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UNIFORM ULTIMATE BOUNDEDNESS OF SOLUTIONS OF THIRD-ORDER NONLINEAR DELAY DIFFERENTIAL EQUATIONS

BY

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Abstract. In this paper, we study the uniform boundedness and uniform ultimate boundedness of solutions of the third-order delay differential equation

$$(*) \qquad \dddot{x} + \varphi(x,\dot{x})\ddot{x} + g(\dot{x}(t-r(t))) + f(x(t-r(t))) = p(t,x,\dot{x},\ddot{x}),$$

where $0 \leq r(t) \leq \gamma, \gamma$ is a positive constant, $\varphi(x,\dot{x}), g(\dot{x}), f(x)$ and $p(t,x,\dot{x},\ddot{x})$ are continuous functions. We obtain some sufficient conditions which ensure that all the solutions of Eq.(*) are uniformly bounded and uniformly ultimately bounded. Our result revise and improve some earlier results.

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1. Introduction

Recently, Tunc [14] considered the delay differential equation

$$\dddot{x} + \varphi(x,\dot{x})\ddot{x} + g(\dot{x}(t-r(t))) + f(x(t-r(t))) = p(t,x,\dot{x},\ddot{x}), \tag{1.1}$$

or the equivalent system form

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= -\varphi(x,y)z - g(y) - f(x) + \int_{t-r(t)}^{t} g'(y(s))z(s)ds \end{aligned} \tag{1.2}$$

$$+\int_{t-r(t)}^{t} f'(x(s))y(s)ds + p(t,x,y,z),$$

where $0 \leq r(t) \leq \gamma, \gamma$ is a positive constant, $\varphi(x,y), g(y), f(x)$ and $p(t,x,y,z)$ are continuous functions; $g(0) = f(0) = 0$. The derivative $\frac{\partial}{\partial x}\varphi(x,y) \equiv \varphi_x(x,y)$ exist and is also continuous. The functions f and g are also assumed differentiable. He obtained the following result.

Theorem A ([14, Theorem 2])**.** *Assume the following conditions are fulfilled. There are positive constants $a, b, c, c_0, m, \gamma, \beta, L$ and M such that*

(*i*) $ab - c > 0$,

(*ii*) $f(x)\operatorname{sgn} x > 0$ *for all* $x \neq 0$, $\sup\{f'(x)\} = c, |f'(x)| \leq L$ *for all* $x, \frac{f(x)}{x} \geq c_0 (x \neq 0)$,

(*iii*) $\frac{g(y)}{y} \geq b$ *for all* $y \neq 0, |g'(y)| \leq M$ *for all* y,

(*iv*) $0 \leq r(t) \leq \gamma, r'(t) \leq \beta, 0 < \beta < 1$,

(*v*) $\varphi(x,y) \geq a, y\varphi_x(x,y) \leq 0$ *for all* x *and* y,

(*vi*) $0 \leq \frac{g(y)}{y} - b \leq \min\left\{\frac{\sqrt{c_0}}{8}, \frac{ab-c}{8a\sqrt{b}}\right\} (y \neq 0)$,

(*vii*) $|p(t,x,y,z)| \leq m$ *for all* $t \in [0,\infty)$, *for all* x, y, z.

Then the solutions of (1.1) *are uniformly bounded and uniformly ultimately bounded, provided that*

$$\gamma < \min\left\{\frac{15c_0}{8(L+M)}, \frac{3(ab-c)(1-\beta)}{4[L(1+\mu+ab-c+a+a^2)+(L+M)(\mu+a^2)(1-\beta)]}, \frac{7(ab-c)(1-\beta)}{8[bM(1+\mu+ab-c+a+a^2+b(L+M)(1+a)(1-\beta)]}\right\}$$

with $\mu = \frac{ab+c}{2b}$.

