ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
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CONVERGENCE THEOREMS OF THE GOULD INTEGRAL WITH RESPECT TO A SUBMEASURE

BY

ALINA EMILIA IOSIF

Abstract. In this paper we establish a Vitaly type theorem of passing to the limit into the Gould integral with respect to a submeasure. As a consequence of the Vitaly type theorem a Lebesgue type theorem is obtained.

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1. Introduction

In [7] PRECUPANU and CROITORU have studied a Gould type integral for a real valued function with respect to a finite additive multimeasure taking its values in the family of all nonvoid, compact, convex sets of a Banach space and have obtained a Lebesgue-type theorem of passing to the limit into this kind of integral. SATCO [8] has established a Vitaly-type theorem of passing to the limit into the Gould integral with respect to a finite additive multimeasure. Following these two works, GAVRILUŢ has obtained in [2] a Vitaly-type theorem and a Lebesgue-type theorem of passing to the limit into the Gould integral with respect to a multisubmeasure taking its values in the family of all nonvoid, compact, convex sets of a Banach space.

In this paper we establish a Vitaly-type convergence theorem for the Gould integral with respect to a submeasure. Also, as a consequence of Vitaly-type theorem we give a Lebesgue-type theorem.

Let T be an abstract, nonvoid set, $\mathcal{A}$ an algebra of subsets of T and $f : T \to R$ a function. We denote by $\mathcal{P}$ the class of all finite partitions of

T by sets from $\mathcal{A}$. All over this paper we assume that $m : \mathcal{A} \mapsto \overline{\mathbb{R}}_+$ is a submeasure in Drewnowski's sense [1].

Let us also consider the following set functions associated to m:

$$\overline{m}(A) = \sup\{\sum_{i=1}^{n} m(A_i)\}, \text{for every } A \in \mathcal{A}, \tag{1}$$

where the supremum is extended over all finite partitions $\{A_i\}_{i=\overline{1,n}}$ of A, called *the variation* of m and

$$\widetilde{m}(A) = \inf\{\overline{m}(B); A \subseteq B, B \in \mathcal{A}\}, \text{ for every } A \subseteq T. \tag{2}$$

We note that $\widetilde{m}(A) = \overline{m}(A)$, for every $A \in \mathcal{A}$ and $\overline{m}$ is a finitely additive set function on $\mathcal{A}$.

Definition 1.1. A submeasure $m : \mathcal{A} \mapsto \overline{\mathbb{R}}_+$ is said to be *of finite variation* if $\overline{m}(T) < \infty$.

Definition 1.2. A submeasure $m : \mathcal{A} \mapsto \overline{\mathbb{R}}_+$ is said to be *absolutely continuous* with respect to a set function $\nu : \mathcal{A} \mapsto \mathbb{R}$, if for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for every $A \in \mathcal{A}$ with $\overline{m}(A) < \delta$, we have $|\nu(A)| < \varepsilon$, denoted by $m \ll \nu$.

We recall from [5] the following notions:

Definition 1.3. i) A set $A \in \mathcal{A}$ is said to be *m-bounded* if

$$\{m(B); B \subseteq A, B \in \mathcal{A}\} \text{ is a bounded set in } \mathbb{R}.$$

ii) The space $(T, \mathcal{A}, m)$ is said to be *non-degenerate* if for any m-unbounded set $A \in \mathcal{A}$, there exists $\alpha > 0$ such that every m-unbounded set $B \in \mathcal{A}$, $B \subseteq A$, contains a m-bounded set $C \in \mathcal{A}$, $C \subseteq B$ such that $m(C) > \alpha$.

