a of R, denoted by (resp. $(a)_l$, $(a)_r$) (a) is the (resp. left, right) ideal (resp. $(\{a\})_l$, $(\{a\})_r$) $(\{a\})$ of G_H-ring R.

Remark 3.5. Since 0_R of a G_H-ring $(R, +, \odot)$ is absorbing in R, (0_R) (resp. $(0_R)_l$, $(0_R)_r$) is $\{0_R\}$ and thus is the smallest (resp. left, right) ideal of R.

Proposition 3.6. *Let $(R, +, \odot)$ be a G_H-ring with condition($\mathcal{R}$) having an i-set $\mathcal{E}$. Then, for any A in $P^*(R)$, $(A)_l = \overline{R \odot A} = \cup\{\sum_{i=1}^{n} x_i \odot a_i : x_i \in R, a_i \in A, n \in \mathbb{N}\}$, $(A)_r = \overline{A \odot R} = \cup\{\sum_{i=1}^{n} a_i \odot x_i : x_i \in R, a_i \in A, n \in \mathbb{N}\}$ and $(A) = \overline{R \odot A \odot R} = \cup\{\sum_{i=1}^{n} x_i \odot a_i \odot y_i : x_i y_i \in R, a_i \in A, n \in \mathbb{N}\}$*

Proof. Let $p, q \in \overline{R \odot A}$ and $x \in R$. Then, $p \in \sum_{i=1}^{n} x_i \odot a_i$ and $q \in \sum_{j=1}^{m} y_j \odot b_j$ for some $x_i, y_j \in R$ and $a_i, b_j \in A$. So, $p - q \in (\sum_{i=1}^{n} x_i \odot a_i) - (\sum_{j=1}^{m} y_j \odot b_j) = \sum_{i=1}^{n} x_i \odot a_i + \sum_{j=1}^{m} -(y_j \odot b_j) = \sum_{i=1}^{n} x_i \odot a_i + \sum_{j=1}^{m} (-y_j) \odot b_j \subseteq \overline{R \odot A}$ and also $r \in x \odot p \subseteq x \odot \sum_{i=1}^{n} x_i \odot a_i \subseteq \sum_{i=1}^{n} x \odot (x_i \odot a_i) = \sum_{i=1}^{n} (x \odot x_i) \odot a_i \Rightarrow r \in \sum_{i=1}^{n} t_i \odot a_i \subseteq \overline{R \odot A}$ for some $t_i \in (x \odot x_i) \subseteq R$. Hence, $\overline{R \odot A}$ is a left ideal of R. Again, for any $a \in A$, and for the i-set $\mathcal{E} = \{e_1, e_1, \ldots, e_n\}$, $a \in \sum_{i=1}^{n} e_i \odot a \subseteq \overline{R \odot a} \Rightarrow a \in \overline{R \odot A}$. So, $A \subseteq \overline{R \odot A}$. Moreover, for any ideal K containing A, clearly $\overline{R \odot A} \subseteq K$. Hence, $(A)_l = \overline{R \odot A}$. The cases for $(A)_r$ and (A) can be proved similarly. $\square$

Definition 3.7. An ideal (resp. left, right) I of a G_H-ring $(R, +, \odot)$ is said to be a maximal ideal (resp. left, right) if $I \neq R$ and for any ideal (resp. left, right) K of R, $I \subsetneq K \subseteq R \Rightarrow K = R$.

Proposition 3.8. *Let $(R, +, \odot)$ be a G_H-ring with an i-set $\mathcal{E}$. Then, every proper ideal in R is contained in a maximal ideal.*

