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ON GENERALIZED ABSOLUTE CESÀRO SUMMABILITY

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Abstract. In this paper, a main theorem dealing with $\mid C,1\mid_k$ summability factors has been generalized under more weaker conditions for $\mid C,\alpha,\beta\mid_k$ summability factors. This theorem also includes some new results.

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1. Introduction

A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence c_n and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). Obviously every increasing sequence is almost increasing. However, the converse need not be true as can be seen by taking an example, say $b_n = ne^{(-1)^n}$. A sequence (d_n) of positive numbers is said to be δ-quasi monotone, if $d_n > 0$ ultimately and $\Delta d_n = d_n - d_{n+1} \geq -\delta_n$, where (δ_n) is a sequence of positive numbers (see [2]). Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by $u_n^{\alpha,\beta}$ and $t_n^{\alpha,\beta}$ the n-th Cesàro means of order (α,β), with $\alpha+\beta > -1$, of the sequence (s_n) and (na_n), respectively, i.e., (see [5])

$$u_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} s_v, \tag{1}$$

$$t_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v, \tag{2}$$

where

(3) $A_n^{\alpha+\beta} = O(n^{\alpha+\beta}),\ \alpha+\beta > -1,\ \ A_0^{\alpha+\beta} = 1$ and $A_{-n}^{\alpha+\beta} = 0$ for $n > 0$.

The series $\sum a_n$ is said to be summable $\mid C, \alpha, \beta \mid_k$, $k \geq 1$, if (see [6])

$$\sum_{n=1}^{\infty} n^{k-1} \mid u_n^{\alpha,\beta} - u_{n-1}^{\alpha,\beta} \mid^k < \infty. \tag{4}$$

Since $t_n^{\alpha,\beta} = n(u_n^{\alpha,\beta} - u_{n-1}^{\alpha,\beta})$ (see [6]), condition (4) can also be written as

$$\sum_{n=1}^{\infty} \frac{1}{n} \mid t_n^{\alpha,\beta} \mid^k < \infty. \tag{5}$$

If we take $\beta = 0$, then $\mid C, \alpha, \beta \mid_k$ summability reduces to $\mid C, \alpha \mid_k$ (see [7]) summability. Also,if we take $\beta = 0$ and $\alpha = 1$, then we have $\mid C, 1 \mid_k$ summability. It should be noted that obviously $(C, \alpha, 0)$ mean is the same as (C, α) mean. Mazhar [8] has obtained the following theorem for $\mid C, 1 \mid_k$ summability factors of infinite series.

Theorem A. *Let (X_n) be a positive non-decreasing sequence such that $\mid \Delta X_n \mid = O(X_n/n)$ and $\lambda_n \to 0$ as $n \to \infty$. Suppose that there exists a sequence of numbers (A_n) such that it is δ-quasi-monotone with $\sum n\delta_n X_n < \infty$, $\sum A_n X_n$ is convergent and $\mid \Delta\lambda_n \mid \leq A_n$ for all n. If*

$$\sum_{n=1}^{m} \frac{1}{n} \mid t_n \mid^k = O(X_m) \quad as \quad m \to \infty, \tag{6}$$

then the series $\sum a_n \lambda_n$ is summable $\mid C, 1 \mid_k$,$k \geq 1$.

2. The main result

The aim of this paper is to generalize Theorem A under more weaker conditions for $\mid C, \alpha, \beta \mid_k$ summability, by taking an almost increasing sequence instead of a positive non-decreasing sequence. We shall prove the following theorem.

Theorem. *Let (X_n) be an almost increasing sequence such that $\mid \Delta X_n \mid = O(X_n/n)$ and $\lambda_n \to 0$ as $n \to \infty$. Suppose that there exists a sequence* *of numbers* (A_n) *such that it is* δ*-quasi-monotone with* $\sum n\delta_n X_n < \infty$, $\sum A_n X_n$ *is convergent and* $\mid \Delta\lambda_n \mid \leq A_n$ *for all* n. *If the sequence* $(\theta_n^{\alpha,\beta})$ *defined by*

$$\theta_n^{\alpha,\beta} = \mid t_n^{\alpha,\beta} \mid, \quad \alpha = 1,\ \beta > -1 \tag{7}$$

$$\theta_n^{\alpha,\beta} = \max_{1\leq v\leq n} \mid t_v^{\alpha,\beta} \mid, \quad 0 < \alpha < 1,\ \beta > -1 \tag{8}$$

satisfies the condition

$$\sum_{n=1}^{m} \frac{1}{n}(\theta_n^{\alpha,\beta})^k = O(X_m) \quad as \quad m \to \infty, \tag{9}$$

then the series $\sum a_n\lambda_n$ *is summable* $\mid C, \alpha, \beta \mid_k$ *for* $0 < \alpha \leq 1$, $\beta > -1$, $\alpha + \beta > 0$ *and* $k \geq 1$.

It should be noted that if we take (X_n) as a positive non-decreasing sequence, $\alpha = 1$ and $\beta = 0$, then we get Theorem A. In this case condition (9) reduces to condition (6).

We need the following lemmas for the proof of our theorem.

