ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
MATEMATICĂ, Tomul LVI, 2010, f.2

ON IDEAL AMENABILITY IN BANACH ALGEBRAS

BY

O.T. MEWOMO

Abstract. We prove some new results on ideal amenability in Banach algebra introduced by Gorgi and Yazdanpanah. In particular, we show that if A and B are ideally amenable Banach algebras with bounded approximate identity, then $A\hat{\otimes}B$ is ideally amenable, thus providing answer to one of the questions raised by Gorgi and Yazdanpanah.

Mathematics Subject Classification 2000: 46H20, 46H10, 46H25.
Key words: ideally amenable, closed two-sided ideal, continuous derivation.

1. Introduction

In [3], Gorgi and Yazdanpanah, introduced two notions of amenability for a Banach algebra A. These are the concepts of I-weak amenability and ideal amenability for Banach algebra, where I is a closed two-sided ideal in A. They related these concepts to weak amenability in Banach algebras, and showed that ideal amenability is different from amenability and weak amenability. They also asked the following questions: (i) If A and B are ideally amenable Banach algebras, then is $A\hat{\otimes}B$ ideally amenable? (ii) Under what conditions is the group algebra $L^1(G)$ ideally amenable for a locally compact group G? It seems that these questions are still open.

In this paper, we shall extend some of the results of the authors in [3] and also provide a partial answer to question (i) above, by giving conditions under which it is true: our proof uses the recent solution by Massoud in [5] the module derivation problem for Banach algebras.

2. Preliminaries

First, we recall some standard notions; for further details, see [1] and [4].

Let A be an algebra and let X be an A-bimodule. A *derivation* from A to X is a linear map $D : A \to X$ such that $D(ab) = Da \cdot b + a \cdot Db \quad (a, b \in A)$. For example, $\delta_x : a \to a \cdot x - x \cdot a$ is a derivation; derivations of this form are called the *inner derivations*.

Let A be a Banach algebra, and let X be an A-bimodule. Then X is a Banach A-bimodule if X is a Banach space and if there is a constant $k > 0$ such that $\|a \cdot x\| \leq k \|a\| \|x\|, \quad \|x \cdot a\| \leq k \|a\| \|x\| \quad (a \in A, x \in X)$. By renorming X, we can suppose that $k = 1$. For example, A itself is Banach A-bimodule, and X', the dual space of a Banach A-bimodule X, is a Banach A-bimodule with respect to the module operations specified for by

$$\langle x, a \cdot \lambda \rangle = \langle x \cdot a, \lambda \rangle, \quad \langle x, \lambda \cdot a \rangle = \langle a \cdot x, \lambda \rangle \quad (x \in X)$$

for $a \in A$ and $\lambda \in X'$; we say that X' is the *dual module* of X. In particular every closed two-sided ideal I of A is Banach A-bimodule and I' the dual space of I is a dual A-bimodule.

Successively, the duals $X^{(n)}$ are Banach A-bimodules; in particular $A^{(n)}$ is a Banach A-bimodule for each $n \in \mathbb{N}$. We take $X^{(0)} = X$.

Let A be a Banach algebra, and let X be a Banach A-bimodule. Then $\mathcal{Z}^1(A, X)$ is the space of all continuous derivations from A into X, $\mathcal{N}^1(A, X)$ is the space of all inner derivations from A into X, and the first cohomology group of A with coefficients in X is the quotient space $\mathcal{H}^1(A, X) = \mathcal{Z}^1(A, X)/\mathcal{N}^1(A, X)$. The Banach algebra A is *amenable* if $\mathcal{H}^1(A, X') = \{0\}$ for each Banach A-bimodule X and *weakly amenable* if $\mathcal{H}^1(A, A') = \{0\}$. Further, as in [2], A is *n-weakly amenable* for $n \in \mathbb{N}$ if $\mathcal{H}^1(A, A^{(n)}) = \{0\}$, and A is *permanently weakly amenable* if it is n-weakly amenable for each $n \in \mathbb{N}$. For instance, each C^*-algebra is permanently weakly amenable [2, Theorem 2.1]. Each group algebra is n-weakly amenable whenever n is odd. Recently, the authors in [3], defined A as *I-weakly amenable* if $\mathcal{H}^1(A, I') = \{0\}$ for a closed two-sided ideal I of A, and ideally amenable if it is I-weakly amenable for every closed two-sided ideal I of A. Clearly, an amenable Banach algebra is ideally amenable and an ideally amenable Banach algebra is weakly amenable.

3. Main results

Let A be a Banach algebra. We recall from [1] that a left multiplier on A is an element L in $L(A)$ such that $L(ab) = L(a)b$, $a, b \in A$ and a right multiplier on A is an element R in $L(A)$ such that $R(ab) = aR(b)$. A multiplier is a pair (L, R) where L and R are left and right multipliers on A respectively and $aL(b) = R(a)b \quad (a, b \in A)$. The set of all multipliers on A is denoted by $\mathcal{M}(A)$. It is called multiplier algebra of A.