We point out that the above Theorem A is not applicable to all equations of the general form (1.1). For example, consider the equation

$$\dddot{x}(t) + \left(2 + e^{-x(t)\dot{x}(t)}\right)\ddot{x}(t) + 4\dot{x}(t - r(t)) + \frac{\dot{x}(t - r(t))}{1 + \dot{x}^2(t - r(t))} + x(t - r(t)) + \frac{x(t - r(t))}{1 + x^2(t - r(t))} = \frac{2}{1 + t^2 + x^2(t) + \dot{x}^2(t) + \ddot{x}^2(t)}$$

or its equivalent system form

$$\begin{aligned}
\dot{x} &= y \\
\dot{y} &= z \\
\dot{z} &= -(2 + e^{xy})\, z - \left(4y + \frac{y}{1 + y^2}\right) - \left(x + \frac{x}{1 + x^2}\right) \\
&\quad + \int_{t-r(t)}^{t} \left(4 + \frac{1 - y^2}{(1 + y^2)^2}\right) z(s)ds + \int_{t-r(t)}^{t} \left(1 + \frac{1 - x^2}{(1 + x^2)^2}\right) y(s)ds \\
&\quad + \frac{1}{1 + t^2 + x^2 + y^2 + z^2}.
\end{aligned}$$

Following Theorem A it is easily verified that with $a = 2$, $b = 4$, $c = 2$, $c_0 = 1$, $L = 2$, $M = 5$, all the hypotheses of Theorem A are true except (vi) because $g(y) = 4y + \frac{4y}{1+y^2}$, then

$$\frac{g(y)}{y} - 4 = \frac{4}{1 + y^2} > \min\left\{\frac{1}{8}, \frac{3}{16}\right\} \text{ for some values of } y,$$

which leads to a contradiction.

The purpose of this paper is to study the uniform boundedness and uniform ultimate boundedness of (1.1). The most effective method to study the uniform boundedness and uniform ultimate boundedness of (1.1) is the Lyapunov's direct (or second) method. See [1-18] and the references therein. Thus, by using a more general Lyapunov function, our result improves some known results in [12] and [15], and revise a result in [14].

Before we state our main result, we shall give an important boundedness criteria for the general non-autonomous delay differential systems.

First consider a system of delay differential equations

$$\dot{\bar{x}} = F(t, \bar{x}_t), \quad \bar{x}_t = \bar{x}(t + \theta), \quad -r \le \theta \le 0, \tag{1.3}$$

where $F : \mathbb{R} \times \mathcal{C}_H \longrightarrow \mathbb{R}^n$ is a continuous mapping, and takes bounded sets into bounded sets. Here $F(t,0) = 0$ and $\mathcal{C}_H := \{\phi \in \mathcal{C}([-r,0],\mathbb{R}^n) : \|\phi\| \leq H\}$. The following lemma is a well-known result obtained by BURTON [3].

Lemma 1.1 ([3]). *Let $V(t,\phi) : \mathbb{R} \times C \longrightarrow \mathbb{R}$ be continuous and locally Lipschitz in ϕ. If*

(*i*) $W(|\bar{x}(t)|) \leq V(t,\bar{x}_t) \leq W_1(|\bar{x}(t)|) + W_2\left(\int_{t-r}^{t} W_3(|\bar{x}(s)|)ds\right)$, *and*

(*ii*) $\dot{V}_{(1.3)} \leq -W_3(|\bar{x}(t)|)+M$, *for some $M > 0$ where $W(r)$, $W_i(i = 1,2,3)$ are wedges then the solutions of* (1.3) *are uniformly bounded and uniformly ultimately bounded for bound B.*

2. Main result

Our main result in this paper is the following theorem which revises and improves a result in [14].

Theorem 2.1. *Further to the basic assumptions on φ, f, g, h and p, assume that the following conditions are satisfied ($a, b, c, m, \epsilon, \beta, \gamma, \delta, \delta_0, L, M$-some positive constants;)*

(*i*) $\varphi(x,y) \geq a + \epsilon, y\varphi_x(x,y) \leq 0$ *for all x and y;*

(*ii*) $\frac{g(y)}{y} \geq b$ *for all $y \neq 0$;*

(*iii*) $\frac{f(x)}{x} \geq \delta_0 \quad (x \neq 0), \quad f'(x) \leq c$;

(*iv*) $ab - c > 0$;

(*v*) $|f'(x)| \leq L, |g'(y)| \leq M$, *for all x, y;*

(*vi*) $0 < \beta < 1, 0 \leq r(t) \leq \gamma, r'(t) \leq \beta$; *and*

(*vii*) $|p(t,x,y,z)| \leq m + \delta(|x| + |y| + |z|)$ *for some $m, \delta > 0$.*