Proof. Let I be an ideal of R such that $I \neq R$ and $\mathcal{P}$ be the set of all ideals J in R such that $I \subseteq J \neq R$. Then, $\mathcal{P}$ is non-empty, since $I \in \mathcal{P}$. With respect to set theoretic inclusion, $\mathcal{P}$ is a poset. We consider any chain $\mathcal{C} = \{K_i : i \in \Lambda\}$ of ideals in the poset $\mathcal{P}$. Let $K = (\{\cup_{i \in \Lambda} K_i\})$ and let $a, b \in K$. Then, there are $i, j \in \Lambda$ such that $a \in K_i$ and $b \in K_j$. Since $\mathcal{C}$ is a chain, we suppose that $K_j \subseteq K_i$; then $a, b \in K_i$ and hence $a - b \in K_i \subseteq K$ and $x \odot a, a \odot x \subseteq K_i \subseteq K$ for all $x \in R$, whence K is an ideal of R. Now, $I \subseteq K_i$, $\forall i \in \Lambda \Rightarrow I \subseteq (\{\cup_{i \in \Lambda} K_i\}) = K$. Again,

$K = R \Rightarrow \mathcal{E} = \{e_1, e_2, \ldots, e_n\} \subseteq K = (\{\cup_{i\in\Lambda} K_i\}) \Rightarrow e_l \in K_{i_l}$ for some $i_l \in \Lambda, l = 1, 2, \ldots n$. Suppose that $i = \{max_l\{i_l\}\}$. Then, $\mathcal{E} \subseteq K_i$ which is not possible, since $K_i \in \mathcal{C}$ and $\mathcal{E}$ is an i-set of R. So, $K \neq R$ and hence $K \in \mathcal{P}$. Thus any chain in $\mathcal{P}$ has an upper bound in $\mathcal{P}$ and so by Zorn's lemma, $\mathcal{P}$ contains a maximal element which is clearly a maximal ideal containing I. □

Corollary 3.9. *In a G_H-ring with an i-set maximal ideal always exists.*

Proof. In a G_H-ring $(R, +, \odot)$, with an i-set $(0_R) \neq R$ and hence by proposition 3.8, there exists in G_H-ring R, a maximal ideal containing (0_R). □

Definition 3.10. Let ρ be an equivalence relation on a G_H-ring $(R, +, \odot)$ and let $A, B \in P^*(R)$. Then we define that, $A\rho B$ if for each $a \in A$ there exists $b' \in B$ such that, $a\rho b'$ holds and for each $b \in B$ there exists $a' \in A$ such that, $a'\rho b$ holds. An equivalence relation ρ on a G_H-ring $(R, +, \odot)$ is called a left (resp. right) congruence on R if for any $a, b, x \in R, a\rho b \Rightarrow$ (i) $(a+x)\rho(b+x)$ and (ii) $(x\odot a)\rho(x\odot b)$ (resp. $(a\odot x)\rho(b\odot x)$). An equivalence relation ρ on a G_H-ring $(R, +, \odot)$ is called a congruence on R if it is both left and right congruence on R.

Proposition 3.11. *Let I be an ideal of a G_H-ring $(R, +, \odot)$ and ρ be a relation on R defined by $a\rho b \Leftrightarrow a - b \in I$. Then, ρ is a congruence on R and the set $R/I = \{a + I : a \in R\}$ of all ρ-congruence classes is a G_H-ring (called the quotient G_H-ring of R induced by the ideal I) with respect to a binary operation $+$ defined by $(a + I) + (b + I) = (a + b) + I$ and a hyperoperation $\odot$ defined by $(a+I)\odot(b+I) = (a\odot b)+I = \{p+I : p \in a\odot b\}$.*

Proof. Since I is an ideal of the G_H-ring $(R, +, \odot)$, so $(I, +)$ is a subgroup of the commutative group $(R, +)$. Hence the relation ρ is a congruence relation on the group $(R, +)$. Now, let $a, b \in R$ be such that $a\rho b$ holds. Then, $a = b + i$ for some $i \in I$. Thus, for any $x \in R, x \odot a = x \odot (b + i) \subseteq x \odot b + x \odot i \subseteq x \odot b + I \Rightarrow$ for any $p \in (x \odot a), \exists q' \in x \odot b$ and $j \in I$ such that $p = q' + j$ i.e., $p\rho q'$ holds. Similarly, for any $q \in (x \odot b), \exists p' \in x \odot a$ such that, $p'\rho q$ holds. Hence , $(x \odot a)\rho(x \odot b)$. So, ρ is a congruence on R.

