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VERTICAL CHERN TYPE CLASSES ON COMPLEX FINSLER BUNDLES

BY

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Abstract. In the present paper, we define vertical Chern type classes on complex Finsler bundles, as an extension of the v-cohomology groups theory on complex Finsler manifolds. These classes are introduced in a classical way by using closed differential forms with respect to the conjugated vertical differential in terms of the vertical curvature form of Chern-Finsler linear connection. Also, some invariance properties of these classes are studied.

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Key words: complex Finsler bundles, Chern-Finsler connection, v-cohomology classes.

1. Introduction and preliminaries

In the real case, the idea of decomposing the exterior derivative for a foliated manifold, into three differential operators and the study of their cohomology is due to VAISMAN (see [12]). There, it is proved a Poincaré type Lemma for the differential operator $d^{''}$, corresponding to $(0,1)$ foliated type. In the case of complex analytic foliated manifolds, this idea has the origin in a paper by VAISMAN [13], where is proved that a (p,q,r,s) differential form can be considered to be of $(p+q,r+s)$ complex type and of $(p,q+r+s)$ mixed type (see also [14]) and some decompositions of the exterior derivative are obtained. Also, it is proved a Poincaré type Lemma for the differential operator $\widetilde{d}^{''}$, corresponding to $(0,1)$ mixed type (for details see [13]). Latter, in [10] is studied a decomposition of the exterior differential for the complex type forms on complex Finsler manifolds and it is proved a Grothendieck-Dolbeault type Lemma for the conjugated vertical differential operator $d^{''v}$ which appears in the decomposition of the operator $d^{''}$ corresponding to $(0,1)$ complex type. Also, the v-cohomology groups of a complex Finsler manifold are defined. Some extensions of these results on the total space of a complex Finsler bundle are made in [7].

The main purpose of this paper is to define vertical Chern type classes on complex Finsler bundles. Also, some invariance properties of these classes are studied.

In the first section, following [2], [4], [6], [9], we briefly recall some basic notions about complex Finsler bundles and we make a short review on the v-cohomology (see [7], [10]) and the Chern-Finsler linear connection (see [3]) of a complex Finsler bundle (E, L). In the second section we define vertical Chern type classes as v-cohomology classes represented by $d^{''v}$-closed forms which are constructed using the vertical part of the curvature of Chern-Finsler connection. Finally, we remark that these classes are not topological invariants, but are metric invariants. We notice that similar horizontal invariants on complex Finsler bundles are studied in [8], but with respect to partial Bott complex connection.

Let $\pi : E \to M$ be a holomorphic vector bundle over a complex manifold M, where $dim_{\mathbb{C}} M = n$ and $rank\, E = m$. Let $(U, z^k),\ k = 1, ..., n$ be a local chart on M and $s = \{s_a\},\ a = 1, ..., m$ be a local frame for the sections of E over U. It is well known that this chart induces canonically another one on E of the form $(\pi^{-1}(U), u = (z^k, \eta^a)),\ k = 1, ..., n,\ a = 1, ..., m$, where $s = \eta^a s_a$ is a section on $E_z = \pi^{-1}(z)$, for all $z \in M$. If $(\pi^{-1}(U^{'}), u^{'} = (z^{'k}, \eta^{'a}))$ is another local chart on E then the transition laws of these coordinates are

$$z^{'k} = z^{'k}(z),\ \eta^{'a} = M^a_b(z)\,\eta^b, \tag{1}$$

where $M^a_b(z),\ a, b = 1, ..., m$ are holomorphic functions and $\det(M^a_b) \neq 0$.

As we already know [2], [6] and [9], the total space E has a structure of $m+n$ dimensional complex manifold because the transition functions $M^a_b(z)$ are holomorphic. Consider the complexified tangent bundle $T_{\mathbb{C}}E$ of the real tangent bundle $T_{\mathbb{R}}E$ and its decomposition $T_{\mathbb{C}}E = T^{'}E \oplus T^{''}E$, where $T^{'}E$ and $T^{''}E = \overline{T^{'}E}$ are the holomorphic and antiholomorphic tangent bundles of E, respectively. The vertical holomorphic tangent bundle $VE = ker\pi_*$ is the relative tangent bundle of the holomorphic projection π. A local frame field on V_uE is $\{\frac{\partial}{\partial \eta^a}\}$, $a = 1, ..., m$. The vertical distribution V_uE is isomorphic to the sections module of E in u.