Definition 1.4. We say that *a sequence of functions* $(f_n)_n$, where $f_n : T \to \mathbb{R}$, for every $n \in N$, *is convergent in submeasure to* f (denoted by $f_n \xrightarrow{m} f$) if for every $\delta > 0, \lim_{n\to\infty} \widetilde{m}(B_n(\delta)) = 0$, where

$$B_n(\delta) = \{t \in T; |f_n(t) - f(t)| \geq \delta\}.$$

Since $\overline{m}$ is additive, using Definition 2.8 [5], we are in the Gould case and $f_n \xrightarrow{m} f$ if and only if $f_n \xrightarrow{\overline{m}} f$.

In the sequel, we recall from [3] and [6] some definitions and results concerning the Gould type integral with respect to a submeasure.

Definition 1.5. I) f is said to be *$\overline{m}$-totally-measurable on* $(T, \mathcal{A}, m)$ if for every $\varepsilon > 0$ there exists a partition $P_\varepsilon = \{A_i\}_{i=\overline{0,n}}$ of T such that:

i) $\overline{m}(A_0) < \varepsilon$ and

ii) $\sup\limits_{t,s \in A_i} |f(t) - f(s)| = osc(f, A_i) < \varepsilon$, for every $i = \overline{1,n}$.

II) f is said to be *$\overline{m}$-totally-measurable on* $B \in \mathcal{A}$ if its restriction $f|_B$ of f to B is $\overline{m}$-totally measurable on $(B, \mathcal{A}_B, m_B)$, where $m_B = m|_{\mathcal{A}_B}$ and $\mathcal{A}_B = \{A \cap B; A \in \mathcal{A}\}$.

To define the notion of m-integrability [3] we assume that $m : \mathcal{A} \mapsto \mathbb{R}_+$ is a submeasure of finite variation and $f : T \to R$ is a bounded function.

Let $\sigma(P) = \sum_{i=1}^{n} f(t_i)m(A_i)$, for every partition $P = \{A_i\}_{i=\overline{1,n}}$ of T and every $t_i \in A_i, i = \overline{1,n}$.

Definition 1.6. I) f is said to be *m-integrable on* T if there exists $I \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists a partition P_ε of T so that for every other partition $P = \{A_i\}_{i=\overline{1,n}}$, with $P \geq P_\varepsilon$ and every choice of points $t_i \in A_i, i = \overline{1,n}$, we have $|\sigma(P) - I| < \varepsilon$.

In this case, I is called *the integral of f on T with respect to the submeasure m*, denoted by $\int_T f dm$.

II) f is said to be *m-integrable on* $B \in \mathcal{A}$ if its restriction $f|_B$ of f to B is *m-integrable* on $(B, \mathcal{A}_B, m_B)$.

We now recall from [6] the notion of generalized Gould integrability with respect to a submeasure.

Let us consider $(T, \mathcal{A}, \overline{m})$ a non-degenerate space and

$$M(f) = \{K \in \mathcal{A}; \overline{m}(K) < \infty \text{ and } f \text{ is bounded on } K\}.$$

Definition 1.7. We say that f is *generalized m-integrable* (briefly, *$g-m$-integrable*) *on* T if the following conditions are satisfied:

i) f is $\overline{m}$-totally-measurable;

ii) there exists $I \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists $K_\varepsilon \in M(f)$ with the property that for any $K \in M(f)$ with $K \supseteq K_\varepsilon$, we have:

$$\left| \int_K f dm - I \right| < \varepsilon,$$

where $\int_K f dm$ is the integral in the sense of Definition 1.6.

I is called the *generalized Gould integral* of f and is denoted by $I = \int_T f dm$.

2. Convergence theorems

We now establish the main result of this paper:

Theorem 2.1. (The Vitali-type Theorem) *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real g-m-integrable functions on T, satisfying the following conditions:*

i) there exists $f : T \to \mathbb{R}$ such that $f_n \overset{m}{\to} f$;

ii) $\int_A f_n dm \ll m$ uniformly in $n \in \mathbb{N}$;

iii) for every $\varepsilon > 0$, there exists $A_\varepsilon \in \mathcal{A}$ such that $\overline{m}(A_\varepsilon) < \infty$ and $\left|\int_A f_n dm\right| < \varepsilon$, for every $A \in \mathcal{A}$, $A \subseteq T\backslash A_\varepsilon$ and every $n \in \mathbb{N}$.