Now, if $a, b, c, d \in R$ be such that $a + I = c + I$ and $b + I = d + I$, then $(a \odot b) + I = (c \odot d) + I$. In fact, $p \in a \odot b = (c + i) \odot (d + j)$ (for some $i, j \in I) \subseteq (c \odot d) + (c \odot j) + (i \odot d) + (i \odot j) \subseteq c \odot d + I$ (since I

is an ideal of R) $\Rightarrow$ for any $p \in a \odot b, \exists q \in c \odot d$ such that $p - q \in I$; so, $(a \odot b) + I \subseteq (c \odot d) + I$. Similarly, $(c \odot d) + I \subseteq (a \odot b) + I$.

Clearly, here $(R/I, +)$ is a commutative group with identity $0_{R/I} = 0_R + I = I$ and $(R/I, \odot)$ is a semihypergroup. Again, for any $a, b, c \in R, (a+I) \odot ((b+I)+(c+I)) = (a+I)\odot((b+c)+I) = (a\odot(b+c))+I \subseteq (a\odot b+a\odot c)+I = ((a\odot b)+I)+((a\odot c)+I) = ((a+I)\odot(b+I))+((a+I)\odot(c+I))$. Moreover, for any $a \in R$, $0_{R/I} \odot (a+I) = (0_R+I) \odot (a+I) = (0_R \odot a)+I = \{0_R\}+I$ (since, $0_R \odot a = \{0_R\}) = \{0_R + I\} = \{I\} = \{0_{R/I}\}$. □

Proposition 3.12. *Let $(R, +, \odot)$ be a commutative G_H-ring with condition $(\mathcal{R})$ and with an i-set $\mathcal{E}$. Then, an ideal I of R is a maximal ideal of R if and only if R/I is a G_H-field.*

Proof. Let $\mathcal{E} = \{e_1, e_2, \ldots e_n\}$ be an i-set of the G_H-ring $(R, +, \odot)$ and I be an ideal of R such that the quotient G_H-ring R/I is a G_H-field. Since, a G_H-field is a non-zero G_H-ring, so, there exists $a \in R$ such that $a + I \neq 0_{R/I} = I$ and hence, $a \notin I$ implying that I is a proper ideal of R. So, there exists at least $e_i \in \mathcal{E}$ such that $e_i \notin I$; hence $e_i + I \neq I = 0_{R/I}$ in R/I for some $e_i \in \mathcal{E}$. Now, for any $a \in R, a \in \sum_{i=1}^{n}(e_i \odot a) \Rightarrow a + I \in \sum_{i=1}^{n}(e_i\odot a)+I = \sum_{i=1}^{n}(e_i+I)\odot(a+I)$. So, $\mathcal{E}_{R/I} = \{e_1+I, e_2+I, \ldots, e_n+I\}$ is an i-set of the quotient G_H-ring R/I of the G_H-ring $(R, +, \odot)$. Let K be an ideal of R such that $I \subsetneq K \subseteq R$. Then, $\exists a \in K$ such that $a \notin I$. Consequently, $a + I \neq I = \{0_{R/I}\}$ in R/I. So, for each $e_i + I \in \mathcal{E}_{R/I}$, there exist $r_{ij} \in R$ such that $e_i+I \in \sum_{j=1}^{m}(r_{ij}+I)\odot(a+I) = \sum_{j=1}^{m}(r_{ij}\odot a)+I \Rightarrow e_i + I = \sum_{j=1}^{m}(s_{ij} + I)$ (for some $s_{ij} \in r_{ij} \odot a \subseteq K$(since $a \in K$ for each $j = 1, 2, \ldots, m$)) $= (\sum_{j=1}^{m} s_{ij}) + I = k_i + I$ (for some $k_i = \sum_{j=1}^{m} s_{ij} \in K$) $\Rightarrow e_i - k_i \in I \subsetneq K \Rightarrow e_i \in K$, whence, $\mathcal{E} \subseteq K$ and thus, $K = R$. So, I is a maximal ideal R.