A supplementary subbundle of VE in $T^{'}E$, i.e. $T^{'}E = HE \oplus VE$ is called a *complex nonlinear connection*, briefly c.n.c. A local base for the horizontal distribution H_uE, called adapted for the c.n.c. is $\{\frac{\delta}{\delta z^k}=\frac{\partial}{\partial z^k}-N_k^a\frac{\partial}{\partial \eta^a}\}$, $k=1,...,n$, where $N_k^a(z,\eta)$ are the coefficients of the c.n.c.

In the following we consider the abreviate notations: $\partial_k=\frac{\partial}{\partial z^k}$, $\delta_k=\frac{\delta}{\delta z^k}$ and $\dot{\partial}_a=\frac{\partial}{\partial \eta^a}$. The adapted bases denoted by $\{\delta_{\overline{k}}:=\frac{\delta}{\delta \overline{z}^k}\}$ and $\{\dot{\partial}_{\overline{a}}:=\frac{\partial}{\partial \overline{\eta}^a}\}$, for the distributions $\overline{HE}$ and $\overline{VE}$ are obtained respectively by conjugation anywhere. Also, we notice that the adapted cobases are given by

$$\{dz^k\},\ \{\delta\eta^a=d\eta^a+N_k^a dz^k\},\ \{d\overline{z}^k\},\ \{\delta\overline{\eta}^a=d\overline{\eta}^a+\overline{N}_k^a d\overline{z}^k\}. \tag{2}$$

Definition 1 ([6])**.** A strictly pseudoconvex complex Finsler structure on E, is a positive real valued smooth function $F^2=L:E\to R_+\cup\{0\}$ which satisfies the following conditions:

(i) L is smooth on $E-\{0\}$;

(ii) $L(z,\eta)\geq 0$ and $L(z,\eta)=0\Leftrightarrow \eta=0$;

(iii) $L(z,\lambda\eta)=|\lambda|^2 L(z,\eta)$, $\forall\ \lambda\in\mathbb{C}$;

(iv) the complex Hessian $(h_{a\overline{b}})=(\dot{\partial}_a\dot{\partial}_{\overline{b}}\ (L))$ is positive definite and determines a Hermitian metric tensor on the fibers of vertical bundle $VE-\{zero\ section\}$.

Definition 2. The pair (E,L) is called a complex Finsler bundle.

According to [2], [9] a c.n.c. on (E,L) related only to the complex Finsler structure L is the canonical (or Chern-Finsler) c.n.c., locally given by

$$\overset{CF}{N_k^a}=h^{\overline{c}a}\partial_k\ \dot{\partial}_{\overline{c}}\ (L). \tag{3}$$

Throughout this paper, we consider only adapted local frames and coframes with respect to the Chern-Finsler c.n.c.

Vertical cohomology. Let us consider $F^{p,q,r,s}(E)$ the set of all (p,q,r,s)-forms with complex values on E, locally defined by

$$\omega=\sum\omega_{IAJB}dz^I\wedge\delta\eta^A\wedge d\overline{z}^J\wedge\delta\overline{\eta}^B, \tag{4}$$

where $I=(i_1,...,i_p)$; $A=(a_1,...,a_q)$; $J=(j_1,...,j_r)$; $B=(b_1,...,b_s)$ and the sum is on the indices $i_1\leq...\leq i_p$; $a_1\leq...\leq a_q$; $j_1\leq...\leq j_r$ and $b_1\leq...\leq b_s$.

