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ON THE FEEDBACK STABILIZATION OF BOUSSINESQ EQUATIONS*

BY

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Abstract. This paper provides feedback stabilization results, preserving the invariance of a given convex set, for the Boussinesq system, in 2D and 3D. The proofs use an existence theorem for weak solutions.

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1. Introduction

Let $T > 0$ and $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be an open and bounded domain, with a smooth boundary $\partial\Omega$ (of class C^2 for instance). Let $Q = \Omega \times (0, T)$, $\Sigma = \partial\Omega \times (0, T)$ and consider the controlled Boussinesq system,

$$(1)\quad \begin{cases} \dfrac{\partial y}{\partial t} - \nu\Delta y + (y \cdot \nabla)y - \gamma(\theta - \theta^0)e_d + \nabla p = f_0 + u_0, & \text{in } Q, \\ \dfrac{\partial \theta}{\partial t} - k\Delta\theta + y \cdot \nabla\theta = \tilde{f}_0 + \tilde{u}_0, & \text{in } Q, \\ \text{div } y = 0, & \text{in } Q, \\ y = 0, \quad \theta = 0, & \text{on } \Sigma, \\ y(\cdot, 0) = y_0, \quad \theta(\cdot, 0) = \theta_0, & \text{in } \Omega, \end{cases}$$

where $y = (y_1, y_2, \ldots, y_d)$ denotes the velocity field, while p and θ are scalar functions, representing the scalar pressure and the temperature of the fluid.

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The densities of external forces are $f_0 = (f_{01}, f_{02}, \ldots, f_{0d})$ and $\tilde{f}_0$. The constants $\nu,\ k > 0$ are the kinematic viscosity coefficient and the thermic diffusivity, respectively. Also, $\gamma = g/\theta^0 > 0$, where g is the gravitational constant and $\theta^0 > 0$ is a constant reference temperature, while $e_d = (0, \ldots, 0, 1)$ is the d^{th} unit vector of $\mathbb{R}^d$. Finally, $u_0 = (u_{01}, u_{02}, \ldots, u_{0d})$ and $\tilde{u}_0$ are controls distributed on Ω.

In this section we recall the functional framework and put the Boussinesq system in an abstract form. In order to give a stability result we have to prove first the *global* existence for the solution of the involved problem. Because in the case $d = 3$ the strong solutions are in general only *local*, in Section 2 we will give an existence result for weak solutions. Section 3 concerns the exponential feedback stabilization of the Boussinesq equations; the solution remains in a closed convex set. The first stability result is global and the control used is distributed on the entire domain. The second result is a local stability one; the control belongs to a finite dimensional space and it is distributed on a subdomain; the idea of proof is based on the method of spectral decomposition, used by Barbu, Triggiani in [3] and by Lefter in [8, 9] to obtain local internal stabilization results for Navier-Stokes and MHD equations, respectively.

The scalar products of $L^2(\Omega)$ and $(L^2(\Omega))^d$ will be denoted by $(\cdot, \cdot)$ and the corresponding norms, by $|\cdot|$. The norms of $H_0^1(\Omega)$ and $(H_0^1(\Omega))^d$ will be denoted by $\|\cdot\| = |\nabla \cdot|$. Recall the standard spaces (see e.g.[5, 10])

$$H = \{y \in (L^2(\Omega))^d; \text{div } y = 0 \text{ in } \Omega, y \cdot n_{\partial\Omega} = 0 \text{ on } \partial\Omega\},$$
$$V = \{y \in (H_0^1(\Omega))^d; \text{div } y = 0 \text{ in } \Omega\}.$$

Let $H_B = H \times L^2(\Omega)$, $V_B = V \times H_0^1(\Omega)$.

It is known that H and H_B are real Hilbert spaces endowed with L^2-type norms $|\cdot|$, while V and V_B are real Hilbert spaces endowed with H_0^1-type norms $\|\cdot\|$. Denoting by V' the dual space of V and identifying H with its own dual, we have $V \subset H \subset V'$ algebraically and topologically, with compact injections. Also, if V_B' is the dual space of V_B and if we identify H_B with its own dual, we have $V_B \subset H_B \subset V_B'$ algebraically and topologically, with compact injections. Denote by $(\cdot, \cdot)$ the scalar products of H and H_B and the pairings between V and V' or between V_B and V_B'; denote by $\|\cdot\|_{V_B'}$ the norm of V_B'.

Let $A \in L(V, V')$, $(Ay, w) = \sum_{i=1}^d \int_\Omega \nabla y_i \cdot \nabla w_i dx$, $\forall y, w \in V$ be the Stokes operator ($L(X, Y)$ is the space of linear continuous operators from

X in Y). We have $(Ay, y) = \|y\|^2,\ \forall y \in V$. Set $D(A) = \{y \in V; Ay \in H\}$ and denote again by A the restriction of A to H.

Let $A_1 \in L(H_0^1(\Omega), H^{-1}(\Omega)),\ A_1 = -\Delta$. Then $(A_1\theta, \theta) = \|\theta\|^2,\ \forall \theta \in H_0^1(\Omega)$. We set $D(A_1) = \{\theta \in H_0^1(\Omega); A_1\theta \in L^2(\Omega)\}$ and we denote again by A_1 the restriction of A_1 to $L^2(\Omega)$.

Let $\mathcal{A} \in L(V_B, V_B')$ be defined by

$$\mathcal{A}z = \begin{pmatrix} \nu Ay \\ kA_1\theta \end{pmatrix},\ \forall\ z = \begin{pmatrix} y \\ \theta \end{pmatrix} \in V_B. \tag{2}$$

Let $D(\mathcal{A}) = D(A) \times D(A_1)$. The operator $\mathcal{A}$ maps $D(\mathcal{A})$ into H_B. Denote by $m_{\nu k} = \min\{\nu, k\}$.

Let $b : V \times V \times V \to \mathbb{R}$ be the trilinear continuous functional defined by

$$b(y, v, w) = \sum_{i,j=1}^{d} \int_\Omega y_i \frac{\partial v_j}{\partial x_i} w_j dx,\ \forall y, v, w \in V.$$

Denoting by $\|\cdot\|_s$ the norm of the Sobolev space $(H^s(\Omega))^d$, the functional b satisfies (see e.g. [5, 10])

$$b(y, w, w) = 0,\ b(y, v, w) = -b(y, w, v),\ \forall y, v, w \in V, \tag{3}$$

$$|b(y, v, w)| \le C_b |y|^{\frac{1}{2}} \|y\|^{\frac{1}{2}} |v|^{\frac{1}{2}} \|v\|^{\frac{1}{2}} \|w\|,\ \forall y, v, w \in V \text{ (for } d = 2), \tag{4}$$

$$|b(y, v, w)| \le C_{b\tau} \|y\| \|v\|_{1+\tau} |w|,\ \forall y, w \in V, v \in D(A) \text{ (for } d = 2) \tag{5}$$

(where $\tau \in (0, 1)$; of course, $C_{b\tau}$ depends on τ),

$$|b(y, v, w)| \le C_b \|y\| \|v\|_{3/2} |w|,\ \forall y, w \in V, v \in D(A) \text{ (for } d = 3), \tag{6}$$

$$|b(y, v, w)| \le C_b |y|^{1/2} \|y\|^{1/2} \|v\| \|w\|,\ \forall y, v, w \in V \text{ (for } d = 3), \tag{7}$$

$$|b(y, v, w)| \le C_b \|y\| \|v\| |w|^{1/2} \|w\|^{1/2},\ \forall y, v, w \in V \text{ (for } d = 3). \tag{8}$$

Let $\bar{b} : V \times H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ be the trilinear continuous functional defined by

$$\bar{b}(y, \theta, \tilde{\theta}) = \sum_{i=1}^{d} \int_\Omega y_i \frac{\partial \theta}{\partial x_i} \tilde{\theta} dx,\ \forall\ y \in V,\ \forall\ \theta, \tilde{\theta} \in H_0^1(\Omega).$$

The application $\bar{b}$ verifies relations analogous to (3)-(8).

In the rest of the paper, we will denote by C_b the positive constants arising in estimations of b or $\bar{b}$ of the type (4)-(8). Other various positive constants will be simply denoted by the symbol C.