Then the solutions of system (1.2) *are uniformly bounded and uniformly ultimately bounded, provided that γ satisfies*

$$\gamma < \min\left\{\frac{\delta_0}{L+M}, \frac{2\mu_1(1-\beta)}{a(1-\beta)(L+M)+L(\alpha ab+a+1)}, \frac{2\epsilon\mu_2(1-\beta)}{\delta_0[(1-\beta)(L+M)+M(\alpha ab+a+1)]}\right\}.$$

Proof. Consider the Lyapunov functional

$$
\begin{aligned}
2V(x_t, y_t, z_t) &= 2a\int_0^x f(\xi)d\xi + 2\int_0^y g(\eta)d\eta + 2a\int_0^y \nu\varphi(x,\nu)d\nu \\
&+ \alpha ab^2x^2 + z^2 - \alpha aby^2 + 2\alpha a^2bxy + 2\alpha abxz + 2ayz + 2yf(x) \\
&+ \lambda\int_{-r(t)}^0\int_{t+s}^t y^2(\theta)d\theta ds + \tau\int_{-r(t)}^0\int_{t+s}^t z^2(\theta)d\theta ds,
\end{aligned} \tag{2.4}
$$

where α is a positive constant satisfying

$$
0 < \alpha < \min\left\{\frac{\delta_0(ab-c)}{ab\left[a\delta_0 + \left(\frac{g(y)}{y} - b\right)^2\right]}, \frac{\delta_0}{ab(\varphi(x,y)-a)}, \frac{1}{a}\right\},
$$

λ and τ are positive constants which will be determined later. Our target is to show that $V(x_t, y_t, z_t)$ satisfies the conditions of Lemma 1.1. First, by (1.2) and (2.1), we have

$$
\begin{aligned}
\frac{d}{dt}V(x_t,y_t,z_t) &= -\alpha abf(x)x - \left\{a\frac{g(y)}{y} - f'(x) - \alpha a^2b\right\}y^2 - \{\varphi(x,y)-a\}z^2 \\
&- \alpha ab\left\{\frac{g(y)}{y} - b\right\}xy - \alpha ab\{\varphi(x,y)-a\}xz + ay\int_0^y \nu\varphi_x(x,\nu)d\nu \\
&+ (\alpha abx + ay + z)\left[\int_{t-r(t)}^t f'(x(s))y(s)ds + \int_{t-r(t)}^t g'(y(s))z(s)ds\right] \\
&+ \lambda y^2r(t) + \tau z^2r(t) - \lambda(1-r'(t))\int_{t-r(t)}^t y^2(\theta)d\theta \\
&- \tau(1-r'(t))\int_{t-r(t)}^t z^2(\theta)d\theta + (\alpha abx + ay + z)p(t,x,y,z).
\end{aligned}
$$

Since $y\varphi_x(x,y) \le 0$, and using conditions (ii), (vi), (vii) of Theorem 2.1 and $2uv \le u^2 + v^2$ we obtain

$$
\begin{aligned}
\frac{d}{dt}V(x_t,y_t,z_t) &\le -\alpha abf(x)x - \left\{ab - c - \alpha a^2b\right\}y^2 - \{\varphi(x,y)-a\}z^2 \\
&- \alpha ab\left\{\frac{g(y)}{y} - b\right\}xy - \alpha ab\{\varphi(x,y)-a\}xz + \frac{1}{2}\alpha ab(L+M)\gamma x^2 \\
&+ \lambda\gamma y^2 + \tau\gamma z^2 + \frac{1}{2}a(L+M)\gamma y^2 + \frac{1}{2}(L+M)\gamma z^2
\end{aligned}
$$

$$+\frac{1}{2}L(\alpha ab+a+1)\int_{t-r(t)}^{t}y^2(s)ds+\frac{1}{2}M(\alpha ab+a+1)\int_{t-r(t)}^{t}z^2(s)ds$$
$$-\lambda(1-\beta)\int_{t-r(t)}^{t}y^2(s)ds-\tau(1-\beta)\int_{t-r(t)}^{t}z^2(s)ds$$
$$+(\alpha ab|x|+a|y|+|z|)|p(t,x,y,z)|.$$