The exterior derivative of $\omega \in F^{p,q,r,s}(E)$ has the decomposition $d = d^{'} + d^{''}$ where $d^{'} = d^{'h} + d^{'v} + \partial_3 + \partial_4$, $d^{''} = d^{''h} + d^{''v} + \partial_1 + \partial_2$ and

$$
\begin{aligned}
&d^{'h} : F^{p,q,r,s} \to F^{p+1,q,r,s}, && d^{'v} : F^{p,q,r,s} \to F^{p,q+1,r,s},\\
&d^{''h} : F^{p,q,r,s} \to F^{p,q,r+1,s}, && d^{''v} : F^{p,q,r,s} \to F^{p,q,r,s+1},\\
&\partial_1 : F^{p,q,r,s} \to F^{p+1,q-1,r+1,s}, && \partial_2 : F^{p,q,r,s} \to F^{p+1,q-1,r,s+1},\\
&\partial_3 : F^{p,q,r,s} \to F^{p+1,q,r+1,s-1}, && \partial_4 : F^{p,q,r,s} \to F^{p,q+1,r+1.s-1}.
\end{aligned}
$$

The conjugated vertical differential operator $d^{''v}$ is locally given by

$$d^{''v}\omega = \sum_a \dot{\partial}_{\overline{a}} \, (\omega_{IAJB})\delta\overline{\eta}^a \wedge dz^I \wedge \delta\eta^A \wedge d\overline{z}^J \wedge \delta\overline{\eta}^B \tag{5}$$

and satisfies $d^{''v}(\omega \wedge \theta) = d^{''v}\omega \wedge \theta + (-1)^{\deg \omega}\omega \wedge d^{''v}\theta$, for any $\omega \in F^{p,q,r,s}$ and $\theta \in F^{p^{'},q^{'},r^{'},s^{'}}$.

This operator has the property $(d^{''v})^2 = 0$ and satisfies a Grothendieck-Dolbeault type Lemma, namely

Theorem 1 ([7])**.** *Let ω be a $d^{''v}$-closed (p, q, r, s)-form defined on a neighborhood U on E and $s \geq 1$. Then there exists a $(p, q, r, s - 1)$ -form θ defined on some neighborhood $U^{'} \subset U$ and such that $d^{''v}\theta = \omega$ on $U^{'}$.*

We notice that the original proof of the above Theorem is inspired by [12] and firstly appeared in [10] for the case when $E = T^{'}M$ is the holomorphic tangent bundle of a complex Finsler manifold (M, F). Also, it may be found in [9], p. 89.

The v-cohomology groups of a complex Finsler bundle with coefficients in the sheaf $\Phi^{p,q,r}$ of germs of $(p, q, r, 0)$-forms are given by

$$H^s(E, \Phi^{p,q,r}) = Z^{p,q,r,s}/d^{''v}F^{p,q,r,s-1}(E), \tag{6}$$

where $Z^{p,q,r,s}$ is the space of $d^{''v}$- closed (p, q, r, s)-forms.

The Chern-Finsler linear connection. Following [3], we identify the holomorphic local frame field $s = \{s_a\}$, $a = 1, ..., m$ of E, with the one of pull-back bundle $\widetilde{E} = \pi^*E$, which is canonically isomorphic to VE by $\pi^*s_a \leftrightarrow \dot{\partial}_a$. Then, $\widetilde{E}$ admits a Hermitian structure h_L induced by the complex Finsler structure L on E, defined by

$$h_L(Z, W) = h_{a\bar{b}}Z^a\overline{W}^b, \; Z, W \in \Gamma(\widetilde{E}). \tag{7}$$

Let $\nabla : \Gamma(\widetilde{E}) \to A^1(\widetilde{E})$ be the Hermitian connection of the bundle $(\widetilde{E}, h_L)$, i.e. $\nabla = \nabla^{'} + \nabla^{''}$ is the unique connection on the bundle $(\widetilde{E}, h_L)$ satisfying the conditions

$$\nabla^{''} = d^{''},\ dh_L(Z,W) = h_L(\nabla Z, W) + h_L(Z, \nabla W),\ Z, W \in \Gamma(\widetilde{E}). \tag{8}$$

Definition 3 ([1])**.** The Hermitian connection ∇ on $(\widetilde{E}, h_L)$ is called the Chern-Finsler linear connection of the bundle (E, L).

With respect to the adapted coframes of the Chern-Finsler c.n.c., the $(1,0)$-connection form $\omega = (\omega^a_b)$ of ∇ is locally defined by

$$\omega^a_b = L^a_{bk} dz^k + C^a_{bc} \delta\eta^c \tag{9}$$

with the local coefficients $L^a_{bk} = h^{\overline{c}a} \delta_k (h_{b\overline{c}})$ and $C^a_{bc} = h^{\overline{d}a}\ \dot{\partial}_c\ (h_{b\overline{d}})$.