Let $B : V_B \to V_B'$ be defined by

$$(Bz, \tilde{z}) = b(y, y, \tilde{y}) + \bar{b}(y, \theta, \tilde{\theta}),\ \forall\ z = \begin{pmatrix} y \\ \theta \end{pmatrix},\ \tilde{z} = \begin{pmatrix} \tilde{y} \\ \tilde{\theta} \end{pmatrix} \in V_B. \tag{9}$$

Let us introduce $\mathcal{R} \in L(H_B, H_B)$ by

$$\mathcal{R}z = \begin{pmatrix} P(-\gamma\theta e_d) \\ 0 \end{pmatrix},\ \forall z = \begin{pmatrix} y \\ \theta \end{pmatrix} \in H_B \tag{10}$$

($P : (L^2(\Omega))^d \to H$ is the Leray projection). Using the notations (2), (9), (10) and denoting also by

$$F = \begin{pmatrix} P(f_0 - \gamma\theta^0 e_d) \\ \tilde{f}_0 \end{pmatrix},\ z_0 = \begin{pmatrix} y_0 \\ \theta_0 \end{pmatrix},\ z = \begin{pmatrix} y \\ \theta \end{pmatrix},\ U = \begin{pmatrix} Pu_0 \\ \tilde{u}_0 \end{pmatrix}, \tag{11}$$

the equations (1) rewrite

$$\begin{cases} \dfrac{dz}{dt}(t) + \mathcal{A}z(t) + Bz(t) + \mathcal{R}z(t) = F(t) + U(t), \quad t \in (0, T) \\ z(0) = z_0. \end{cases} \tag{12}$$

Let $z_e = \begin{pmatrix} y_e \\ \theta_e \end{pmatrix} \in D(\mathcal{A})$ be fixed and define the linear continuous operator

$$\begin{aligned} &\mathcal{A}_0 : V_B \to V_B', \quad (\mathcal{A}_0 z, \tilde{z}) = b(y, y_e, \tilde{y}) + b(y_e, y, \tilde{y}) \\ &+ \bar{b}(y, \theta_e, \tilde{\theta}) + \bar{b}(y_e, \theta, \tilde{\theta}),\ \forall z = \begin{pmatrix} y \\ \theta \end{pmatrix},\ \tilde{z} = \begin{pmatrix} \tilde{y} \\ \tilde{\theta} \end{pmatrix} \in V_B. \end{aligned} \tag{13}$$

Hence $\mathcal{A}_0$ maps $D(\mathcal{A})$ into H_B.

In order to be useful for the stability of the Boussinesq equations, an existence result must be stated for a slightly more complicated system, such as:

$$\begin{cases} \dfrac{dz}{dt}(t) + (\mathcal{A} + B + \mathcal{A}_0 + \mathcal{R} + \mathcal{G})z(t) + \Phi(z(t)) \ni F(t),\ t \in (0, T) \\ z(0) = z_0, \end{cases} \tag{14}$$

where $\mathcal{G} : H_B \to H_B$ is a linear continuous operator (of norm M) and $\Phi \subset H_B \times H_B$ is a maximal monotone operator. The term $\mathcal{G} + \Phi$ usually arises as a nonlinear feedback controller.

In the sequel, the symbol $\rightharpoonup$ will be used to denote the convergence in the weak topology, while the strong convergence will be indicated by $\to$.

2. An existence result for weak solutions

2.1. An auxiliary proposition

Let us state first a proposition which will be used in the proof of the existence theorem for weak solutions.

Following the idea in [2], we introduce, for $N \in \mathbb{N}^*$, the truncated operator

$$B_N : V_B \to V_B', \; B_N z = \begin{cases} Bz, & \text{if } \|z\| \le N \\ \left(\dfrac{N}{\|z\|}\right)^2 Bz, & \text{if } \|z\| > N. \end{cases}$$

Let $\Phi_\lambda : H_B \to H_B$, $\Phi_\lambda = \frac{1}{\lambda}[I - (I + \lambda\Phi)^{-1}]$, $\lambda > 0$, be the Yosida approximation of the maximal monotone operator Φ ($I : H_B \to H_B$ is the identity application).

Proposition 2.1. *Let $N \in \mathbb{N}^*$ be fixed. Assume that $\Phi \subset H_B \times H_B$ is a maximal monotone operator, with $0 \in D(\Phi)$, and that there exist two constants $\alpha_0 \ge 0$, $\alpha \in (0,1)$ such that*

$$(\mathcal{A}h, \Phi_\lambda(h)) \ge -\alpha_0(1 + \|h\|^2) - \alpha|\Phi_\lambda(h)|^2, \; \forall \lambda > 0, \forall h \in D(\mathcal{A}). \tag{15}$$

Let $\mathcal{G} \in L(H_B, H_B)$, $z_0 \in D(\mathcal{A}) \cap D(\Phi)$ and $F \in W^{1,1}(0,T;H_B)$. Then for all $\lambda > 0$ there exists an unique strong solution $z_N^\lambda \in W^{1,\infty}(0,T;H_B) \cap L^\infty(0,T;D(\mathcal{A})) \cap C([0,T];V_B)$ for the problem

$$\begin{cases} \dfrac{dz_N^\lambda}{dt}(t) + \mathcal{A}z_N^\lambda(t) + B_N z_N^\lambda(t) + \mathcal{A}_0 z_N^\lambda(t) + \mathcal{R}z_N^\lambda(t) \\ \qquad\qquad + \mathcal{G}z_N^\lambda(t) + \Phi_\lambda(z_N^\lambda(t)) = F(t), \quad \text{a.e. } t \in (0,T) \\ z_N^\lambda(0) = z_0. \end{cases} \tag{16}$$

Moreover, z_N^λ is right differentiable, $\frac{d^+}{dt}z_N^\lambda(t)$ is right continuous and

$$\begin{aligned} &\frac{d^+}{dt}z_N^\lambda(t) + \mathcal{A}z_N^\lambda(t) + B_N z_N^\lambda(t) + \mathcal{A}_0 z_N^\lambda(t) + \mathcal{R}z_N^\lambda(t) \\ &\qquad + \mathcal{G}z_N^\lambda(t) + \Phi_\lambda(z_N^\lambda(t)) = F(t), \; \forall t \in [0,T). \end{aligned} \tag{17}$$

Proof. The operator $\Upsilon_N = \mathcal{A}+B_N+\mathcal{A}_0+\mathcal{R}+\alpha_N I$ is maximal monotone in $H_B \times H_B$, for a positive constant α_N large enough, (see [7]), the Yosida approximation Φ_λ is demicontinuous monotone, while the linear continuous operator $\mathcal{G} : H_B \to H_B$ is obviously hemicontinuous and bounded in H_B. Hence ([1, 4]) $\Upsilon_N+\Phi_\lambda+\mathcal{G}$ is maximal monotone in $H_B \times H_B$. By [1], Thms. 1.4, 1.6, p. 214-216, (16) has a unique solution $z_N^\lambda \in W^{1,\infty}(0,T;H_B)$ verifying relation (17). From $\mathcal{G} \in L(H_B, H_B)$, we also get $\mathcal{G}z_N^\lambda \in L^\infty(0,T;H_B)$ and consequently $(\mathcal{A}+B_N+\mathcal{A}_0+\mathcal{R})z_N^\lambda + \Phi_\lambda(z_N^\lambda) = F - \frac{dz_N^\lambda}{dt} - \mathcal{G}z_N^\lambda \in L^\infty(0,T;H_B)$. From here, the proof of Proposition 2.1 follows the one of Proposition 3.4 in [7]. □

2.2. Weak solutions

Let K be a set satisfying the hypotheses

(h_K) $K \subset H_B$ is closed and convex, $0 \in K$ and

$$(I+\lambda\mathcal{A})^{-1}(K) \subset K, \ \forall \lambda > 0. \tag{18}$$

Introduce the indicator function of the set K,

$$I_K : H_B \to H_B, \ I_K(h) = \begin{cases} 0, & \text{for } h \in K \\ +\infty, & \text{for } h \notin K, \end{cases}$$

whose subdifferential is

$$\partial I_K(h) = \begin{cases} \emptyset, & h \notin K \\ \{0\}, & h \in \text{int } K \\ N_K(h) = \{z \in H_B; (z, h-h') \geq 0, \ \forall h' \in K\}, & h \in \partial K \end{cases}$$