Then f is g-m-integrable function on T and

$$\lim_{n\to\infty} \int_A f_n dm = \int_A f dm,$$

uniformly with respect to $A \in \mathcal{A}$.

Proof. First, we prove that f is $\overline{m}$-totally-measurable.

Since $f_n \overset{m}{\to} f$ we have $\lim_{n\to\infty} \widetilde{m}(B_n(\delta)) = 0$, for every $\delta > 0$. Let be $\varepsilon > 0$. Hence, for $\delta = \frac{\varepsilon}{3}$ there exist $n_\varepsilon \in \mathbb{N}$ and $A_\varepsilon \in \mathcal{A}$ with $\overline{m}(A_\varepsilon) < \frac{2\varepsilon}{3}$ such that $A_\varepsilon \supset \{t \in T; |f_n(t) - f(t)| \geq \frac{\varepsilon}{3}\}$. Because f_{n_ε} is $\overline{m}$-totally-measurable on T there exists a partition $\{B_i\}_{i=\overline{0,p_\varepsilon}}$ of T such that $\overline{m}(B_0) < \frac{\varepsilon}{3}$ and $\sup_{t,s\in B_i} |f_{n_\varepsilon}(t) - f_{n_\varepsilon}(s)| < \frac{\varepsilon}{3}$, for every $i = \overline{1, p_\varepsilon}$.

Let us consider the partition $P_\varepsilon = \{B_0 \cup A_\varepsilon, B_i\backslash A_\varepsilon\}, \forall i = \overline{1, p_\varepsilon}$ of T. We have $\overline{m}(B_0 \cup A_\varepsilon) < \varepsilon$ and $\sup_{t,s\in B_i\backslash A_\varepsilon} |f(t) - f(s)| < \varepsilon$, which means that f is $\overline{m}$-totally-measurable on T.

We shall prove the conclusion of the theorem in two steps.

Step I. $\overline{m}(T) < \infty$

Let $\varepsilon > 0$ and $M = \overline{m}(T) \neq 0$. (if $\overline{m}(T) = 0$, then $m(A) = 0$, $\forall A \in \mathcal{A}$ and the theorem holds).

I. 1. We shall show that for every $\varepsilon > 0$, there exists $\delta_1(\varepsilon) = \delta_1 > 0$ such that

$$\left|\int_A f dm\right| < \varepsilon, \text{ for every } A \in M(f) \text{ with } \overline{m}(A) < \delta_1. \tag{3}$$

Indeed, from ii) there exists $\delta(\varepsilon) = \delta > 0$ such that

$$\left|\int_A f_n dm\right| < \frac{\varepsilon}{9}, \text{ for every } A \in \mathcal{A} \text{ with } \overline{m}(A) < \delta \text{ and for every } n \in \mathbb{N}. \tag{4}$$

Let us take $A \in \mathcal{A}$ with $\overline{m}(A) < \delta_1 = \min\{\delta, 1\} > 0$ and on which f is bounded. According to Theorem 2.10 [4] there is $\delta_2(\varepsilon, A) = \delta_2 > 0$ such that

$$\left|\int_B f dm\right| < \frac{\varepsilon}{9}, \text{for every } B \in \mathcal{A}, B \subseteq A \text{ with } \overline{m}(B) < \delta_2. \tag{5}$$

Since $f_n \overset{m}{\to} f$, for every $\varepsilon > 0$ there exist $n_0(\varepsilon, A) = n_0 \in \mathbb{N}$ and $C_0(\varepsilon, A) = C_0 \in \mathcal{A}, C_0 \subset A$ with $\overline{m}(C_0) < \delta_2$ such that

$$|f_{n_0}(t) - f(t)| < \frac{\varepsilon}{9M}, \text{ for every } t \in T \backslash C_0. \tag{6}$$