Thus, we have the decomposition $\omega = \omega^h + \omega^v$, where

$$(\omega^a_b)^h = L^a_{bk} dz^k,\ (\omega^a_b)^v = C^a_{bc} \delta\eta^c. \tag{10}$$

The $(1,1)$-curvature form $R = (R^a_b)$ of ∇ is locally given by

$$R^a_b = d^{''} \omega^a_b. \tag{11}$$

Using the decomposition $d^{''} = d^{''h} + d^{''v} + \partial_1 + \partial_2$, a straightforward calculus leads to the following decomposition of the curvature R

$$R = R^{h\overline{h}} + R^{h\overline{v}} + R^{v\overline{h}} + R^{v\overline{v}} \tag{12}$$

i.e. in a horizontal component $R^{h\overline{h}}$, two mixed components $R^{h\overline{v}}$, $R^{v\overline{h}}$ and a vertical component $R^{v\overline{v}}$. Also, we notice that with respect to a holomorphic local frame field $s = \{s_a\}$ the vertical part of the curvature is given by

$$R^{v\overline{v}} s_b = (R^a_{bc\overline{d}} \delta\eta^c \wedge \delta\overline{\eta}^d) s_a, \tag{13}$$

where $R^a_{bc\overline{d}} = -\ \dot{\partial}_{\overline{d}}\ (C^a_{bc})$. It will be important in our study.

2. Vertical Chern type classes

Using a similar argument from real Finsler geometry (see [11]), let us consider $(\omega^a_b)^v = C^a_{bc}\delta\eta^c \in F^{0,1,0,0}(E)$ the vertical part of $(1,0)$-connection form of ∇ and $(R^a_b)^{v\overline{v}} \in F^{0,1,0,1}(E)$ the vertical part of $(1,1)$-curvature form of ∇. According to the local expression of $(R^a_b)^{v\overline{v}}$, we obtain $(R^a_b)^{v\overline{v}} = R^a_{bc\overline{d}} \delta\eta^c \wedge \delta\overline{\eta}^d = -\ \dot{\partial}_{\overline{d}}\ (C^a_{bc}) \delta\eta^c \wedge \delta\overline{\eta}^d = d^{''v} (\omega^a_b)^v$.

Taking into account the relation $(d^{''v})^2 = 0$, we have

Proposition 1. *$R^{v\overline{v}}$ is a d''^{v}-closed differential form.*

Let $gl(m,\mathbb{C}) \approx \mathbb{C}^{m\times m}$ be the Lie algebra of the linear general group $GL(m,\mathbb{C})$.

Definition 4 ([15])**.** A symmetric polynomial $f \in S^j(Gl(m,\mathbb{C}))$ is invariant if $f(a^{-1}X_1 a, ..., a^{-1}X_j a) = f(X_1, ..., X_j)$ for any $a \in Gl(m,\mathbb{C})$ and $X_j \in gl(m,\mathbb{C})$.

Proposition 2 ([15])**.** *The algebra of invariant symmetric polynomials on $gl(m,\mathbb{C})$ is generated by the elementary symmetric functions $f_j \in S^j(Gl(m,\mathbb{C}))$ given by*

$$\det(I_m - \frac{1}{2\pi i}X) = \sum_{j=0}^{m} f_j(X) = 1 - \frac{1}{2\pi i}trX + ... + (-\frac{1}{2\pi i})^m \det X. \tag{14}$$

Proposition 3. *With respect to an adapted local frame of E, the vertical curvature form of Chern-Finsler linear connection transforms at local changes coordinates by*

$$R'^{v\overline{v}} = M^{-1}R^{v\overline{v}}M\,,\ M \in Gl(m,\mathbb{C}). \tag{15}$$

Proof. From [9], we know that the change rules of the vertical vector fields and of the vertical differentials are respectively

$$\dot{\partial}_a = M_a^b(z)\ \dot{\partial}'_b,\ \dot{\partial}_{\overline{a}} = \overline{M_a^b}(z)\ \dot{\partial}'_{\overline{b}},\ \delta\eta^a = M_b^a(z)\delta\eta'^b\,,\ \delta\overline{\eta}^a = \overline{M}_b^a(z)\delta\overline{\eta}'^b, \tag{16}$$

where $M_a^b(z)$ are holomorphic functions, which represent the transition functions of E.