($N_K(h)$ is the normal cone at K in h). Then $D(\partial I_K) = K$ and $0 \in D(\partial I_K)$. The Yosida approximation of ∂I_K is

$$(\partial I_K)_\lambda(h) = \frac{1}{\lambda}(h - P_K h) = \frac{1}{\lambda}Q_K h, \ \forall h \in H_B, \ \forall \lambda > 0,$$

where $P_K : H_B \to K$ is the orthogonal projection operator on K and $Q_K = I - P_K$. We recall that $(\partial I_K)_\lambda$ coincides with the Fréchet derivative of the regularisation of I_K,

$$(I_K)_\lambda(h) = \frac{1}{2\lambda}|h - P_K h|^2 = \frac{1}{2\lambda}|Q_K h|^2, \ \lambda > 0, \ h \in H_B.$$

Definition 2.1. Let $z_0 \in H_B$ and $F \in L^2(0,T;V_B')$. By a weak solution for the perturbed Boussinesq equations

$$(19)\quad \begin{cases} \dfrac{dz(t)}{dt} + (\mathcal{A} + B + \mathcal{A}_0 + \mathcal{R} + \mathcal{G})z(t) + \partial I_K(z(t)) \ni F(t), t \in (0,T) \\ z(0) = z_0 \end{cases}$$

we mean a function

$$z = \begin{pmatrix} y \\ \theta \end{pmatrix} \in L^2(0,T;V_B) \cap C_w([0,T];H_B), \text{ with } \frac{dz}{dt} \in L^1_{loc}(0,T;V_B'),$$

for which there exists a selection $\eta \in L^2(0,T;V_B')$, $\eta(t) \in \partial I_K(z(t))$, a.e. $t \in (0,T)$, such that

$$(20)\quad \begin{cases} \left(\dfrac{dz}{dt}(t), v\right) + \nu(y(t), v_1)_V + k(\theta(t), v_2)_{H_0^1(\Omega)} + b(y(t), y(t), v_1) \\ +\bar{b}(y(t), \theta(t), v_2) + b(y(t), y_e, v_1) + b(y_e, y(t), v_1) + \bar{b}(y(t), \theta_e, v_2) \\ +\bar{b}(y_e, \theta(t), v_2) - \gamma(\theta(t) e_d, v_1) + (\mathcal{G}z(t), v) + (\eta(t), v) \\ \qquad = (F(t), v), \text{ a.e. } t \in (0,T), \ \forall v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in V_B \\ z(0) = z_0. \end{cases}$$

By $(\cdot,\cdot)_V$, $(\cdot,\cdot)_{H_0^1(\Omega)}$ we have denoted the scalar products in V and $H_0^1(\Omega)$, respectively.

The space $C_w([0,T];H_B)$ is a subspace of $L^\infty(0,T;H_B)$ consisting of weakly continuous functions; in particular the initial condition in (20) is taken in this sense.

The existence of the weak solutions is ensured by the following theorem,

Theorem 2.1. *Let $T > 0$ and let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be an open, bounded set, with a smooth boundary and let $K \subset H_B$ satisfy conditions (h_K). Suppose, in addition, that*

(h_P) *if $d = 2$ then there exists a constant $\tau \in (0,1)$ such that P_K takes the bounded sets of H_B in bounded sets in $(H^{1+\tau}(\Omega))^3$ and if $d = 3$ then P_K takes the bounded sets of H_B in bounded sets in $(H^{3/2}(\Omega))^4$.*

Let $\mathcal{G} \in L(H_B, H_B)$. Let $z_0 \in K$ and $F \in L^2(0,T;H_B)$. Then problem (19) admits at least a weak solution $z \in L^2(0,T;V_B) \cap C_w([0,T];H_B)$. Moreover, $\frac{dz}{dt} \in L^2(0,T;V_B')$ for $d = 2$, $\frac{dz}{dt} \in L^{4/3}(0,T;V_B')$ for $d = 3$, and $z(t) \in K$, $t \in [0,T)$. If $d = 2$ the weak solution is unique.

Proof. Because the Yosida approximation of ∂I_K has a more convenient form, let us analyze first an approximate problem, involving B_N, $(\partial I_K)_{\frac{1}{j}}$, with $N, j \in \mathbb{N}^*$ and having more regular initial data z_0^j, F_j. Here $(\partial I_K)_{\frac{1}{j}} = jQ_K$ is the Yosida approximation $(\partial I_K)_\lambda$ for $\lambda = 1/j$; the sequence $1/j \searrow 0$. Passing to the limit, first with j, then with N, we will get a weak solution for problem (19).

Let $z_0 \in K = \overline{D(\mathcal{A}) \cap K}^{H_B}$ and $F \in L^2(0,T;H_B)$. There exist two sequences $(z_0^j)_{j\in\mathbb{N}^*} \subset D(\mathcal{A}) \cap K$, $(F_j)_{j\in\mathbb{N}^*} \subset W^{1,1}(0,T;H_B)$, such that $z_0^j \to z_0$ in H_B and $F_j \to F$ in $L^2(0,T;H_B)$.

Recalling (18) and applying [4], Prop. 4.5, p. 131 for the single valued maximal monotone operator $\mathcal{A} \subset H_B \times H_B$, with $\overline{D(\mathcal{A})}^{H_B} = H_B$, we obtain

$$(\mathcal{A}h, (\partial I_K)_\lambda(h)) \geq 0, \ \forall h \in D(\mathcal{A}), \ \forall \lambda > 0. \tag{21}$$

Assume that $j, N \in \mathbb{N}^*$ are fixed. From Proposition 2.1 it results that the problem

$$\begin{cases} \dfrac{dz_N^j(t)}{dt} + \mathcal{A}z_N^j(t) + B_N z_N^j(t) + \mathcal{A}_0 z_N^j(t) + \mathcal{R}z_N^j(t) \\ \qquad +\mathcal{G}z_N^j(t) + (\partial I_K)_{\frac{1}{j}}(z_N^j(t)) = F_j(t), \text{ a.e. } t \in (0,T) \\ z_N^j(0) = z_0^j \end{cases} \tag{22}$$

admits an unique solution

$$z_N^j = \begin{pmatrix} y_N^j \\ \theta_N^j \end{pmatrix} \in W^{1,\infty}(0,T;H_B) \cap L^\infty(0,T;D(\mathcal{A})) \cap C([0,T];V_B).$$

Let us first multiply equation (22) scalarly by $z_N^j(t)$ and integrate it on $(0,t)$. Taking into account that the operator $(\partial I_K)_{\frac{1}{j}}$ is monotone and then applying (3), (4) for $d = 2$ and (6) for $d = 3$, we get

$$\frac{1}{2}|z_N^j(t)|^2 + \nu \int_0^t \|y_N^j(s)\|^2 ds + k \int_0^t \|\theta_N^j(s)\|^2 ds \leq \frac{1}{2}|z_0^j|^2 + \int_0^t (F_j(s), z_N^j(s)) ds$$
$$- \int_0^t (\mathcal{G}z_N^j(s), z_N^j(s)) ds + \int_0^t \left(\gamma \theta_N^j(s) e_d, y_N^j(s)\right) ds - \int_0^t b(y_N^j(s), y_e, y_N^j(s)) ds$$
$$- \int_0^t \bar{b}(y_N^j(s), \theta_e, \theta_N^j(s)) ds \leq \frac{1}{2}|z_0^j|^2 + \int_0^t |F_j(s)||z_N^j(s)| ds$$
$$+ \left(M + \frac{\gamma}{2}\right) \int_0^t |z_N^j(s)|^2 ds + C_b \int_0^t |y_N^j(s)| \|y_e\|_a \|y_N^j(s)\| ds$$

$$+ C_b \begin{cases} \displaystyle\int_0^t |y_N^j(s)|^{\frac{1}{2}} \|y_N^j(s)\|^{\frac{1}{2}} |\theta_N^j(s)|^{\frac{1}{2}} \|\theta_N^j(s)\|^{\frac{1}{2}} \|\theta_e\| ds, & d = 2 \\ \displaystyle\int_0^t \|y_N^j(s)\| \|\theta_e\|_{3/2} |\theta_N^j(s)| ds, & d = 3 \end{cases}$$

$$\leq \frac{1}{2}|z_0^j|^2 + \frac{1}{2}\int_0^t |F_j(s)|^2 ds + \left(M + \frac{\gamma+1}{2}\right)\int_0^t |z_N^j(s)|^2 ds$$

$$+ C\int_0^t |z_N^j(s)| \|z_e\|_a \|z_N^j(s)\| ds, \text{ where } a = 1 \text{ if } d = 2 \text{ and } a = 3/2 \text{ if } d = 3.$$