According to Theorem 2.12 [3] we have

$$\int_A f dm = \int_{A \cap C_0} f dm + \int_{A \backslash C_0} f dm. \tag{7}$$

Consequently, using successively (7), (5), (4), we obtain

$$\begin{aligned} \left|\int_A f dm\right| &\leq \left|\int_{A \cap C_0} f dm\right| + \left|\int_{A \backslash C_0} f dm\right| \\ &< \frac{\varepsilon}{9} + \left|\int_{A \backslash C_0} f_{n_0} dm\right| + \left|\int_{A \backslash C_0} f_{n_0} dm - \int_{A \backslash C_0} f dm\right| \\ &< \frac{2\varepsilon}{9} + \left|\int_{A \backslash C_0} f_{n_0} dm - \int_{A \backslash C_0} f dm\right|. \end{aligned} \tag{8}$$

Because f is bounded on A there exists $N > 0$ such that

$$\sup_{t \in A \backslash C_0} |f(t)| \leq N.$$

Then $|f_{n_0}(t)| \leq |f_{n_0}(t) - f(t)| + |f(t)| < \frac{\varepsilon}{9} + N$, $\forall t \in A \backslash C_0$. Hence, f_{n_0} is bounded on $A \backslash C_0$. Applying the definition 1.5 of m-integrability for f and

f_{n_0} there exists a common partition $P_\varepsilon = \{B_i\}_{i=\overline{1,p}}$ of $A\backslash C_0$ such that for every $t_i \in B_i, i = \overline{1,p}$ we have

$$\left| \int_{A\backslash C_0} f dm - \sum_{i=1}^{p} f(t_i) m(B_i) \right| < \frac{\varepsilon}{9}$$

and

$$\left| \int_{A\backslash C_0} f_{n_0} dm - \sum_{i=1} p f_{n_0}(t_i) m(B_i) \right| < \frac{\varepsilon}{9}.$$

Consequently, using (6), we have

$$\begin{aligned} &\left| \int_{A\backslash C_0} f_{n_0} dm - \int_{A\backslash C_0} f dm \right| < \frac{2\varepsilon}{9} + \sum_{i=1}^{p} |f_{n_0}(t_i) - f(t_i)| \, m(B_i) \\ &< \frac{2\varepsilon}{9} + \frac{\varepsilon}{9M} \overline{m}(A\backslash C_0) < \frac{2\varepsilon}{9} + \frac{\varepsilon}{9M} M = \frac{\varepsilon}{3}, \end{aligned} \tag{9}$$

that used in (8) leads us to

$$\left| \int_A f dm \right| < \frac{5\varepsilon}{9} < \varepsilon. \tag{10}$$

I. 2. To prove that f is g-m-integrable we shall use Theorem 2.5 [6].

From the hypothesis i) for $\varepsilon > 0$ there exists $n_0(\varepsilon) = n_0 \in \mathbb{N}$ and such that for every $n \in \mathbb{N}$, $n \geq n_0$ there is a set $A(\varepsilon, n) = A \in \mathcal{A}$ with $\overline{m}(A) < \delta_1$ such that

$$|f_n(t) - f(t)| < \frac{\varepsilon}{9M}, \text{ for every } t \in T\backslash A. \tag{11}$$

Because f_{n_0} is g-m-integrable, according to Theorem 2.5 [6] there is a set $A'(\varepsilon, n_0) = A' \in M(f_{n_0})$ such that for every $K \in M(f_{n_0})$ with $K \cap A' = \emptyset$ we have

$$\left| \int_K f_{n_0} dm \right| < \frac{\varepsilon}{9}. \tag{12}$$

Let $K_\varepsilon = A'\backslash A$. We observe that

$$|f(t)| \leq |f_{n_0}(t) - f(t)| + |f_{n_0}(t)| < \frac{\varepsilon}{9M} + \sup_{t \in A'} |f_{n_0}(t)|,$$

for every $t \in A^{'}\backslash A$. Hence, $K_\varepsilon \in M(f)$.