Conversely, let I be a maximal ideal of the G_H-ring $(R, +, \odot)$. Then, I being a proper ideal of G_H-ring R, there exists $e_i \in \mathcal{E}$ such that $e_i \notin I$ and hence, $e_i + I \neq I = 0_{R/I}$ in R/I. Thus R/I is a non-zero G_H-ring and $\mathcal{E}_{R/I} = \{e_1+I, e_2+I, \ldots, e_n+I\}$ is an i-set of R/I. Let, $a+I \in R/I\backslash\{0_{R/I}\}$. Then, $a \notin I$ (since, $0_{R/I} = I$). Suppose that, $K = (I, a)$ be the ideal of R generated by $I\cup\{a\}$. Clearly, $I \subsetneq K$ and hence $K = R$ (since I is a maximal ideal). So, $\mathcal{E} \subseteq K$. Now, since R is a commutative G_H-ring with condition $(\mathcal{R})$, so by proposition 3.6, $K = \cup\{\alpha+\sum_{j=1}^{m} x_j \odot a : \alpha \in I, x_j \in R, m \in \mathbb{N}\}$. Thus, for each $e_i \in \mathcal{E}, e_i \in \alpha + \sum_{j=1}^{m} r_{ij} \odot a$, for some $\alpha \in I$ and $r_{ij} \in R$ (since, $\mathcal{E} \subseteq K$). So, $e_i + I \in (\alpha + I) + \sum_{j=1}^{m}(r_{ij} \odot a + I) = I + \sum_{j=1}^{m}(r_{ij} \odot$

$a+I) = \sum_{j=1}^{m}(r_{ij} \odot a + I)$, (since, $I = 0_{R/I}$) $= \sum_{j=1}^{m}(r_{ij}+I) \odot (a+I)$. Hence, every non-zero element of R/I is a unit in R/I with respect to the i-set $\mathcal{E}_{R/I}$. Thus, R/I is a G_H-field. $\square$

Definition 3.13. Let $(R, +, \odot)$ and $(S, \oplus, \otimes)$ be two G_H-rings. A mapping $f : R \to S$ is called a G_H-ring homomorphism if for any $x, y \in R$, (i)$f(x+y) = f(x) \oplus f(y)$, (ii)$f(x \odot y) \subseteq f(x) \otimes f(y)$. A surjective (resp. injective) G_H-ring homomorphisms called an epimorphism (resp. monomorphism). A G_H-ring isomorphism is a bijective G_H-ring homomorphism.

Example 3.14. Let us consider the G_H-rings $(G, +, \odot)$ and $(G, +, \otimes)$ stated respectively in the examples 2.3(a) and 3.3(b). This is immediate to show here that the mapping $f : (G, +, \odot) \to (G, +, \otimes)$ defined by $f(a) = -a$, $\forall a \in G$ is an isomorphism from the G_H-ring $(G, +, \odot)$ to the G_H-ring $(G, +, \otimes)$.

Proposition 3.15. *Let, $f : R \to S$ be a (G_H-ring) homomorphism from a G_H-ring $(R, +, \odot)$ to a G_H-ring $(S, \oplus, \otimes)$. Then, the kernel of the G_H-ring R (i.e., the set ker. $(f) = \{x \in R : f(x) = 0_S\}$) is an ideal of the G_H-ring R.*

Proof. Since, here $f : R \to S$ is a G_H-ring homomorphism, so, $f : (R, +) \to (S, \oplus)$ is a group homomorphism and thus, $f(0_R) = 0_S \Rightarrow 0_R \in$ Ker.(f). and also for any $a, b \in$ Ker.(f), $a - b \in$ Ker.(f). Now, $a \in$ Ker.$(f) \Rightarrow f(a) = 0_S \Rightarrow f(x \odot a) \subseteq f(x) \otimes f(a) = \{0_S\} \Rightarrow x \odot a \subseteq$ Ker.(f). Similarly, $a \odot x \subseteq$ Ker.(f). Thus, Ker.(f) is an ideal of R. $\square$