Hence,

$$\frac{1}{2}|z_N^j(t)|^2 + m_{\nu k}\int_0^t \|z_N^j(s)\|^2 ds \leq \frac{1}{2}|z_0^j|^2 + \frac{1}{2}\int_0^t |F_j(s)|^2 ds$$
$$+ C\int_0^t |z_N^j(s)|^2 ds + \frac{m_{\nu k}}{2}\int_0^t \|z_N^j(s)\|^2 ds,$$

which yields

$$|z_N^j(t)|^2 + m_{\nu k}\int_0^t \|z_N^j(s)\|^2 ds \leq |z_0^j|^2 + \int_0^T |F_j(s)|^2 ds + C\int_0^t |z_N^j(s)|^2 ds.$$

By Gronwall's inequality,

$$|z_N^j(t)|^2 + m_{\nu k}\int_0^t \|z_N^j(s)\|^2 ds$$
$$\leq C\left(\|F_j\|^2_{L^2(0,T;H_B)}, |z_0^j|^2\right) \leq C, \text{ a.e. } t \in (0,T), \tag{23}$$

where C does not depend on N, j.

Let us now multiply (22) scalarly by $(\partial I_K)_{\frac{1}{j}}(z_N^j(t)) = \nabla(I_K)_{\frac{1}{j}}(z_N^j(t))$. Recalling (21), after integrating on $(0,t)$ we obtain that

$$(I_K)_{\frac{1}{j}}(z_N^j(t)) - (I_K)_{\frac{1}{j}}(z_0^j) + \int_0^t (B_N z_N^j(s), (\partial I_K)_{\frac{1}{j}}(z_N^j(s)))ds$$
$$+ \int_0^t (\mathcal{A}_0 z_N^j(s), (\partial I_K)_{\frac{1}{j}}(z_N^j(s)))ds + \int_0^t (\mathcal{R} z_N^j(s), (\partial I_K)_{\frac{1}{j}}(z_N^j(s)))ds$$
$$+ \int_0^t (\mathcal{G} z_N^j(s), (\partial I_K)_{\frac{1}{j}}(z_N^j(s)))ds + \int_0^t |(\partial I_K)_{\frac{1}{j}}(z_N^j(s))|^2 ds \tag{24}$$
$$= \int_0^t (F_j(s), (\partial I_K)_{\frac{1}{j}}(z_N^j(s)))ds.$$

But $(I_K)_{\frac{1}{j}}(h) = (j/2)|h - P_K h|^2 = (j/2)|Q_K h|^2$, $\forall h \in H_B$, $\forall j \in \mathbb{N}^*$, particularly, $(I_K)_{\frac{1}{j}}(z_0^j) = 0$ because $z_0^j \in K$. Using also the relation $(\partial I_K)_{\frac{1}{j}} = jQ_K$ and dividing by j in relation (24), we get

$$\begin{aligned}
&\frac{1}{2}|Q_K z_N^j(t)|^2 + j\int_0^t |Q_K z_N^j(s)|^2 ds = \int_0^t (F_j(s), Q_K z_N^j(s))ds \\
&- \int_0^t (\mathcal{G} z_N^j(s), Q_K z_N^j(s))ds - \int_0^t (\mathcal{R} z_N^j(s), Q_K z_N^j(s))ds \\
&- \int_0^t (B_N z_N^j(s), Q_K z_N^j(s))ds - \int_0^t (\mathcal{A}_0 z_N^j(s), Q_K z_N^j(s))ds.
\end{aligned} \tag{25}$$

But

$$(F_j(s), Q_K z_N^j(s)) \leq \frac{3}{j}|F_j(s)|^2 + \frac{j}{12}|Q_K z_N^j(s)|^2$$

and

$$(\mathcal{G} z_N^j(s), Q_K z_N^j(s)) \leq M|z_N^j(s)||Q_K z_N^j(s)| \leq \frac{3M^2}{j}|z_N^j(s)|^2 + \frac{j}{12}|Q_K z_N^j(s)|^2.$$

Let us denote by $[Q_K z_N^j(s)]_1 \in H$, $[Q_K z_N^j(s)]_2 \in L^2(\Omega)$ the components of $Q_K z_N^j(s) \in H_B$. We have

$$\begin{aligned}
&-(\mathcal{R} z_N^j(s), Q_K z_N^j(s)) = \gamma\left(\theta_N^j(s)e_d, \left[Q_K z_N^j(s)\right]_1\right) \\
&\leq \gamma|z_N^j(s)||Q_K z_N^j(s)| \leq \frac{3\gamma^2}{j}|z_N^j(s)|^2 + \frac{j}{12}|Q_K z_N^j(s)|^2.
\end{aligned}$$

Denote also by $[P_K z_N^j(s)]_1 \in H$, $[P_K z_N^j(s)]_2 \in L^2(\Omega)$ the components of $P_K z_N^j(s) \in K \subset H_B$. From $P_K z_N^j(s) + Q_K z_N^j(s) = z_N^j(s) = \begin{pmatrix} y_N^j(s) \\ \theta_N^j(s) \end{pmatrix}$, we obviously obtain that $y_N^j(s) = [P_K z_N^j(s)]_1 + [Q_K z_N^j(s)]_1$ and $\theta_N^j(s) = [P_K z_N^j(s)]_2 + [Q_K z_N^j(s)]_2$.

Using (3), (5), (6) and the hypothesis (h_P), we infer

$$\begin{aligned}
&|(B_N z_N^j(s), Q_K z_N^j(s))| \leq \left|b\left(y_N^j(s), y_N^j(s), \left[Q_K z_N^j(s)\right]_1\right)\right| \\
&+ \left|\bar{b}\left(y_N^j(s), \theta_N^j(s), \left[Q_K z_N^j(s)\right]_2\right)\right| = \left|b\left(y_N^j(s), \left[P_K z_N^j(s)\right]_1, \left[Q_K z_N^j(s)\right]_1\right)\right| \\
&+ \left|\bar{b}\left(y_N^j(s), \left[P_K z_N^j(s)\right]_2, \left[Q_K z_N^j(s)\right]_2\right)\right| \leq C_b\|y_N^j(s)\|\left(\left\|\left[P_K z_N^j(s)\right]_1\right\|_a\right.
\end{aligned}$$

$$\cdot \left|\left[Q_K z_N^j(s)\right]_1\right| + \left\|\left[P_K z_N^j(s)\right]_2\right\|_a \left|\left[Q_K z_N^j(s)\right]_2\right|\Big)$$
$$\leq C\|z_N^j(s)\| \left\|P_K z_N^j(s)\right\|_a \left|Q_K z_N^j(s)\right| \leq \frac{C}{j}\|z_N^j(s)\|^2 + \frac{j}{12}\left|Q_K z_N^j(s)\right|^2,$$

where $a = 3/2$ if $d = 3$ and $a = 1+\tau$, with $\tau \in (0,1)$ given by (h_P), if $d = 2$ (τ being fixed, we have renoted $C_{b\tau} = C_b$ in (5)). C is a positive constant independent of j, N.

