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ON A CHARACTERIZATION OF HILBERTIAN NORMS

BY

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Abstract. In this paper we prove that a characterization of hilbertian norms established in smooth normed spaces X with $\dim X \geq 3$ also holds in two dimensional spaces.

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1. Introduction

Let X be a real smooth linear normed space, that is there exists

$$\lim_{t \to 0} \frac{\|x+ty\|^2 - \|x\|^2}{2t}$$

for all $x, y \in X$. Usually, this limit is denoted by $n'(x;y)$. It is well known, that $n(x;\cdot)$ is a linear continuous functional on X and $n'(x;x) = \|x\|^2$ for every $x \in X$. Therefore, X is a smooth space (see [1], [2]). Using an euclidean property of heights of a triangle, in [4] it is established the following characterization of hilbertian norms:

Theorem 1. *In a smooth real linear normed space X with $dim X \geq 3$ the norm is hilbertian if and only*

$$\begin{aligned}(1)\qquad & n'(y;x)[n'(x;y)n'(x-y;x) - \|x\|^2 n'(x-y;y)] \\ & = n'(x;y)[\|y\|^2 n'(x-y,x) - n'(y;x)n'(x-y;y)],\ \forall x,y \in X.\end{aligned}$$

In the present paper we show that this result is also true if $\dim X = 2$. Our goals is to find the smooth norms of a two dimensional space X which have the property (1). For this we use an idea of [3] taking into account that $n'(x;\cdot)$ is a linear functional for any $x \in X$. Thus, if $x = (x_1, x_2)$, $y = (y_1, y_2)$, $x_1, x_2, y_1, y_2 \in \mathbb{R}$, then

$$n'(x;y) = A(x_1,x_2)y_1 + B(x_1,x_2)y_2, \tag{2}$$

and

$$\|x\|^2 = A(x_1,x_2)x_1 + B(x_1,x_2)x_2, \tag{3}$$

where A, B are two real function on $\mathbb{R}^2$. Moreover, since $n'(tx;y) = tn'(x;y)$, for all $t \in \mathbb{R}$, we also have

$$\begin{aligned} &A(tx_1,tx_2) = tA(x_1,x_2),\\ &B(tx_1,tx_2) = tB(x_1,x_2), \quad \text{for all } t \in \mathbb{R}. \end{aligned} \tag{4}$$

2. Proof of Theorem 1 in two dimensional spaces

Taking into account the equalities (2) and (3), the property (1) becomes

$$\begin{aligned} &(A(y_1,y_2)x_1 + B(y_1,y_2)x_2)[(A(x_1,x_2)y_1 + B(x_1,x_2)y_2)\\ &(A(x_1-y_1,x_2-y_2)x_1 + B(x_1-y_1,x_2-y_2)x_2) - (A(x_1,x_2)x_1\\ &+ B(x_1,x_2)x_2)(A(x_1-y_1,x_2-y_2)y_1 + B(x_1-y_1,x_2-y_2)y_2)]\\ &= (A(x_1,x_2)y_1 + B(x_1,x_2)y_2)[(A(y_1,y_2)y_1 + B(y_1,y_2)y_2)\\ &(A(x_1-y_1,x_2-y_2)x_1 + B(x_1-y_1,x_2-y_2)x_2) - (A(y_1,y_2)x_1\\ &+ B(y_1,y_2)x_2)(A(x_1-y_1,x_2-y_2)y_1 + B(x_1-y_1,x_2-y_2)y_2)]. \end{aligned}$$

If $x = (1,t)$, $y = (0,t)$, $t \in \mathbb{R}$, we obtain

$$\begin{aligned} &(A(0,t) + tB(0,t))[tB(1,t)(A(1,0) + tB(1,0)) - tB(1,0)(A(1,t) + tB(1,t))]\\ &= tB(1,t)[tB(0,t)(A(1,0) + tB(1,0)) - tB(1,0)(A(0,t) + tB(0,t))]. \end{aligned}$$

By (4) we get

$$\begin{aligned} &(A(0,1) + tB(0,1))[B(1,t)(A(1,0) + tB(1,0)) - B(1,0)(A(1,t) + tB(1,t))]\\ &= tB(1,t)[B(0,1)(A(1,0) + tB(1,0)) - B(1,0)(A(0,1) + tB(0,1))], \end{aligned}$$

that is,

$$\begin{aligned} &(A(0,1) + tB(0,1))(A(1,0)B(1,t) - B(1,0)A(1,t))\\ &= tB(1,t)(A(1,0)B(0,1) - A(0,1)B(1,0)). \end{aligned} \tag{5}$$