Using conditions (iii) and (vi) of Theorem 2.1, after some rearrangement, we obtain

$$\frac{d}{dt}V(x_t,y_t,z_t)\leq-\frac{1}{2}\alpha ab\left\{\delta_0-(L+M)\gamma\right\}x^2$$
$$-\frac{1}{4}\alpha ab\delta_0\left\{[x+2\delta_0^{-1}(\varphi(x,y)-a)z]^2+\left[x+2\delta_0^{-1}\left(\frac{g(y)}{y}-b\right)y\right]^2\right\}$$
$$-\delta_0^{-1}(\varphi(x,y)-a)[\delta_0-\alpha ab(\varphi(x,y)-a)]z^2$$
$$+\left\{\frac{1}{2}a(L+M)\gamma+\lambda\gamma\right\}y^2+\left\{\frac{1}{2}a(L+M)\gamma+\tau\gamma\right\}z^2$$
$$+\left\{\frac{1}{2}L(\alpha ab+a+1)-\lambda(1-\beta)\right\}\int_{t-r(t)}^{t}y^2(s)ds$$
$$+\left\{\frac{1}{2}M(\alpha ab+a+1)-\tau(1-\beta)\right\}\int_{t-r(t)}^{t}z^2(s)ds$$
$$+(\alpha ab|x|+a|y|+|z|)[m+\delta(|x|+|y|+|z|)]$$
$$-\left\{ab-c-\alpha ab\left(a+\delta_0^{-1}\left[\frac{g(y)}{y}-b\right]^2\right)\right\}y^2.$$

If we choose

$$0<\alpha<\min\left\{\frac{\delta_0(ab-c)}{ab\left[a\delta_0+\left(\frac{g(y)}{y}-b\right)^2\right]},\frac{\delta_0}{ab(\varphi(x,y)-a)}\right\}$$

there exits some positive constants μ_1 and μ_2 such that

$$\frac{d}{dt}V(x_t,y_t,z_t)\leq-\frac{1}{2}\alpha ab\left\{\delta_0-(L+M)\gamma\right\}x^2-\mu_1y^2-\epsilon\mu_2\delta_0^{-1}z^2$$
$$+\frac{1}{2}\left\{a(L+M)+2\lambda\right\}\gamma y^2+\frac{1}{2}\left\{L+M+2\tau\right\}\gamma z^2$$

$$+\frac{1}{2}\left\{L(\alpha ab+a+1)-2\lambda(1-\beta)\right\}\int_{t-r(t)}^{t}y^2(s)ds$$
$$+\frac{1}{2}\left\{M(\alpha ab+a+1)-2\tau(1-\beta)\right\}\int_{t-r(t)}^{t}z^2(s)ds$$
$$+(\alpha ab|x|+a|y|+|z|)[m+\delta(|x|+|y|+|z|)].$$

Further simplification yields

$$\begin{aligned}\frac{d}{dt}V(x_t,y_t,z_t)\leq&-\frac{1}{2}\alpha ab\left\{\delta_0-(L+M)\gamma\right\}x^2\\&-\left\{\mu_1-\frac{1}{2}\left[a(L+M)+2\lambda\right]\gamma\right\}y^2\\&-\left\{\epsilon\mu_2\delta_0^{-1}-\frac{1}{2}\left[L+M+2\tau\right]\gamma\right\}z^2\\&+\frac{1}{2}\left\{L(\alpha ab+a+1)-2\lambda(1-\beta)\right\}\int_{t-r(t)}^{t}y^2(s)ds\\&+\frac{1}{2}\left\{M(\alpha ab+a+1)-2\tau(1-\beta)\right\}\int_{t-r(t)}^{t}z^2(s)ds\\&+m(\alpha ab|x|+a|y|+|z|)+\delta(\alpha ab|x|+a|y|+|z|)(|x|+|y|+|z|).\end{aligned}$$