Analogously, we get

$$-(\mathcal{A}_0 z_N^j(s), Q_K z_N^j(s)) = -b\left(y_N^j(s), y_e, \left[Q_K z_N^j(s)\right]_1\right)$$
$$-b\left(y_e, y_N^j(s), \left[Q_K z_N^j(s)\right]_1\right) - \bar{b}\left(y_N^j(s), \theta_e, \left[Q_K z_N^j(s)\right]_2\right)$$
$$-\bar{b}\left(y_e, \theta_N^j(s), \left[Q_K z_N^j(s)\right]_2\right) = -b\left(y_N^j(s), y_e, \left[Q_K z_N^j(s)\right]_1\right)$$
$$-\bar{b}\left(y_N^j(s), \theta_e, \left[Q_K z_N^j(s)\right]_2\right) - b\left(y_e, \left[P_K z_N^j(s)\right]_1, \left[Q_K z_N^j(s)\right]_1\right)$$
$$-\bar{b}\left(y_e, \left[P_K z_N^j(s)\right]_2, \left[Q_K z_N^j(s)\right]_2\right) \leq C_b\|y_N^j(s)\|\left(\|y_e\|_a \left|\left[Q_K z_N^j(s)\right]_1\right|\right.$$
$$\left.+\|\theta_e\|_a \left|\left[Q_K z_N^j(s)\right]_2\right|\right) + C_b\|y_e\|\left(\left\|\left[P_K z_N^j(s)\right]_1\right\|_a \left|\left[Q_K z_N^j(s)\right]_1\right|\right.$$
$$\left.+\left\|\left[P_K z_N^j(s)\right]_2\right\|_a \left|\left[Q_K z_N^j(s)\right]_2\right|\right) \leq C\|z_N^j(s)\|\|z_e\|_a \left|Q_K z_N^j(s)\right|$$
$$+C\|z_e\|\left\|P_K z_N^j(s)\right\|_a \left|Q_K z_N^j(s)\right| \leq \frac{j}{6}\left|Q_K z_N^j(s)\right|^2 + \frac{C}{j}(\|z_N^j(s)\|^2 + 1)$$

(a has the same meaning as in the estimation of $|(B_N z_N^j(s), Q_K z_N^j(s))|$).

Then relation (25) becomes

$$\frac{1}{2}|Q_K z_N^j(t)|^2 + j\int_0^t |Q_K z_N^j(s)|^2 ds \leq \frac{j}{2}\int_0^t |Q_K z_N^j(s)|^2 ds$$
$$+\frac{3}{j}\int_0^t |F_j(s)|^2 ds + \frac{3(M^2+\gamma^2)}{j}\int_0^t |z_N^j(s)|^2 ds + \frac{C}{j}\int_0^t \|z_N^j(s)\|^2 ds$$
$$+\frac{C}{j}\int_0^t ds \leq \frac{C}{j} + \frac{j}{2}\int_0^t |Q_K z_N^j(s)|^2 ds, \text{ (by (23))}.$$

By consequence,

$$\frac{1}{2}|Q_K z_N^j(t)|^2 + \frac{j}{2}\int_0^t |Q_K z_N^j(s)|^2 ds \leq \frac{C}{j}.$$

Dropping the term $(1/2)|Q_K z_N^j(t)|^2 \geq 0$ and multiplying by j, we infer

$$\int_0^t |(\partial I_K)_{\frac{1}{j}}(z_N^j(s))|^2 ds \leq C, \text{ a.e } t \in (0,T), \tag{26}$$

where C is a positive constant independent of j, N.

Relation (23) also implies

$$\int_0^t |\mathcal{G} z_N^j(s)|^2 ds \leq M^2 \int_0^t |z_N^j(s)|^2 ds \leq C, \text{ a.e } t \in (0,T), \tag{27}$$

$$\int_0^t |\mathcal{R} z_N^j(s)|^2 ds \leq \gamma^2 \int_0^t |z_N^j(s)|^2 ds \leq C, \text{ a.e. } t \in (0,T), \tag{28}$$

where C is a positive constant independent of j, N.

For $d = 2$, using (3), (4) and (23), we obtain

$$\|B_N z_N^j(t)\|_{V_B'} \leq C|z_N^j(t)|\|z_N^j(t)\| \leq C\|z_N^j(t)\|, \text{ hence}$$

$$\int_0^T \|B_N z_N^j(t)\|_{V_B'}^2 dt \leq C, \text{ if } d = 2. \tag{29}$$

If $d = 3$, from (7) and (23), it results that

$$\|B_N z_N^j(t)\|_{V_B'} \leq C|z_N^j(t)|^{1/2}\|z_N^j(t)\|^{3/2} \leq C\|z_N^j(t)\|^{3/2}, \text{ so}$$

$$\int_0^T \|B_N z_N^j(t)\|_{V_B'}^{4/3} dt \leq C, \text{ if } d = 3. \tag{30}$$

From (26)-(30) and from the continuity of the operators $\mathcal{A}, \mathcal{A}_0 : V_B \to V_B'$ we get

$$\int_0^T \left\| \frac{dz_N^j}{dt}(t) \right\|_{V_B'}^2 dt \leq C \quad \text{for } d = 2; \tag{31}$$

$$\int_0^T \left\| \frac{dz_N^j}{dt}(t) \right\|_{V_B'}^{4/3} dt \leq C \quad \text{for } d = 3. \tag{32}$$

On a subsequence, renoted $(z_N^j)_j$, we obtain for $j \to \infty$,

$$
\begin{aligned}
&z_N^j \rightharpoonup z_N \text{ in } L^2(0,T;V_B),\\
&\mathcal{A}z_N^j \rightharpoonup \mathcal{A}z_N \text{ and } \mathcal{A}_0 z_N^j \rightharpoonup \mathcal{A}_0 z_N \text{ in } L^2(0,T;V_B'),\\
&\mathcal{G}z_N^j \rightharpoonup \mathcal{G}z_N \text{ and } \mathcal{R}z_N^j \rightharpoonup \mathcal{R}z_N \text{ in } L^2(0,T;H_B),\\
&\frac{dz_N^j}{dt} \rightharpoonup \frac{dz_N}{dt} \text{ in } \begin{cases} L^2(0,T;V_B'), & \text{if } d=2\\ L^{4/3}(0,T;V_B'), & \text{if } d=3, \end{cases}\\
&B_N z_N^j \rightharpoonup \beta_N \text{ in } \begin{cases} L^2(0,T;V_B'), & \text{if } d=2\\ L^{4/3}(0,T;V_B'), & \text{if } d=3, \end{cases}
\end{aligned}
$$

$$
(\partial I_K)_{\frac{1}{j}}(z_N^j) \rightharpoonup \eta_N \text{ in } L^2(0,T;H_B). \tag{33}
$$

In order to prove that $\beta_N = B_N z_N$ and $\eta_N \in \partial I_K(z_N)$, we use that for a positive constant C_N (depending on N, which is fixed as we pass to the limit with j) large enough, B_N satisfies the inequality (see [11], (2.6) p. 491)

$$
\begin{aligned}
&|(B_N w - B_N v, w - v)| \le \frac{\nu}{2}\|w_1 - v_1\|^2 + \frac{k}{2}\|w_2 - v_2\|^2 + C_N|w - v|^2,\\
&\forall w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in V_B,\ \forall v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in V_B
\end{aligned}
$$

and we get that, for $j, k \in \mathbb{N}^*$,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{ds}|z_N^j(s) - z_N^k(s)|^2 + \frac{\nu}{2}\|y_N^j(s) - y_N^k(s)\|^2 + \frac{k}{2}\|\theta_N^j(s) - \theta_N^k(s)\|^2\\
&\le C_N|z_N^j(s) - z_N^k(s)|^2 + |(\mathcal{G}(z_N^j(s) - z_N^k(s)), z_N^j(s) - z_N^k(s))|\\
&+|(\mathcal{R}(z_N^j(s) - z_N^k(s)), z_N^j(s) - z_N^k(s))| + |b(y_N^j(s) - y_N^k(s), y_e, y_N^j(s) - y_N^k(s))|\\
&+ |\bar{b}(y_N^j(s) - y_N^k(s), \theta_e, \theta_N^j(s) - \theta_N^k(s))| + |F_j(s) - F_k(s)||z_N^j(s) - z_N^k(s)|\\
&\le C_b\|y_N^j(s) - y_N^k(s)\|\|y_e\|_a|y_N^j(s) - y_N^k(s)|\\
&+ C_b \begin{cases} |y_N^j(s) - y_N^k(s)|^{\frac{1}{2}}\|y_N^j(s) - y_N^k(s)\|^{\frac{1}{2}}|\theta_N^j(s) - \theta_N^k(s)|^{\frac{1}{2}}\\ \qquad\qquad \cdot\|\theta_N^j(s) - \theta_N^k(s)\|^{\frac{1}{2}}\|\theta_e\|, & d=2\\ \|y_N^j(s) - y_N^k(s)\|\|\theta_N^j(s) - \theta_N^k(s)\|\|\theta_e\|_{3/2}, & d=3 \end{cases}\\
&+ \frac{1}{2}|F_j(s) - F_k(s)|^2 + C|z_N^j(s) - z_N^k(s)|^2 \le \frac{1}{2}|F_j(s) - F_k(s)|^2\\
&+ C|z_N^j(s) - z_N^k(s)|^2 + C\|z_N^j(s) - z_N^k(s)\|\|z_N^j(s) - z_N^k(s)\|\|z_e\|_a,
\end{aligned}
$$

where $a = 1$ if $d = 2$ and $a = 3/2$ if $d = 3$. This implies

$$\frac{1}{2}\frac{d}{ds}|z_N^j(s) - z_N^k(s)|^2 + \frac{m_{\nu k}}{2}\|z_N^j(s) - z_N^k(s)\|^2 \le \frac{1}{2}|F_j(s) - F_k(s)|^2 + C|z_N^j(s) - z_N^k(s)|^2 + \frac{m_{\nu k}}{4}\|z_N^j(s) - z_N^k(s)\|^2,$$

which is

$$\frac{d}{ds}|z_N^j(s) - z_N^k(s)|^2 + \frac{m_{\nu k}}{2}\|z_N^j(s) - z_N^k(s)\|^2 \le |F_j(s) - F_k(s)|^2 + C|z_N^j(s) - z_N^k(s)|^2.$$