Therefore, if $B(1,t) \neq 0$, we obtain

$$\frac{A(1,t)}{B(1,t)} = \frac{A(0,1)(tB(1,0)+A(1,0))}{B(1,0)(tB(0,1)+A(0,1))},$$

whenever $t \neq -\frac{A(0,1)}{B(0,1)}$, $B(0,1) \neq 0$. Hence, the function $t \to \frac{A(1,t)}{B(1,t)}$, $t \in \mathbb{R}$, is a real function of homographic type, i.e.

$$\frac{A(1,t)}{B(1,t)} = \frac{\alpha t+\beta}{\gamma t+\alpha}, \text{ if } B(1,t) \neq 0,\ t \neq -\frac{\alpha}{\gamma},$$

where $\alpha = A(0,1)A(1,0), \beta = A(0,1)B(1,0)$, $\gamma = B(0,1)B(1,0)$.

On the other hand if $x_1 \neq 0, x_2 \neq 0$ and $B(1,\frac{x_2}{x_1}) \neq 0$ we have

$$n'(x,y)=x_1B\left(1,\frac{x_2}{x_1}\right)\left[\frac{A(1,\frac{x_2}{x_1})}{B(1,\frac{x_2}{x_1})}y_1+y_2\right]=x_1B\left(1,\frac{x_2}{x_1}\right)\left[\frac{\alpha x_2+\beta x_1}{\gamma x_2+\alpha x_1}y_1+y_2\right].$$

By (3) it follows

$$\|x\|^2 = x_1B(1,\frac{x_2}{x_1})\frac{\beta x_1^2+2\alpha x_1x_2+\gamma x_2^2}{\alpha x_1+\gamma x_2}. \tag{6}$$

Since the norm is differentiable it is well known that

$$A(x_1,x_2) = \frac{\partial}{\partial x_1}\left(\frac{1}{2}\|x\|^2\right),\quad B(x_1,x_2) = \frac{\partial}{\partial x_2}\left(\frac{1}{2}\|x\|^2\right),$$

and so, denoting $B(1,t) = f(t)$, $t \in \mathbb{R}$, by usual computations we find

$$\begin{aligned} x_1f(\frac{x_2}{x_1}) = \frac{\partial}{\partial x_2}\left(\frac{1}{2}\|x\|^2\right) &= \frac{1}{2}x_1f\left(\frac{x_2}{x_1}\right)\frac{(2\alpha^2-\beta\gamma)x_1^2+2\alpha\gamma x_1x_2+\gamma^2x_2^2}{(\alpha x_1+\gamma x_2)^2} \\ &+ \frac{1}{2}f'\left(\frac{x_2}{x_1}\right)\frac{\beta x_1^2+2\alpha x_1x_2+\gamma x_2^2}{\alpha x_1+\gamma x_2}, \end{aligned}$$

i.e.

$$\begin{aligned} &x_1f\left(\frac{x_2}{x_1}\right)\frac{2(\alpha x_1+\gamma x_2)^2-(2\alpha^2-\beta\gamma)x_1^2-2\alpha\gamma x_1x_2-\gamma x_2^2}{\alpha x_1+\gamma x_2} \\ &= f'\left(\frac{x_2}{x_1}\right)(\beta x_1^2+2\alpha x_1x_2+\gamma x_2^2). \end{aligned}$$

Dividing by $\beta x_1^2 + 2\alpha x_1 x_2 + \gamma x_2^2$ we obtain the following simple differential equation $\frac{f'(t)}{f(t)} = \frac{\gamma}{\gamma t + \alpha}$. Thus, it follows that $f(t) = (\gamma t + \alpha)k$, where k is a real constant. In fact, $k = \|(1,0)\|^{-2}$.

Now, we return to the relation (6) and so we get

$$\|x\|^2 = k(\beta x_1^2 + 2\alpha x_1 x_2 + \gamma x_2^2). \tag{7}$$

If $B(1,t) = 0$, then from (3) it follows that $\|(1,t)\|^2 = A(1,t) > 0$. In this case we consider the function $g(t) = A(1,t)$, $t \in \mathbb{R}$. Similarly, we obtain the same equality (7). Also, since the norm is a continuous function, the equality (7) also holds if $x_1 = 0$ or $x_2 = 0$ or $\frac{x_2}{x_1} = -\frac{\alpha}{\gamma}$.

Finally, we established (7) for all $(x_1, x_2) \in \mathbb{R}^2$.

Therefore, the norm is obviously hilbertian.

Remark. Since a norm is hilbertian if its restriction to every two dimensional linear subspace is hilbertian, the above proof can be considered as a new proof of Theorem 4 in [4] for any linear normed space.

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