Now, let us take $K \in M(f)$ with $K \cap K_\varepsilon = \emptyset$. Applying Theorem 2.12 [3] we get that

$$\int_K f dm = \int_{K\cap A} f dm + \int_{K\backslash A} f dm.$$

Then, from (10), we obtain

$$\left|\int_K f dm\right| = \left|\int_{K\cap A} f dm + \int_{K\backslash A} f dm\right| \leq \left|\int_{K\cap A} f dm\right| + \left|\int_{K\backslash A} f dm\right| < \frac{5\varepsilon}{9} + \left|\int_{K\backslash A} f dm\right|$$

and since $K\backslash A = [(K \cap A^{'}) \cup (K\backslash A^{'})]\backslash A = [K \cap (A^{'}\backslash A)] \cup [(K\backslash A^{'})\backslash A] = (K\backslash A^{'})\backslash A$, applying successively (12) and (9) on the set $(K\backslash A^{'})\backslash A$, it results

$$\left|\int_K f dm\right| < \frac{5\varepsilon}{9} + \left|\int_{(K\backslash A')\backslash A} f dm\right| < \frac{5\varepsilon}{9} + \left|\int_{(K\backslash A')\backslash A} f_{n_0} dm\right| + \left|\int_{(K\backslash A')\backslash A} f_{n_0} dm - \int_{(K\backslash A')\backslash A} f dm\right| < \frac{5\varepsilon}{9} + \frac{\varepsilon}{9} + \frac{\varepsilon}{3} = \varepsilon.$$

I. 3. We now prove that $\lim_{n\to\infty} \int_A f_n dm = \int_A f dm$, uniformly with respect to $A \in \mathcal{A}$.

Indeed, since the m-g-integral of f is absolutely continuous with respect to $\overline{m}$ (Theorem 3.2 -iii) [6]), for every $\varepsilon > 0$, there exists $\widetilde{\delta}(\varepsilon) = \widetilde{\delta} > 0$ such that $\left|\int_B f dm\right| < \frac{\varepsilon}{5}$, for every $B \in \mathcal{A}$, with $\overline{m}(B) < \widetilde{\delta} \leq \delta$. Then, as in Step I.2, there is $n_0(\varepsilon) = n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, there exists $C_n \in \mathcal{A}$ with $\overline{m}(C_n) < \widetilde{\delta}$ and $C_n \supseteq \{t \in T; |f_n(t) - f(t)| \geq \frac{\varepsilon}{5\overline{m}(T)}\}$. Hence, for every $n \geq n_0$ and every $A \in \mathcal{A}$, we obtain

$$\left|\int_A f_n dm - \int_A f dm\right| \leq \left|\int_{A\cap C_n} f_n dm - \int_{A\cap C_n} f dm\right| + \left|\int_{A\backslash C_n} f_n dm - \int_{A\backslash C_n} f dm\right| \leq \left|\int_{A\cap C_n} f_n dm\right|$$

$$+\left|\int_{A\cap C_n} f dm\right| + \left|\int_{A\setminus C_n} f_n dm - \int_{A\setminus C_n} f dm\right|$$
$$< \frac{\varepsilon}{9} + \frac{\varepsilon}{5} + \left|\int_{A\setminus C_n} f_n dm - \int_{A\setminus C_n} f dm\right|.$$

But, as in the Step I.1 (see (9)), we have $\left|\int_{A\setminus C_n} f_n dm - \int_{A\setminus C_n} f dm\right| < \frac{\varepsilon}{5}$. Consequently,

$$\left|\int_A f_n dm - \int_A f dm\right| < \frac{\varepsilon}{9} + \frac{2\varepsilon}{5} < \varepsilon,$$

as claimed.