After integrating on $(0, t)$ and applying the Gronwall lemma, we get

$$|z_N^j(t) - z_N^k(t)|^2 + \frac{m_{\nu k}}{2}\int_0^t \|z_N^j(s) - z_N^k(s)\|^2 ds \le C\left(|z_0^j - z_0^k|^2 + \int_0^T |F_j(s) - F_k(s)|^2 ds\right), \text{ for all } j, k \in \mathbb{N}^*.$$

Hence, for $j \to \infty$,

$$z_N^j \to z_N \text{ in } L^2(0,T;V_B) \cap C([0,T];H_B). \tag{34}$$

Taking into account that

$$\|B_N z_N^j(s) - B_N z_N(s)\|_{V_B'} \le C\|z_N^j(s) - z_N(s)\|(\|z_N^j(s)\| + \|z_N(s)\|),$$

relation (34) leads to $B_N z_N^j \to B_N z_N$ in $L^1(0,T;V_B')$.

On the other side, from (33), (34) and the maximal monotony of ∂I_K we obtain $\eta_N(t) \in \partial I_K(z_N(t))$, a.e. $t \in (0,T)$.

By consequence, the problem

$$\begin{cases} \dfrac{dz_N(t)}{dt} + \mathcal{A}z_N(t) + B_N z_N(t) + \mathcal{A}_0 z_N(t) + \mathcal{R}z_N(t) \\ \qquad + \mathcal{G}z_N(t) + \partial I_K(z_N(t)) \ni F(t), \quad \text{a.e. } t \in (0,T) \\ z_N(0) = z_0 \end{cases} \tag{35}$$

admits, for every $N \in \mathbb{N}^*$, a solution

$$z_N \in L^2(0,T;V_B) \cap C([0,T];H_B), \text{ with}$$
$$\frac{dz_N}{dt},\ B_N z_N \in \begin{cases} L^2(0,T;V_B'), & \text{if } d = 2 \\ L^{4/3}(0,T;V_B'), & \text{if } d = 3 \end{cases},$$
$$\mathcal{R}z_N, \mathcal{G}z_N, \eta_N \in L^2(0,T;H_B), \text{ where}$$

$\eta_N(t)=F(t)-\frac{dz_N}{dt}(t)-(\mathcal{A}+B_N+\mathcal{A}_0+\mathcal{R}+\mathcal{G})z_N(t)\in\partial I_K(z_N(t))$, a.e. $t\in(0,T)$. Moreover, z_N satisfies the estimates

$$|z_N(t)|^2+m_{\nu k}\int_0^t\|z_N(s)\|^2ds\le C,\ \text{a.e. } t\in(0,T), \tag{36}$$

$$\int_0^t|\eta_N(s)|^2ds+\int_0^t|\mathcal{R}z_N(s)|^2ds+\int_0^t|\mathcal{G}z_N(s)|^2ds\le C,\ \text{a.e. } t\in(0,T), \tag{37}$$

$$\int_0^T\left(\|B_Nz_N(t)\|^2_{V_B'}+\left\|\frac{dz_N}{dt}(t)\right\|^2_{V_B'}\right)dt\le C,\ \text{if } d=2, \tag{38}$$

$$\int_0^T\left(\|B_Nz_N(t)\|^{4/3}_{V_B'}+\left\|\frac{dz_N}{dt}(t)\right\|^{4/3}_{V_B'}\right)dt\le C,\ \text{if } d=3, \tag{39}$$

where C is a positive constant, independent of N. Then, on a subsequence renoted $(z_N)_N$, we obtain for $N\to\infty$ that

$$z_N\rightharpoonup z \text{ in } L^2(0,T;V_B),\ z_N\rightharpoonup z \text{ weakly * in } L^\infty(0,T;H_B), \tag{40}$$

$$\begin{aligned}
&\mathcal{A}z_N\rightharpoonup\mathcal{A}z \text{ and } \mathcal{A}_0z_N\rightharpoonup\mathcal{A}_0z \text{ in } L^2(0,T;V_B'),\\
&\mathcal{R}z_N\rightharpoonup\mathcal{R}z \text{ and } \mathcal{G}z_N\rightharpoonup\mathcal{G}z \text{ in } L^2(0,T;H_B),\\
&\frac{dz_N}{dt}\rightharpoonup\frac{dz}{dt} \text{ in } \begin{cases}L^2(0,T;V_B'), & \text{if } d=2\\ L^{4/3}(0,T;V_B'), & \text{if } d=3,\end{cases}\\
&B_Nz_N\rightharpoonup\beta \text{ in } \begin{cases}L^2(0,T;V_B'), & \text{if } d=2\\ L^{4/3}(0,T;V_B'), & \text{if } d=3,\end{cases}\\
&\eta_N\rightharpoonup\eta \text{ in } L^2(0,T;H_B).
\end{aligned} \tag{41}$$

From (40), (41) and the compactness theorem of Aubin it follows that

$$z_N\to z \text{ in } L^2(0,T;H_B). \tag{42}$$

The properties of ∂I_K ensure that $\eta(t)\in\partial I_K(z(t))$, a.e. $t\in(0,T)$. In order to prove that $\beta(t)=Bz(t)$, a.e. $t\in(0,T)$, consider $E_N=\{t\in[0,T];\|z_N(t)\|\le N\}$. Clearly, $B_Nz_N=Bz_N$ in E_N and the Lebesgue measure $m([0,T]\setminus E_N)=m(\{t\in[0,T];\|z_N(t)\|>N\})\le\frac{C}{N^2}$. Let $\psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\in L^\infty(0,T;\mathcal{V}_\mathcal{B})$, where $\mathcal{V}_\mathcal{B}=\{\zeta\in(C_0^\infty(\Omega))^d;\text{div}\zeta=0\}\times C_0^\infty(\Omega)$; $\mathcal{V}_\mathcal{B}$