On the other hand, at coordinates changes, the vertical coefficients C_{bc}^a transform by the rule $C'^a_{bc} = M_p^a M_b^q M_c^r C_{qr}^p$. Thus, we have

$$\begin{aligned}(R'^p_q)^{v\overline{v}} &= -\ \dot{\partial}'_{\overline{s}}\ (C'^p_{qr})\delta\eta'^r \wedge \delta\overline{\eta}'^s \\ &= -M_{\overline{s}}^{\overline{d}}\ \dot{\partial}_{\overline{d}}\ (M_a^p M_q^b M_r^c C_{bc}^a)M_c^r\delta\eta^c \wedge M_{\overline{d}}^{\overline{s}}\delta\overline{\eta}^d \\ &= -M_q^b\ \dot{\partial}_{\overline{d}}\ (C_{bc}^a)\delta\eta^c \wedge \delta\overline{\eta}^d M_a^p \\ &= M_q^b(R_b^a)^{v\overline{v}}M_a^p\end{aligned}$$

and this end the proof. □

In the following we consider the $(0,j,0,j)$-forms $\overset{v}{\gamma}_j\,(E,L) = f_j(R^{v\overline{v}})$ with complex values on E, defined by the relation

$$\det\left(I_m - \frac{1}{2\pi i}R^{v\overline{v}}\right) = \sum_{j=0}^{m} \overset{v}{\gamma}_j\,(E,L). \tag{17}$$

According to Proposition 2, we have

$$\overset{v}{\gamma}_j\,(E,L) = \frac{(-1)^j}{(2\pi i)^j j!}\sum_{a,b}\delta^{b_1\ldots b_j}_{a_1\ldots a_j}(R^{a_1}_{b_1})^{v\overline{v}} \wedge \ldots \wedge (R^{a_j}_{b_j})^{v\overline{v}}. \tag{18}$$

Occurs

Proposition 4. *The differential forms $\overset{v}{\gamma}_j\,(E,L)$ are d''^v-closed.*

The differential forms given by (18) are called *vertical Chern type forms* of the complex Finsler bundle (E,L).

According to Proposition 4 and (6), these forms define the following vertical cohomology classes

$$\overset{v}{c}_j\,(E,L) = [\overset{v}{\gamma}_j\,(E,L)] \in H^j(E,\Phi^{0,j,0}) \tag{19}$$

and these are called *vertical Chern type classes* of (E,L).

In particular, the first vertical Chern type class of (E,L) is represented by the $(0,1,0,1)$-form

$$\overset{v}{\gamma}_1\,(E,L) = -\frac{1}{2\pi i}R^a_{ac\overline{d}}\delta\eta^c \wedge \delta\overline{\eta}^d. \tag{20}$$

Given a complex Finsler structure L on E, according to [5], we consider $h'_L = \mu h_L$ a new Hermitian structure on $\widetilde{E}$, where μ is a positive real valued function on E. With respect to a fixed local holomorphic frame field $s = \{s_a\}$, $a = 1,...,m$ of $\widetilde{E}$, we calculate the vertical $(1,0)$- connection forms $\omega^v = (\omega^a_b)^v$ and $\omega'^v = (\omega'^a_b)^v$ of h_L and h'_L, respectively. Taking into account (9) it follows $(\omega^a_b)^v = h^{\overline{c}a}d'^v h_{b\overline{c}}$ and $(\omega'^a_b)^v = \frac{1}{\mu}h^{\overline{c}a}d'^v(\mu h_{b\overline{c}}) = h^{\overline{c}a}d'^v h_{b\overline{c}} + \frac{d'^v\mu}{\mu}\delta^a_b$. Thus, we have

$$(\omega'^a_b)^v = (\omega^a_b)^v + d'^v(log\mu)\delta^a_b. \tag{21}$$

The relation between the vertical $(1,1)$-curvature forms $R^{v\overline{v}} = (R^a_b)^{v\overline{v}}$ and $R'^{v\overline{v}} = (R'^a_b)^{v\overline{v}}$ of h_L and h'_L, respectively, can be obtained by applying d''^v, namely

$$(R'^a_b)^{v\overline{v}} = (R^a_b)^{v\overline{v}} + d''^v d'^v(\log\mu)\delta^a_b. \tag{22}$$