Step II. $\overline{m}(T) = \infty$.

From the hypothesis iii), for every $\varepsilon > 0$, there exists $A_\varepsilon \in \mathcal{A}$ such that $\overline{m}(A_\varepsilon) < \infty$ and

$$\left|\int_A f_n dm\right| < \frac{\varepsilon}{7}, \text{ for every } A \in \mathcal{A}, A \subseteq T\setminus A_\varepsilon \text{ and every } n \in \mathbb{N}. \tag{13}$$

Since f is m-integrable on A_ε (from Step I), by virtue of Theorem 2.5 [6] one can find $\widetilde{A}_\varepsilon \in M_{A_\varepsilon}(f)$ such that

$$\left|\int_A f dm\right| < \frac{\varepsilon}{7}, \text{ for every } A \in M_{A_\varepsilon}(f) \text{ with } A \cap \widetilde{A}_\varepsilon = \emptyset. \tag{14}$$

To prove the g-m-integrability of f we shall again use Theorem 2.5 [6] . With this end in mind, let $K \in M(f), K \cap \widetilde{A}_\varepsilon = \emptyset$. By virtue of Theorem 3.2 -iii) [6] for every $\varepsilon > 0$ there is $\delta(\varepsilon, K) = \delta > 0$ such that

$$\left|\int_B f dm\right| < \frac{\varepsilon}{7}, \text{ for every } B \subset K, B \in \mathcal{A} \text{ with } \overline{m}(B) < \delta. \tag{15}$$

Now, from i) there exist $n_0 \in \mathbb{N}$ and $C_{n_0} \in \mathcal{A}$ with $\overline{m}(C_{n_0}) < \delta$ such that $C_{n_0} \supset \{t \in T; |f_{n_0}(t) - f(t)| \geq \frac{\varepsilon}{7\overline{m}(K)}\}$ Hence, holds

$$|f_{n_0}(t) - f(t)| < \frac{\varepsilon}{7\overline{m}(K)}, \text{ for every } t \in T\setminus C_{n_0}.$$

According to Theorem 2.12 [3] we get that

$$\int_K f dm = \int_{K\setminus(A_\varepsilon\setminus\widetilde{A}_\varepsilon)} f dm + \int_{K\cap(A_\varepsilon\setminus\widetilde{A}_\varepsilon)} f dm \tag{16}$$

and since $K\backslash(A_\varepsilon\backslash\widetilde{A_\varepsilon}) = K\cap c(A_\varepsilon\backslash\widetilde{A_\varepsilon}) = K\cap[c(A_\varepsilon\cap c\widetilde{A_\varepsilon})] = K\cap(cA_\varepsilon\cup\widetilde{A_\varepsilon}) = (K\backslash A_\varepsilon)\cup(K\cap\widetilde{A_\varepsilon}) = (K\backslash A_\varepsilon)\cup\emptyset = K\backslash A_\varepsilon$, from (16) it results

$$\left|\int_K f dm\right| \leq \left|\int_{K\backslash A_\varepsilon} f dm\right| + \left|\int_{K\cap(A_\varepsilon\backslash\widetilde{A}_\varepsilon)} f dm\right|. \tag{17}$$

By virtue of Theorem 2.12 [3] we obtain

$$\int_{K\backslash A_\varepsilon} f dm = \int_{(K\backslash A_\varepsilon)\cap C_{n_0}} f dm + \int_{K\backslash A_\varepsilon\backslash C_{n_0}} f dm.$$