is dense in V_B. Using (7) and (3) coupled with (8), we obtain

$$\begin{aligned}
&\int_0^T |(B_N z_N(t) - Bz(t), \psi(t))|dt \\
&\leq \int_{E_N} [|b(y_N(t) - y(t), y_N(t), \psi_1(t))|dt + |b(y(t), y_N(t) - y(t), \psi_1(t))|]dt \\
&+ \int_{E_N} [|\bar{b}(y_N(t) - y(t), \theta_N(t), \psi_2(t))| + |\bar{b}(y(t), \theta_N(t) - \theta(t), \psi_2(t))|]dt \\
&+ \int_{[0,T]\setminus E_N} [|(B_N z_N(t), \psi(t))| + |(Bz(t), \psi(t))|]dt \\
&\leq C_b \left\{ \int_0^T |y_N(t) - y(t)|^{1/2} \|y_N(t) - y(t)\|^{1/2} \left[\|y_N(t)\| + \|y(t)\|\right] \|\psi_1(t)\|dt \right. \\
&+ \int_0^T |y_N(t) - y(t)|^{1/2} \|y_N(t) - y(t)\|^{1/2} \|\theta_N(t)\| \|\psi_2(t)\|dt \\
&\left. + \int_0^T \|y(t)\| |\theta_N(t) - \theta(t)|^{1/2} \|\theta_N(t) - \theta(t)\|^{1/2} \|\psi_2(t)\|dt \right\} \\
&+ C_b \int_{[0,T]\setminus E_N} \left[\|\psi_1(t)\| \left(|y_N(t)|^{1/2} \|y_N(t)\|^{3/2} + |y(t)|^{1/2} \|y(t)\|^{3/2} \right) \right. \\
&\left. + \|\psi_2(t)\| \left(|y_N(t)|^{1/2} \|y_N(t)\|^{1/2} \|\theta_N(t)\| + |y(t)|^{1/2} \|y(t)\|^{1/2} \|\theta(t)\| \right) \right] dt \\
&\leq C \int_0^T |z_N(t) - z(t)|^{1/2} \|z_N(t) - z(t)\|^{1/2} \left[\|z_N(t)\| + \|z(t)\|\right] \|\psi(t)\|dt \\
&+ C \int_{[0,T]\setminus E_N} \|\psi(t)\| \left(|z_N(t)|^{1/2} \|z_N(t)\|^{3/2} + |z(t)|^{1/2} \|z(t)\|^{3/2} \right) dt \\
&\leq C\|\psi\|_{L^\infty(0,T;V_B)} \left(\int_0^T |z_N(t) - z(t)|^2 dt \right)^{1/4} \left(\int_0^T \|z_N(t) - z(t)\|^2 dt \right)^{1/4} \\
&\cdot \left[\int_0^T \left(\|z_N(t)\|^2 + \|z(t)\|^2 \right) dt \right]^{1/2} + C\|\psi\|_{L^\infty(0,T;V_B)} \left[\left(\int_0^T \|z_N(t)\|^2 dt \right)^{3/4} \right. \\
&\left. \cdot \left(\int_{[0,T]\setminus E_N} |z_N(t)|^2 dt \right)^{1/4} + \left(\int_0^T \|z(t)\|^2 dt \right)^{3/4} \left(\int_{[0,T]\setminus E_N} |z(t)|^2 dt \right)^{1/4} \right].
\end{aligned}$$

Recalling (40), we obtain that

$$\int_0^T |(B_N z_N(t) - Bz(t), \psi(t))|dt$$

$$\leq C\|\psi\|_{L^\infty(0,T;V_B)}\left[\left(\int_0^T |z_N(t)-z(t)|^2dt\right)^{1/4}+\frac{1}{\sqrt{N}}\right]$$

and by (42),

$$\lim_{N\to\infty}\int_0^T (B_N z_N(t)-Bz(t),\psi(t))dt=0,\ \forall\psi\in L^\infty(0,T;\mathcal{V}_\mathcal{B}).$$

Then $\beta(t)=Bz(t)$, a.e. $t\in(0,T)$, which concludes the existence part of Theorem 2.1.

If $d=2$, the solution is unique. Indeed, if problem (19) has had two solutions $z_1=\begin{pmatrix}y_1\\ \theta_1\end{pmatrix}$, $z_2=\begin{pmatrix}y_2\\ \theta_2\end{pmatrix}$, from the monotony of ∂I_K and the properties of b and $\bar{b}$ it would follow that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|z_1(t)-z_2(t)|^2+m_{\nu k}\|z_1(t)-z_2(t)\|^2\\
&\leq -b(y_1(t)-y_2(t),y_1(t),y_1(t)-y_2(t))\\
&-\bar{b}(y_1(t)-y_2(t),\theta_1(t),\theta_1(t)-\theta_2(t))-b(y_1(t)-y_2(t),y_e,y_1(t)-y_2(t))\\
&-\bar{b}(y_1(t)-y_2(t),\theta_e,\theta_1(t)-\theta_2(t))-(\mathcal{R}(z_1(t)-z_2(t)),z_1(t)-z_2(t))\\
&-(\mathcal{G}(z_1(t)-z_2(t)),z_1(t)-z_2(t))\leq C|z_1(t)-z_2(t)|\|z_1(t)-z_2(t)\|\|z_1(t)\|\\
&+C|z_1(t)-z_2(t)|\|z_1(t)-z_2(t)\|\|z_e\|+\left(\frac{\gamma}{2}+M\right)|z_1(t)-z_2(t)|^2\\
&\leq C\left(\|z_1(t)\|^2+1\right)|z_1(t)-z_2(t)|^2+\frac{m_{\nu k}}{2}\|z_1(t)-z_2(t)\|^2.
\end{aligned}$$

Hence, $\begin{cases}\dfrac{d}{dt}|z_1(t)-z_2(t)|^2\leq C\left(\|z_1(t)\|^2+1\right)|z_1(t)-z_2(t)|^2\\ (z_1-z_2)(0)=0\end{cases}$ and Gronwall's lemma gives $z_1\equiv z_2$. □

3. Feedback stabilization

3.1. Stability on closed convex sets

Let us consider the controlled system

$$\begin{cases}\dfrac{dz}{dt}(t)+\mathcal{A}z(t)+Bz(t)+\mathcal{R}z(t)=F_e+U(t), & t>0\\ z(0)=z_0,\end{cases}\tag{43}$$

where $z(t) = \begin{pmatrix} y(t) \\ \theta(t) \end{pmatrix}$ and $z_0 = \begin{pmatrix} y_0 \\ \theta_0 \end{pmatrix} \in V_B$, $F_e \in H_B$. Let $z_e = \begin{pmatrix} y_e \\ \theta_e \end{pmatrix} \in D(\mathcal{A})$ be a steady state solution for (43), i.e. z_e satisfies

$$\mathcal{A}z_e + Bz_e + \mathcal{R}z_e = F_e. \tag{44}$$

Let $K \subset H_B$ fullfill conditions (h_K), Section 2.2. The interest is in finding a feedback controller U such that $z(t) - z_e \in K$, $\forall t \geq 0$ and $\lim_{t\to\infty} |z(t) - z_e| = 0$ exponentially.

We set $Z = z - z_e = \begin{pmatrix} y - y_e \\ \theta - \theta_e \end{pmatrix}$. Then the problem (43) transforms into

$$\begin{cases} \dfrac{dZ}{dt}(t) + \mathcal{A}Z(t) + BZ(t) + \mathcal{A}_0 Z(t) + \mathcal{R}Z(t) = U(t), \quad t > 0 \\ Z(0) = z_0 - z_e, \end{cases}$$

where the operator $\mathcal{A}_0 \in L(V_B, V_B')$ is defined by relation (13), for z_e in (44).

The following global stability result takes place,

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be an open, bounded set, with a smooth boundary. Let $F_e \in H_B$, and let $z_e \in D(\mathcal{A})$ be a solution to* (44). *Let K satisfy the hypothesis (h_K). Assume that $z_0 - z_e \in K$ and that P_K verifies relation (h_P), page 291. Then, for $\sigma > 0$ large enough, the problem*

$$\begin{cases} \dfrac{dZ}{dt}(t) + (\mathcal{A} + B + \mathcal{A}_0 + \mathcal{R})Z(t) + \partial I_K(Z(t)) + \sigma Z(t) \ni 0, t > 0 \\ Z(0) = Z_0 = z_0 - z_e, \end{cases} \tag{45}$$

has at least a weak solution $Z \in L^2(0, T; V_B) \cap C_w([0, T]; H_B)$, with

$$\frac{dZ}{dt} \in \begin{cases} L^2(0, T; V_B'), & \text{if } d = 2 \\ L^{4/3}(0, T; V_B'), & \text{if } d = 3 \end{cases}, \text{ for all } T > 0.$$

In addition, $Z(t) \in K$, $t \geq 0$ and there exists $\delta = \delta(\sigma) > 0$ such that $\lim_{t\to\infty} |Z(t)|e^{\delta t} = 0$. If $d = 2$, the weak solution Z is unique.

Proof. Let, for the moment, $\sigma > 0$ be any fixed constant; the exact interval in which σ takes values in order to accomplish stability will be precised later. We apply Theorem 2.1 with $Z_0 = z_0 - z_e$, $F = 0$ and $\mathcal{G} \in L(H_B, H_B)$, $\mathcal{G} = \sigma I$. Then problem (45) has for any $T > 0$ at least one

weak solution Z from the demanded spaces, and $Z(t) \in K,\ t \in [0,T)$. In the bidimensional case the solution is unique.