The relation (22) leads to

Proposition 5. *The first vertical Chern type class of (E, L) is invariant by conformal changes on the pull-back bundle $\widetilde{E}$.*

In fact the above proposition remains valid for all higher vertical Chern type classes, namely we can state the following

Proposition 6. *The vertical Chern type classes $\overset{v}{c}_j\,(E, L)$ are invariant by conformal changes on the pull-back bundle $\widetilde{E}$.*

Proof. It is classical that for a square matrix A of order p we have

$$\det(A - \lambda I) = \sum_{j=0}^{p}(-1)^j \Delta_{p-j}(A)\lambda^j,$$

where $\Delta_j(A) = \frac{1}{j!}\sum_{\alpha,\beta}\delta^{\beta_1\ldots\beta_j}_{\alpha_1\ldots\alpha_j} a^{\alpha_1}_{\beta_1}\cdot\ldots\cdot a^{\alpha_j}_{\beta_j}$ is the sum of the principal minors of order j of the matrix A (see for instance [15], p. 235). If we denote by I the identity endomorphism of $\widetilde{E}$ then, taking into account $d''^v\Delta_j(R^{v\overline{v}}) = 0$, by direct calculations we have

$$\begin{aligned} & f_j(R'^{v\overline{v}}) = f_j(R^{v\overline{v}} + d''^v d'^v(\log\mu)\otimes I) \\ & = f_j(R^{v\overline{v}}) + \frac{(-1)^j}{(2\pi i)^j j!}\sum_{k=1}^{j}\Delta_{j-k}(R^{v\overline{v}})\wedge(d''^v d'^v(\log\mu))^k \\ & = f_j(R^{v\overline{v}}) + d''^v\left(\frac{(-1)^j}{(2\pi i)^j j!}\sum_{k=1}^{j}\Delta_{j-k}(R^{v\overline{v}})\wedge d'^v\log\mu\wedge(d''^v d'^v(\log\mu))^{k-1}\right) \end{aligned}$$

which says that $f_j(R'^{v\overline{v}})$ and $f_j(R^{v\overline{v}})$ are in the same d''^v-cohomology class.$\square$

Finally, we consider a more general situation. Let $L_t : E \to R_+$, $0 \leq t \leq 1$, be a family of complex Finsler structures on the bundle E. To each Finsler structure we associate the Chern-Finsler connection. We have

Theorem 2. *The vertical Chern type classes* $\overset{v}{c}_j\,(E,L)$ *do not depend on the choice of the complex Finsler structure in the family* $\{L_t\}_{0\leq t\leq 1}$.

Proof. According to the first section, each complex Finsler structure L_t on E, induces a Hermitian structure h_{L_t} on pull-back bundle $\widetilde{E}$. If we consider $\mathcal{C}_j(\widetilde{E},h_L)=f_j(R)$ the classical Chern form of order j of the Hermitian bundle $\widetilde{E}$, a straightforward calculus (see [16] p. 21), leads to

$$\mathcal{C}_j(\widetilde{E},h_{L_1})-\mathcal{C}_j(\widetilde{E},h_{L_0})=\int_0^1\frac{d}{dt}(\mathcal{C}_j(\widetilde{E},h_{L_t}))dt=d^{'}d^{''}\alpha \tag{23}$$

for a $(j-1,\,j-1)$-form α on E.

Using (12) and (18), it is easy to check that $\overset{v}{\gamma}_j\,(E,L)=v\overline{v}\mathcal{C}_j(\widetilde{E},h_L)$. Now, taking into account the decompositions of $d^{'}$ and $d^{''}$ and the relation $d^{'v}d^{''v}+d^{''v}d^{'v}=0$ (see [7], [10]), if we consider in the relation (23) all differential forms being (p,q,r,s)-forms on E by equating the $v\,\overline{v}$-type components, we get

$$\overset{v}{\gamma}_j\,(E,L_1)-\overset{v}{\gamma}_j\,(E,L_0)=d^{'v}d^{''v}\beta=-d^{''v}d^{'v}\beta,$$

where β is a $(0,j-1,0,j-1)$-form on E, and this end the proof. $\square$

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