Consequently, from (17), using successively (14) and (15), we have

$$\begin{aligned}\left|\int_K f dm\right| &< \left|\int_{(K\backslash A_\varepsilon)\cap C_{n_0}} f dm\right| + \left|\int_{(K\backslash A_\varepsilon)\backslash C_{n_0}} f dm\right| + \frac{\varepsilon}{7}\\ &< \frac{\varepsilon}{7} + \frac{\varepsilon}{7} + \left|\int_{(K\backslash A_\varepsilon\backslash)C_{n_0}} f_{n_0} dm\right|\\ &\quad + \left|\int_{(K\backslash A_\varepsilon)\backslash C_{n_0}} f_{n_0} dm - \int_{(K\backslash A_\varepsilon)\backslash C_{n_0}} f dm\right|.\end{aligned}$$

Then, applying Theorem 2.7-iii) [3], we have

$$\begin{aligned}\left|\int_K f dm\right| &< \frac{3\varepsilon}{7} + \sup_{t\in(K\backslash A_\varepsilon)\backslash C_{n_0}} |f_{n_0}(t) - f(t)|\,\overline{m}(K)\\ &< \frac{3\varepsilon}{7} + \frac{\varepsilon}{7\overline{m}(K)}\overline{m}(K) = \frac{4\varepsilon}{7} < \varepsilon,\end{aligned} \tag{18}$$

which assures that f is g-m-integrable on T.

We now prove the second assertion of the theorem.

From Step I) we have that for every $\varepsilon > 0$ there is $n_0(\varepsilon) = n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$

$$\left|\int_B f_n dm - \int_B f dm\right| < \frac{\varepsilon}{7}, \text{ for each } B \subseteq A_\varepsilon, \text{ with } B \in \mathcal{A}.$$

Then, for every $A \in \mathcal{A}$, we obtain

$$\left|\int_A f_n dm - \int_A f dm\right| = \left|\int_{A\cap A_\varepsilon} f_n dm - \int_{A\cap A_\varepsilon} f dm\right|$$
$$+ \left|\int_{A\setminus A_\varepsilon} f_n dm - \int_{A\setminus A_\varepsilon} f dm\right| < \frac{\varepsilon}{7} + \left|\int_{A\setminus A_\varepsilon} f_n dm\right| + \left|\int_{A\setminus A_\varepsilon} f dm\right|$$
$$< \frac{2\varepsilon}{7} + \left|\int_{A\setminus A_\varepsilon} f dm\right|.$$

Because f is g-m-integrable on $A\setminus A_\varepsilon$ there exists $A^\varepsilon \in M(f)$, $A^\varepsilon \subseteq A\setminus A_\varepsilon$, such that

$$\left|\int_{A\setminus A_\varepsilon} f dm - \int_{A^\varepsilon} f dm\right| < \frac{\varepsilon}{7}. \tag{19}$$

We observe that $A^\varepsilon \cap \widetilde{A_\varepsilon} = \emptyset$ and using (18) for A^ε, we have

$$\left|\int_{A^\varepsilon} f dm\right| < \frac{4\varepsilon}{7}. \tag{20}$$

Using (19) and (20), we obtain

$$\left|\int_{A\setminus A_\varepsilon} f dm\right| \leq \left|\int_{A\setminus A_\varepsilon} f dm - \int_{A^\varepsilon} f dm\right| + \left|\int_{A^\varepsilon} f dm\right| < \frac{\varepsilon}{7} + \frac{4\varepsilon}{7} = \frac{5\varepsilon}{7}.$$

Consequently,

$$\left|\int_A f_n dm - \int_A f dm\right| < \frac{2\varepsilon}{7} + \frac{5\varepsilon}{7} = \varepsilon, \text{ for every } n \geq n_0.$$

Thus the proof is complete. □

Corollary 2.3. (Lebesgue) *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real $\overline{m}$-totally measurable functions on T, satisfying the following conditions:*

i) there exists $f : T \to \mathbb{R}$ such that $f_n \xrightarrow{m} f$;

ii) there exists $g : T \to \mathbb{R}_+$ a bounded g-m-integrable function on T such that $|f_n(t)| \leq g(t), (\forall)\ n \in \mathbb{N}$, $(\forall)t \in T$ and $|f(t)| \leq g(t)$, $(\forall)t \in T$.