Now, if we choose

$$\lambda=\frac{L(\alpha ab+a+1)}{2(1-\beta)}>0,\quad \tau=\frac{M(\alpha ab+a+1)}{2(1-\beta)}>0$$

and

$$\begin{aligned}\gamma<\min\Bigg\{&\frac{\delta_0}{L+M},\frac{2\mu_1(1-\beta)}{a(1-\beta)(L+M)+L(\alpha ab+a+1)},\\&\frac{2\epsilon\mu_2(1-\beta)}{\delta_0[(1-\beta)(L+M)+M(\alpha ab+a+1)]}\Bigg\},\end{aligned}$$

we get

$$\begin{aligned}\frac{d}{dt}V(x_t,y_t,z_t)&\leq-\eta(x^2+y^2+z^2)+m(\alpha ab|x|+a|y|+|z|)\\&\quad+\delta(\alpha ab|x|+a|y|+|z|)(|x|+|y|+|z|)\\&\leq-(\eta-\delta\Delta)(x^2+y^2+z^2)+m(\alpha ab|x|+a|y|+|z|),\end{aligned}$$

where $\Delta = \frac{1}{2}\max\{4\alpha ab + a + 1, \alpha ab + 4a + 1, \alpha ab + a + 3\}$. If we choose $\delta < \frac{\eta}{\Delta}$, then there is some $\theta > 0$ such that

$$\begin{aligned}
&\frac{d}{dt}V(x_t, y_t, z_t) \leq -\theta(x^2 + y^2 + z^2) + k\theta(|x| + |y| + |z|)\\
&= -\frac{\theta}{2}(x^2 + y^2 + z^2) - \frac{\theta}{2}\left\{(|x| - k)^2 + (|y| - k)^2 + (|z| - k)^2\right\} + \frac{3\theta}{2}k^2\\
&\leq -\frac{\theta}{2}\left(x^2 + y^2 + z^2\right) + \frac{3\theta}{2}k^2, \text{ for some } k, \theta > 0.
\end{aligned}$$

Thus, condition (ii) of Lemma 1.1 is satisfied by taking $W_3(r) = \left(\frac{\theta}{2}\right) r^2$ and $M = \left(\frac{3\theta}{2}\right) k^2$. Next, we show that condition (i) of Lemma 1.1 is satisfied. We note that by conditions imposed on f, g and φ we have

$$\begin{aligned}
&2a\int_0^x f(\xi)d\xi + 2\int_0^y g(\eta)d\eta + 2a\int_0^y \nu\varphi(x,\nu)d\nu\\
&+ \alpha ab^2x^2 + z^2 - \alpha aby^2 + 2\alpha a^2bxy + 2\alpha abxz + 2ayz + 2yf(x)\\
&= \alpha ab(1 - \alpha a)x^2 + a\left\{2\int_0^x f(\xi)d\xi - \frac{1}{ab}f^2(x)\right\}\\
&+ ab\left\{a^{-\frac{1}{2}}y + a^{-\frac{1}{2}}b^{-1}f(x)\right\}^2 + \left\{2\int_0^y g(\nu)d\nu - by^2\right\}\\
&+ a\left\{2\int_0^y v\varphi(x,\nu)d\nu - ay^2\right\} + (\alpha abx + ay + z)^2\\
&\geq \alpha ab(1 - \alpha a)x^2 + a\left\{2\int_0^x \left[1 - \frac{1}{ab}f'(\xi)\right] f(\xi)d\xi - \frac{1}{ab}f^2(0)\right\}\\
&+ 2\int_0^y \left(\frac{g(v)}{v} - b\right) vdv + 2a\int_0^y (\varphi(x,v) - a)vdv + (\alpha abx + ay + z)^2.
\end{aligned}$$

Using conditions (i), (ii), (iii) and (iv) of Theorem 2.1, we obtain

$$\begin{aligned}
V(x_t, y_t, z_t) &\geq \frac{1}{2}\alpha ab(1 - \alpha a)x^2 + a\delta_0\left(1 - \frac{c}{ab}\right)x^2\\
&+ \epsilon ay^2 + (\alpha abx + ay + z)^2.
\end{aligned}$$

But $1 - \alpha a > 0$ since we can choose α such that $0 < \alpha < \frac{1}{a}$. Also, by condition (iv) of Theorem 2.1, $1 - \frac{c}{ab} > 0$. Hence we can easily check that $V(x_t, y_t, z_t)$ satisfies condition (i) of Lemma 1.1. This completes the proof of Theorem 2.1.

Remark 2.1. It is clear that Theorem 2.1 is an improvement and extension of Theorem A. In particular, from our theorem we see that hypothesis (vi) assumed in Theorem A is not necessary for the uniform boundedness and uniform ultimate boundedness of solutions of (1.1).

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