Let A be a Banach algebra, let $B(A)$ be the Banach algebra of bounded linear operators on A and let $\mathcal{M}(A)$ be multiplier algebra of A. That is, $\mathcal{M}(A) = \{(L, R) : L, R \in L(A), L(ab) = L(a)b, R(ab) = aR(b), aL(b) = R(a)b, a, b \in A\}$. As norm closed subalgebra of $B(A) \times B(A)^{op}$ (where $B(A)^{op}$ is the opposite algebra of $B(A)$), $\mathcal{M}(A)$ is a Banach algebra.

For $a \in A, L_a, R_a$ will denote the linear maps $b \to ab$ and $b \to ba$ on A, respectively. Then $(L_a, R_a) \in \mathcal{M}(A)$ with $\|L_a\| \leq \|a\|$ and $\|R_a\| \leq \|a\|$. It is easy to see that $a \to La$ (Resp. $a \to R_a$) is injective if and only if A is left (Resp. right) faithful. In particular, if A has a bounded approximate identity $\{e_\alpha\}$ of bound m, then $\|L_a\| \geq m^{-1}\|a\|$ and $\|R_a\| \geq m^{-1}\|a\|$ for all $a \in A$. In this case, A is identified with a norm closed ideal in $\mathcal{M}(A)$.

For each Banach A-bimodule X, $\mathcal{M}(A)$ acts on X through

$$T \cdot x = \lim_\alpha L(e_\alpha) \cdot x, \quad x \cdot T = \lim_\alpha x \cdot R(e_\alpha), \quad x \in X, T = (L, R) \in \mathcal{M}(A),$$

thus X is a Banach $\mathcal{M}(A)$-bimodule.

Also, A embeds in $\mathcal{M}(A)$ through $a \to (L_a, R_a)$, in the case where A is faithful, A is an ideal in $\mathcal{M}(A)$. Given a continuous derivation $D : \mathcal{M}(A) \to X'$, the restriction $\tilde{D}$ of D to A is also a continuous derivation.

The following result is very useful and important.

Proposition 3.1. *Let A be a Banach algebra.*

1. *If I is an ideal in A with an approximate identity, then every ideal in I is an ideal in A.*
2. *If A is faithful and has a bounded approximate identity, then every ideal in A is an ideal in $\mathcal{M}(A)$.*

Proposition 3.2. *Let A be a Banach algebra with a bounded approximate identity, and let I be a closed two-sided in A, then*

$$\mathcal{H}^1(\mathcal{M}(A), I') = \mathcal{H}^1(A, I'). \tag{3.1}$$

Proof. I is a closed two-sided ideal in A imply that I is a Banach A-bimodule and also a Banach $\mathcal{M}(A)$-bimodule with the module operation given above. It suffices to show that if the right hand side of (3.1) is $\{0\}$, then so is the left hand side.

Let $D : \mathcal{M}(A) \to I'$ be a continuous derivation and let $\tilde{D} : A \to I'$ be the restriction of D to A. Then, since we assume the right hand side of (3.1) is $\{0\}$ then $\tilde{D} = D_x$ for some $x \in I'$. Let $D_\alpha(T) = L(e_\alpha) \cdot x - x \cdot R(e_\alpha)$, for each index α and $T = (L, R) \in \mathcal{M}(A)$, then the net D_α is uniformly norm bounded, and so, passing to a subnet, we may assume that it converges to some $D_0 : \mathcal{M}(A) \to I'$ which is an A-module morphism in $\sigma(I', \mathcal{M}(A))$. It is easy to show that $D_0 = D$ and since clearly $D_0 = D_x$ on A, so by the continuity of these derivations in the strict topology on $\mathcal{M}(A)$ and density of A in $\mathcal{M}(A)$ in the strict topology, we get $D = D_x$ on $\mathcal{M}(A)$, and so the left hand side of (3.1) is $\{0\}$. □

Corollary 3.1. *Let A be a Banach algebra with a bounded approximate identity and let I be a closed two-sided ideal in A. If A is I-weakly amenable and faithful, then $\mathcal{M}(A)$ is I- weakly amenable.*

Proof. Since A is faithful and has a bounded approximate identity then every ideal in A is also an ideal $\mathcal{M}(A)$ by Proposition 3.1. Thus the result follows from Proposition 3.2. □

Example 3.1. *Let G be a locally compact group. If $L^1(G)$ is I-weakly amenable for any closed two-sided ideal I in $L^1(G)$, then $\mathcal{M}(L^1(G))$ is I weakly amenable.*

Proof. Since $L^1(G)$ is left and right faithful, it is an ideal in $\mathcal{M}(L^1(G))$, and since $L^1(G)$ has a bounded approximate identity, then every ideal in $L^1(G)$ is an ideal in $\mathcal{M}(L^1(G))$ by Proposition 3.1, and so the result follows from Corollary 3.1. □

Corollary 3.2. *Let A be a Banach algebra and let I be a closed two-sided ideal in A. If I has a bounded approximate identity and A is ideally amenable, then I is also ideally amenable.*

Proof. Let J be an arbitrary closed two-sided ideal in I, then J is also an ideal in A since I has a bounded approximate identity by Proposition 3.1 (1). J is a Banach I-bimodule and also Banach $\mathcal{M}(I)$-bimodule.