Then for every n, f_n and f are g-m-integrable and $\lim_{n\to\infty} \int_A f_n dm = \int_A f dm$, uniformly with respect to $A \in \mathcal{A}$.

Proof. We shall show that the hypothesis of the Vitali-type theorem are satisfied.

Since g is g-m-integrable on T for every $\varepsilon > 0$ there exists $K_\varepsilon \in M(g)$ such that for every $K \in M(g)$ with $K \cap K_\varepsilon = \emptyset$, we have

$$\left|\int_K g dm\right| = \int_K g dm < \frac{\varepsilon}{3}.$$

According to Theorems 2.17 and 2.7-v) from [3] we get that

$$\left|\int_K f_n dm\right| \leq \int_K |f_n|\, dm \leq \int_K g dm < \frac{\varepsilon}{3}, \text{ for every } n \in \mathbb{N} \tag{21}$$
and for every $K \in M(f_n)$ with $K \cap K_\varepsilon = \emptyset$.

Hence, f_n are g-m-integrable.

Now we prove that the condition ii) holds. Indeed, since the g-m-integral is absolutely continuous with respect to $\overline{m}$, for every $\varepsilon > 0$ there is $\delta(\varepsilon) = \delta > 0$ such that $\left|\int_{K'} g dm\right| < \frac{\varepsilon}{3}$, for every $K' \subseteq K_\varepsilon$ with $\overline{m}(K') < \delta$. Hence, from (21) we have $\left|\int_{K'} f_n dm\right| < \frac{\varepsilon}{3}$, for every $n \in \mathbb{N}$.

Let us consider $K \in \mathcal{A}$ with $\overline{m}(K) < \delta$. First, we prove that for every $C \in \mathcal{A}$ with $C \cap K_\varepsilon = \emptyset$ we have $\left|\int_C f_n dm\right| < \frac{2\varepsilon}{3}$. Indeed, let $C \in \mathcal{A}$ be such that $C \cap K_\varepsilon = \emptyset$. Since for every $n \in \mathbb{N}$, f_n is g-m-integrable on C for every $\varepsilon > 0$ there exists $\widetilde{K_\varepsilon} \in M_C(f_n)$ such that for every $K_1 \in M_C(f_n)$, $K_1 \supseteq \widetilde{K_\varepsilon}$,

$$\left|\int_{K_1} f_n dm - \int_C f_n dm\right| < \frac{\varepsilon}{3}\ .$$

Since $C \cap K_\varepsilon = \emptyset$ then $K_1 \cap K_\varepsilon = \emptyset$ and we obtain

$$\left|\int_C f_n dm\right| \leq \left|\int_{K_1} f_n dm - \int_C f_n dm\right| + \left|\int_{K_1} f_n dm\right| < \frac{2\varepsilon}{3}. \tag{22}$$

Taking $C = K \backslash K_\varepsilon$ in (22) we have

$$\left|\int_{K \backslash K_\varepsilon} f_n dm\right| < \frac{2\varepsilon}{3}. \tag{23}$$

Hence, from (22) and (23) we obtain

$$\begin{aligned}\left|\int_K f_n dm\right| &= \left|\int_{K\cap K_\varepsilon} f_n dm + \int_{K\setminus K_\varepsilon} f_n dm\right| \\ &\leq \left|\int_{K\cap K_\varepsilon} f_n dm\right| + \left|\int_{K\setminus K_\varepsilon} f_n dm\right| < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon.\end{aligned}$$

Consequently, the hypothesis of the Vitali-type theorem are satisfied and the proof is finished.

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Petroleum-Gas University of Ploiesti,
Bd. Bucureşti, no. 39, Ploieşti,
ROMANIA
pet_aly@yahoo.com