Theorem 3.16 (Isomorphism theorem on G_H-rings)**.** *Let $f : R \to S$ be an epimorphism from a G_H-ring R to a G_H-ring S. Then, R / Ker.$(f) \simeq S$.*

Proof. Let $f : (R, +, \odot) \to (S, \oplus, \otimes)$ be a G_H-ring-epimorphism. Then, $f : (R, +) \to (S, \oplus)$ is a group-epimorphism. So, the mapping $\Psi : (R/\text{ker.}(f), +) \to (S, \oplus)$ defined by $\Psi(a+ \text{ker.}(f)) = f(a)$ is a group-isomorphism. Now, for any $a, b \in R$, $\Psi((a+ \text{ker.}(f)) \odot (b+ \text{ker.}(f))) = \Psi((a \odot b)+ \text{ker.}(f)) = f(a \odot b) \subseteq f(a) \otimes f(b) = \Psi(a+ \text{ker.}(f)) \otimes \Psi(b+ \text{ker.}(f))$. Thus, Ψ is a G_H-ring-isomorphism. $\square$

4. G_H-rings over a ring

Let $(R,+,\cdot)$ be a ring and $A \in P^*(R)$ with $|A| \geq 2$. On R, $\odot$ is a hyperoperation defined by $x \odot y = x \cdot A \cdot y = \{x \cdot a \cdot y : a \in A\}$ for all $x, y \in R$. Then, $(R,+,\odot)$ is a G_H-ring which is called the G_H-ring over the ring $(R,+,\cdot)$ induced by A and is denoted by R_A. This is to be noted that, for any ring $(R,+,\cdot)$ and any $A \in P^*(R)$ with $|A| \geq 2$, $x \odot (-y) = \{xa(-y) : a \in A\} = \{-(xay) : a \in A\} = -(x \odot y)$ and similarly $(-x) \odot y = -(x \odot y)$, for all $x, y \in R_A$. Thus, the G_H-ring over a ring is a G_H-ring with condition $(\mathcal{R})$ and hence it is a multiplicative hyperring with absorbing zero. We shall study in this section some properties of G_H-rings R_A over some rings R induced by $A \in P^*(R)$. At the on-set of the study of R_A over the ring $(R,+,\cdot)$, we restrict the symbol $L(A)$ (resp. $R(A)$) to denote the set of all those left (resp. right) zero divisors in the ring $(R,+,\cdot)$, that are taken from a non-empty subset A of R. $L(A), R(A)$ may be empty in some cases. Throughout this section,unless otherwise stated, $(R,+,\cdot)$ will stand for a ring and for the sake of brevity, for any $x, y \in R$, the element $x \cdot y$ will be written simply by juxtaposition xy.

Proposition 4.1. *The G_H-ring R_A over $(R,+,\cdot)$ has a zero divisor if and only if at least one of the following holds true:*

(i) $0_R \in A$;

(ii) $L(A) \cup R(A) \neq \phi$;

(iii) $L(xA) \neq \phi$ *and* $R(Ax) \neq \phi$ *for some* $x \in R$.

Proof. Let x be a zero divisor of the G_H-ring R_A. Then $\exists y, z \in R^*$ such that $0_{R_A} \in x \odot y \cap z \odot x$. So, $\exists a, b \in A$ such that $0_R = xay = zbx$. Suppose that $0_R \notin A$. Then we have the following possibilities. Case-1, at least one of the elements xa, zb, ay, bx of R is 0_R. If one of first two is true then a or b is in $L(A)$ (since, $0_R \notin A$), i.e., $L(A) \neq \phi$. If one of next two is true then, $R(A) \neq \phi$. Case-2, any of xa, zb, ay, bx is not 0_R. Then, $xa \in L(xA), zb \in L(zA), ay \in R(Ay), bx \in R(Ax)$ (since, $(xa)y = (zb)x = x(ay) = z(bx) = 0_R$). Thus for $x \in R, L(xA) \neq \phi$ and $R(Ax) \neq \phi$.