Let $D : \mathcal{M}(I) \to J'$ be a continuous derivation. By the definition of $\mathcal{M}(I)$, there is an A-module morphism $\varphi : A \to \mathcal{M}(I)$ and $D \circ \varphi : A \to J'$ is a derivation on A, and since A is ideally amenable, then $D \circ \varphi$ is inner. Hence D is inner on I. By using the same argument as above D is inner on $\mathcal{M}(I)$ and so the result follows from Proposition 3.2. □

Remark. Corollary 3.2 is the same as Theorem 1.9 of [3], but our proof is new and independent to that given in [3].

Let A and B be Banach algebras, the projective tensor product of A and B is denoted by $(A\hat{\otimes}B, \|.\|_\pi)$. Each $z \in (A\hat{\otimes}B)$ has a representation $z = \sum_{j=1}^{\infty} a_j \otimes b_j$, where $a_j \in A$ and $b_j \in B$ for each $j \in \mathbb{N}$ and $\sum_{j=1}^{\infty} \|a_j\|\|b_j\| < \infty$; further, $\|z\|$ is equal to the infimum of $\sum_{j=1}^{\infty} \|a_j\|\|b_j\|$ over all such representations. $(A\hat{\otimes}B, \|.\|_\pi)$ is a Banach algebra and a Banach A-bimodule. For further detail, see [1, p. 165].

In our next result, we give conditions under which $(A\hat{\otimes}B)$ is ideally amenable.

Theorem 3.1. *Let A and B be ideally amenable Banach algebras. Suppose A and B both have bounded approximate identity. Then $A\hat{\otimes}B$ is ideally amenable.*

Proof. A and B have bounded approximate identities imply $A\hat{\otimes}B$ has bounded approximate identities. Let I be an arbitrary closed two-sided ideal in $A\hat{\otimes}B$, then I is a Banach $A\hat{\otimes}B$-bimodule. The mappings $a \to \sigma_a$ and $b \to \tau_b$ defined by $\sigma_a(c \otimes d) = ac \otimes d, \tau_b(c \otimes d) = c \otimes bd$, extend and then lift to commuting continuous Banach algebra homomorphisms and A-module, B-module morphisms of A and B onto closed subalgebras A_1 and B_1 of $\mathcal{M}(A\hat{\otimes}B)$, and I is then a Banach A_1-module and Banach B_1-module. Now each continuous derivation $D : A\hat{\otimes}B \to I'$ gives rise to some continuous derivation $\tilde{D} : \mathcal{M}(A\hat{\otimes}B) \to I'$, whose restriction D_1 to B_1 is inner, say $\tilde{D} = D_x$ on B_1, for some $x \in I'$. Consider $d : I' \to L_A(B_1, I')$, (where $L_A(B_1, I')$ is the space of all A-module morphisms from B_1 to I'), which send each $y \in I'$ to the restriction of D_y to B_1. Then the fact that $\tilde{D} - D_x$ is zero on B_1 implies that $\tilde{D} - D_x$ is sending A_1 into $Im(d)^\perp \cong (I/Im(d))'$. So $Y = I'/Im(d)$ is an A_1-module, and also $\tilde{D} - D_x = D_y$ on A_1, for some $y \in Y$. $\tilde{D} - D_x$ and D_y are equal to zero on B_1, and so using the fact that $A_1 \cup B_1$ is dense in $\mathcal{M}(A\hat{\otimes}B)$, $\tilde{D}$ is inner on $\mathcal{M}(A\hat{\otimes}B)$, so D is inner on $(A\hat{\otimes}B)$. Thus, $(A\hat{\otimes}B)$ is ideally amenable. □

Remark. The above theorem provide a partial answer to the question raised by the authors in [3].

REFERENCES

1. Dales, H.G. – *Banach Algebras and Automatic Continuity,* London Mathematical Society Monographs, New Series, 24, Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 2000.

2. Dales, H.G.; Ghahramani, F.; Grønbæk, N. – *Derivations into iterated duals of Banach algebras,* Studia Math., 128 (1998), 19–54.

3. Gorgi, M.E.; Yazdanpanah, T. – *Derivations into duals of ideals of Banach algebras,* Proc. Indian Acad. Sci. Math. Sci., 114 (2004), 399–408.

4. Palmer, T.W. – *Banach Algebras and the General Theory of* **-Algebras,* Algebras and Banach algebras. Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1994.

5. Massoud, A. – *Module amenability for semigroup algebras,* arXiv:math/0205256v1.

Received: 14.I.2009
Revised: 27.III.2009

Department of Mathematics,
University of Agriculture,
Abeokuta,
NIGERIA
mewomoot@unaab.edu.ng