Lemma 1 ([3]). *Let* (X_n) *be an almost increasing sequence such that* $n \mid \Delta X_n \mid = O(X_n)$. *If* (A_n) *is a* δ*-quasi-monotone with* $\sum n\delta_n X_n < \infty$ *and* $\sum A_n X_n$ *is convergent, then*

$$nA_nX_n = O(1) \quad as \quad n \to \infty, \tag{10}$$

$$\sum_{n=1}^{\infty} nX_n \mid \Delta A_n \mid < \infty. \tag{11}$$

Lemma 2 ([4]). *If* $0 < \alpha \leq 1$, $\beta > -1$ *and* $1 \leq v \leq n$, *then*

$$\mid \sum_{p=0}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} a_p \mid \leq \max_{1\leq m\leq v} \mid \sum_{p=0}^{m} A_{m-p}^{\alpha-1} A_p^{\beta} a_p \mid . \tag{12}$$

3. Proof of the theorem

Let $(T_n^{\alpha,\beta})$ be the n-th (C, α, β) mean of the sequence $(na_n\lambda_n)$. Then, by (2), we have

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \lambda_v.$$

First, applying Abel's transformation and then using Lemma 2, we have that

$$
\begin{aligned}
T_n^{\alpha,\beta} &= \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p + \frac{\lambda_n}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v, \\
\mid T_n^{\alpha,\beta} \mid &\leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \mid \Delta\lambda_v \mid\mid \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p \mid + \frac{\mid \lambda_n \mid}{A_n^{\alpha+\beta}} \mid \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \mid \\
&\leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} A_v^{\alpha} A_v^{\beta} \theta_v^{\alpha,\beta} \mid \Delta\lambda_v \mid + \mid \lambda_n \mid \theta_n^{\alpha,\beta} \\
&= T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta}, \quad \text{say.}
\end{aligned}
$$

Since $\mid T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta} \mid^k \leq 2^k(\mid T_{n,1}^{\alpha,\beta} \mid^k + \mid T_{n,2}^{\alpha,\beta} \mid^k)$, in order to complete the proof of the theorem, by (5), it is sufficient to show that

$$
\sum_{n=1}^{\infty} \frac{1}{n} \mid T_{n,r}^{\alpha,\beta} \mid^k < \infty \quad \text{for} \quad r = 1, 2.
$$

Whenever $k > 1$, we can apply Hölder's inequality with indices k and k', where $\frac{1}{k} + \frac{1}{k'} = 1$, we get that

$$
\begin{aligned}
&\sum_{n=2}^{m+1} \frac{1}{n} \mid T_{n,1}^{\alpha,\beta} \mid^k \leq \sum_{n=2}^{m+1} \frac{1}{n} \mid \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} A_v^{\alpha} A_v^{\beta} \theta_v^{\alpha,\beta} \Delta\lambda_v \mid^k \\
&= O(1) \sum_{n=2}^{m+1} \frac{1}{n^{1+(\alpha+\beta)k}} \left\{ \sum_{v=1}^{n-1} v^{\alpha k} v^{\beta k} A_v (\theta_v^{\alpha,\beta})^k \right\} \times \left\{ \sum_{v=1}^{n-1} A_v \right\}^{k-1} \\
&= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} A_v (\theta_v^{\alpha,\beta})^k \sum_{n=v+1}^{m+1} \frac{1}{n^{1+(\alpha+\beta)k}} \\
&= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} A_v (\theta_v^{\alpha,\beta})^k \int_v^{\infty} \frac{dx}{x^{1+(\alpha+\beta)k}} \\
&= O(1) \sum_{v=1}^{m} A_v (\theta_v^{\alpha,\beta})^k = O(1) \sum_{v=1}^{m} v A_v \frac{1}{v} (\theta_v^{\alpha,\beta})^k \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v A_v) \sum_{p=1}^{v} \frac{1}{p} (\theta_p^{\alpha,\beta})^k + O(1) m A_m \sum_{v=1}^{m} \frac{1}{v} (\theta_v^{\alpha,\beta})^k
\end{aligned}
$$

$$= O(1) \sum_{v=1}^{m-1} v \mid \Delta A_v \mid X_v + O(1) \sum_{v=1}^{m-1} A_v X_v + O(1) m A_m X_m$$
$$= O(1) \quad \text{as} \quad m \to \infty,$$

in view of hypotheses of the theorem and Lemma 1. Similarly, we have that

$$\sum_{n=1}^{m} \frac{1}{n} \mid T_{n,2}^{\alpha,\beta} \mid^k = O(1) \sum_{n=1}^{m} \frac{\mid \lambda_n \mid}{n} (\theta_n^{\alpha,\beta})^k$$
$$= O(1) \sum_{n=1}^{m} \frac{1}{n} (\theta_n^{\alpha,\beta})^k \sum_{v=n}^{\infty} \mid \Delta \lambda_v \mid$$
$$= O(1) \sum_{v=1}^{\infty} \mid \Delta \lambda_v \mid \sum_{n=1}^{v} \frac{1}{n} (\theta_n^{\alpha,\beta})^k$$
$$= O(1) \sum_{v=1}^{\infty} \mid \Delta \lambda_v \mid X_v = O(1) \sum_{v=1}^{\infty} A_v X_v < \infty.$$

Therefore, we get that

$$\sum_{n=1}^{\infty} \frac{1}{n} \mid T_{n,r}^{\alpha,\beta} \mid^k < \infty \quad \text{for} \quad r = 1, 2, \quad \text{by} \quad (5).$$

This completes the proof of the theorem. If we take $\beta = 0$, then we get a new result for $\mid C, \alpha \mid_k$ summability factors. Also, if we take $\beta = 0$, $\alpha = 1$ and $X_n = \log n$, then we obtain a result of Mazhar [8] dealing with $\mid C, 1 \mid_k$ summability factors.

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