Conversely, let (i) is true. Then, every nonzero element of R is a zero divisor in the G_H-ring R_A (since, then $0_R \in A \Rightarrow 0_{R_A} = x0_Rx \in x \odot x$). Suppose (ii) is true and $a \in L(A)$. Then, $\exists x \in R^*$ such that $ax = 0_R$ and so, $xax = 0_R$ which implies that $0_{R_A} \in x \odot x$ (since, $a \in L(A) \Rightarrow a \in A$) whence x is a zero divisor in the G_H-ring R_A. By similar argument, $R(A) \neq \phi \Rightarrow R_A$

has a zero divisor. If (iii) is true i.e., if $L(xA) \neq \phi$ and $R(Ax) \neq \phi$ for some $x \in R$, then $\exists a, b \in A$ and $y, z \in R^*$ such that $xay = 0_R$ and $zbx = 0_R$ and so, $0_{R_A} \in x \odot y \cap z \odot x$, i.e., x is a zero divisor in the G_H-ring R_A. $\square$

Proposition 4.2. *Let I be a non-trivial proper ideal of $(R, +, \cdot)$. Then, the G_H-ring R_I does not have any of the left and the right i-set (and thus does not contain any hyperidentity).*

Proof. Straightforward. $\square$

Proposition 4.3. *Let $(R, +, \cdot)$ be a ring with unity 1_R and $A \in P^*(R)$ with $|A| \geq 2$. Then R_A is a G_H-ring with hyperidentity e if and only if e is a unit in $(R, +, \cdot)$ and $e^{-1} \in A$.*

Proof. e is a hyperidentity of $R_A \Rightarrow 1_R \in e \odot 1_R \cap 1_R \odot e \Rightarrow \exists a, b \in A$ such that $1_R = ea = be \Rightarrow a = b = e^{-1} \Rightarrow e$ is a unit of $(R, +, \cdot)$ and $e^{-1} \in A$. Conversely let e be a unit of $(R, +, \cdot)$ and $e^{-1} \in A$. Then for any $x \in R, x = ee^{-1}x = xe^{-1}e \Rightarrow x \in e \odot x \cap x \odot e \Rightarrow e$ is a hyperidentity of R_A.

It follows from proposition 4.3 that, for a ring $(R, +, \cdot)$ with unity 1_R, the G_H-ring R_A has 1_R as its hyperidentity if and only if $1_R \in A$. Proposition 4.3 also reveals the fact that the G_H-ring over the ring of integers $(\mathbb{Z}, +, \cdot)$ induced by a set $A \in P^*(\mathbb{Z}), (|A| \geq 2)$ cannot have any hyperidentity, unless $A \cap \{1 - 1\} \neq \phi$ (since, 1 and -1 are the only two units in the ring of integers). However, the following theorem states that the G_H-ring Z_A over the ring of integers $(\mathbb{Z}, +, \cdot)$ may have an i-set under some conditions imposed on A. $\square$

Proposition 4.4. *Let $(\mathbb{Z}, +, \cdot)$ be the ring of integers and A be a subsemigroup of the group $(\mathbb{Z}, +)$. Then, $\mathbb{Z}_A$ is a (commutative) G_H-ring with an i-set if and only if some elements a_j $(j \in \{1, 2, \cdots, m+2\})$, not all necessarily distinct, can be so chosen from A that the equation $\{\sum_{j=1}^{m} a_j x_j + x \cdot gcd(a_{m+1}, a_{m+2}) = 1\}$ is solvable in the ring $(\mathbb{Z}, +, \cdot)$ for x and x_j.*

Proof. Suppose, for some suitably chosen elements $a_j (j \in \{1, 2, \cdots, m+2\})$, not all necessarily distinct, of the set A, the equation $\{\sum_{j=1}^{m} a_j x_j\} + x \cdot gcd(a_{m+1}, a_{m+2}) = 1$ is solvable in the ring $(\mathbb{Z}, +, \cdot)$. Then, $\exists (c, b_1, \cdots, b_m) \in \mathbb{Z}^m$ such that $\{\sum_{j=1}^{m} a_j b_j\} + c \cdot gcd(a_{m+1}, a_{m+2}) = 1$. Thus $gcd(a_{m+1}, a_{m+2})$

divides the number $1 - \{\sum_{j=1}^{m} a_j b_j\}$. Note that $gcd(a_{m+1}, a_{m+2})$ being here well-defined, a_{m+1} and a_{m+2} are not both equal to zero. So, the equation $a_{m+1}y + a_{m+2}z = 1 - \{\sum_{j=1}^{m} a_j b_j\}$ must have an integral solution. Thus, $\exists(b_{m+1}, b_{m+2}) \in \mathbb{Z}^2$ such that $a_{m+1}b_{m+1} + a_{m+2}b_{m+2} + \{\sum_{j=1}^{m} a_j b_j\} = 1 \ldots (\alpha)$.Suppose only $k(\leq m+2)$ number of the elements $b_j(j = 1, 2, \cdots, m+2)$ are distinct and those are $b_{i_1}, b_{i_1}, \ldots, b_{i_k}$. Then, from (α), $\exists a_{i_1}, a_{i_1}, \ldots, a_{i_k} \in A$ such that $\{\sum_{p=1}^{k} a_{i_p} b_{i_p}\} = 1$ (In fact, if in (α), $b_l = b_q = b_{i_t}$ for some $l \neq q$ in $\{1, 2, \ldots, m+2\}$ and for some t in $\{1, 2, \ldots, k\}$ then, we write $a_l b_l + a_q b_q = (a_l + a_q)b_{i_t} = a_{i_t} b_{i_t}$ where $a_l + a_q = a_{i_t} \in A$, since, A is a subsemigroup of the group $(\mathbb{Z}, +)$). Hence, $x = \{\sum_{p=1}^{k}\} x a_{i_p} b_{i_p}$ i.e.,$x \in \{\sum_{p=1}^{k}\} x \odot b_{i_p}$, $\forall x \in \mathbb{Z}$. Hence,the G_H-ring $\mathbb{Z}_A$ has an i-set $B = \{b_{i_p} : p = 1, 2, \ldots, k\}$.

Conversely, let $\mathbb{Z}_A$ be a G_H-ring with an i-set $B = \{b_1, b_2, \cdots b_{m+2}\}$. Then, $1 \in \{\sum_{j=1}^{m+2} 1 \odot b_j\} \Rightarrow \{\sum_{j=1}^{m+2} a_j b_j\} = 1$ for some $a_j \in A$ (not necessarily all distinct for different $j = 1, 2, \cdots, m+2$). Clearly then, all of $a_j(j = 1, 2, \ldots m+2)$ can never be zero. Without any loss of generality, suppose that $a_{m+2} \neq 0$. Then, $a_{m+1}b_{m+1} + a_{m+2}b_{m+2} = 1 - \{\sum_{j=1}^{m} a_j b_j\} \Rightarrow$ the equation $a_{m+1}x_{m+1} + a_{m+2}x_{m+2} = 1 - \{\sum_{j=1}^{m} a_j b_j\}$ has integral solution. Thus for $a_{m+1}, a_{m+2} \in A, gcd(a_{m+1}, a_{m+2})$ divides the number $(1 - \{\sum_{j=1}^{m} a_j b_j\})$. So, $\exists c \in \mathbb{Z}$ such that $c \cdot gcd(a_{m+1}, a_{m+2}) = 1 - \{\sum_{j=1}^{m} a_j b_j\}$. Hence, the equation $\{\sum_{j=1}^{m} a_j x_j\} + x \cdot gcd(a_{m+1}, a_{m+2}) = 1$ is solvable in the ring $(\mathbb{Z}, +, \cdot)$. □

Proposition 4.5. *For the ring of integers* $(\mathbb{Z}, +, \cdot)$ *and for any subsemigroup* A *of the group* $(\mathbb{Z}, +)$*, the* G_H*-ring* Z_A *has an i-set if* A *has at least two relatively prime integers.*

Proof. It follows directly from the proposition 4.4. □

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