For proving the exponential stability, we consider first the approximative problem

$$(46)\quad \begin{cases} \dfrac{dZ_N}{dt}(t) + (\mathcal{A} + B_N + \mathcal{A}_0 + \mathcal{R} + \sigma I)Z_N(t) + \partial I_K(Z_N(t)) \ni 0, t>0 \\ Z_N(0) = Z_0 = z_0 - z_e, \end{cases}$$

with $N \in \mathbb{N}^*$, which admits for any $T > 0$ a solution (see the proof of Theorem 2.1)

$$Z_N = \begin{pmatrix} Y_N \\ \Theta_N \end{pmatrix} \in L^2(0,T;V_B) \cap C([0,T];H_B),$$
$$\frac{dZ_N}{dt} \in \begin{cases} L^2(0,T;V_B'), & \text{if } d=2 \\ L^{4/3}(0,T;V_B'), & \text{if } d=3 \end{cases}.$$

Moreover, on a subsequence, renoted $(Z_N)_N$, we have $Z_N \rightharpoonup Z$ in $L^2(0,T;V_B)$, $Z_N \rightharpoonup Z$ weakly * in $L^\infty(0,T;H_B)$, $Z_N \to Z$ in $L^2(0,T;H_B)$, Z being a weak solution for problem (45).

Let us multiply scalarly equation (46) by $Z_N(t)$ and use (3), (4), (6) and the fact that ∂I_K is a monotone operator, with $0 \in \partial I_K(0)$ $(0 \in K)$. We obtain

$$\frac{1}{2}\frac{d}{dt}|Z_N(t)|^2 + \nu\|Y_N(t)\|^2 + k\|\Theta_N(t)\|^2 + \sigma|Z_N(t)|^2 \le \gamma(\Theta_N(t)e_d, Y_N(t))$$
$$- b(Y_N(t), y_e, Y_N(t)) - \bar{b}(Y_N(t), \theta_e, \Theta_N(t)) \le \frac{\gamma}{2}|Z_N(t)|^2 + C_b\|Y_N(t)\|$$
$$\cdot |Y_N(t)|\|y_e\|_a + C_b \begin{cases} |Y_N(t)|^{\frac{1}{2}}\|Y_N(t)\|^{\frac{1}{2}}|\Theta_N(t)|^{\frac{1}{2}}\|\Theta_N(t)\|^{\frac{1}{2}}\|\theta_e\|, & d=2 \\ \|Y_N(t)\||\Theta_N(t)|\|\theta_e\|_{3/2}, & d=3 \end{cases}$$
$$\le C\|Z_N(t)\||Z_N(t)|\|z_e\|_a + \frac{\gamma}{2}|Z_N(t)|^2, \text{ where } a = \begin{cases} 1, & d=2 \\ 3/2, & d=3. \end{cases}$$

This yields

$$\frac{1}{2}\frac{d}{dt}|Z_N(t)|^2 + m_{\nu k}\|Z_N(t)\|^2 + \sigma|Z_N(t)|^2 \le \frac{m_{\nu k}}{2}\|Z_N(t)\|^2$$
$$+ \left(\frac{C^2\|z_e\|_a^2}{2m_{\nu k}} + \frac{\gamma}{2}\right)|Z_N(t)|^2.$$

It results that

$$\frac{d}{dt}|Z_N(t)|^2 \leq -2\left(\sigma - \frac{C^2\|z_e\|_a^2}{2m_{\nu k}} - \frac{\gamma}{2}\right)|Z_N(t)|^2, \text{ a.e. } t > 0.$$

By consequence,

$$|Z_N(t)|^2 \leq e^{-2\delta_1 t}|Z_0|^2, \text{ a.e. } t \geq 0, \text{ for all } N \in \mathbb{N}^*,$$

where we take σ large enough, such that $\delta_1 = \sigma - \frac{C^2\|z_e\|_a^2}{2m_{\nu k}} - \frac{\gamma}{2} > 0$.

Passing to the limit with $N \to \infty$, it results that

$$|Z(t)|^2 \leq e^{-2\delta_1 t}|z_0 - z_e|^2, \text{ a.e. } t \geq 0.$$

Choosing $\delta \in (0, \delta_1)$, it follows that $\lim_{t\to\infty}|Z(t)|e^{\delta t} = 0$, which concludes the proof. □

3.1.1 Particular case: Stability on finite dimensional sets

Let $(f_i)_{i\in\mathbb{N}^*} \subset V$ be an orthonormate basis of H composed of eigenfunctions of the Stokes operator A. Let $(\lambda_i)_{i\in\mathbb{N}^*}$ be the corresponding eigenvalues. We have $Af_i = \lambda_i f_i,\ i \in \mathbb{N}^*$.

Let $(\vartheta_i)_{i\in\mathbb{N}^*} \subset H_0^1(\Omega)$ be an orthonormate basis of $L^2(\Omega)$ composed of eigenfunctions of the operator $A_1 = -\Delta$. Let $(\mu_i)_{i\in\mathbb{N}^*}$ be the corresponding eigenvalues; hence $A_1\vartheta_i = \mu_i\vartheta_i,\ i \in \mathbb{N}^*$.

Then $E_i^1 = \begin{pmatrix} f_i \\ 0 \end{pmatrix} \in V_B,\ i \in \mathbb{N}^*$ are eigenfunctions for the operator $\mathcal{A}$, corresponding to the eigenvalues $\nu\lambda_i$; also, $E_i^2 = \begin{pmatrix} 0 \\ \vartheta_i \end{pmatrix} \in V_B,\ i \in \mathbb{N}^*$ are eigenfunctions for the operator $\mathcal{A}$, corresponding to the eigenvalues $k\mu_i$. The set $\mathbf{B} = \left\{E_i^1, E_i^2;\ i \in \mathbb{N}^*\right\}$ is an orthonormate basis in H_B and $\nu\lambda_i,\ k\mu_i > 0$, with $i \in \mathbb{N}^*$ are all the eigenvalues of $\mathcal{A}$.

Let $E_1, E_2, \ldots, E_r \in \mathbf{B}$, with $r \in \mathbb{N}^*$ and let $K = \text{linspan}\,\{E_i;\ i = \overline{1,r}\} \subset H_B$. Then the properties (h_K), (h_P) are satisfied. Moreover, $\partial I_K(\cdot) = K^\perp$ in the elements of $K = \partial K$.

Consequently, the stabilization Theorem 3.1 holds. □

3.2. Finite dimensional, compactly supported stabilizing feedback controllers

Let

$$K = \text{linspan}\,\{E_i;\ i = \overline{1,r}\} \subset H_B, \tag{47}$$

where $E_1, E_2, \ldots, E_r \in \mathbf{B}$, with $r \in \mathbb{N}^*$ are eigenfunctions for the operator $\mathcal{A}$.

Rearrange (if necessary) the eigenfunctions of the operator $\mathcal{A}$ such that their sequence, $(E_i)_{i\in\mathbb{N}^*}$, has on the first r positions the generators of K. Then any element $W \in H_B$ may be written as

$$W = \sum_{i=1}^{\infty} W_i E_i,\ W_i = (W, E_i) \in \mathbb{R},\ \text{while } P_K W = \sum_{i=1}^{r} W_i E_i \in K.$$

Let $(\chi_i)_{i\in\mathbb{N}^*}$ be the eigenvalues corresponding to the eigenfunctions $(E_i)_{i\in\mathbb{N}^*}$ We have $\chi_i > 0$ and $\mathcal{A}E_i = \chi_i E_i,\ i \in \mathbb{N}^*$.

For the set K, given by (47), the hypotheses (h_K) and (h_P) are verified.

Consider the controlled system

$$\begin{cases} \dfrac{dz}{dt}(t) + \mathcal{A}z(t) + Bz(t) + \mathcal{R}z(t) = F_e + U(t), \quad t > 0 \\ z(0) = z_0, \end{cases} \tag{48}$$

where $z_0 \in V_B$, $F_e \in H_B$. Let $z_e \in D(\mathcal{A})$ be a stationary solution of (48), i.e. z_e satisfies (44). We intend to find a feedback controller U supported in a subdomain of Ω, such that $z(t) - z_e \in K,\ t \geq 0$ and $\lim_{t\to\infty} |z(t) - z_e| = 0$ exponentially.

Let $Z = z - z_e$. Then problem (48) transforms into

$$\begin{cases} \dfrac{dZ}{dt}(t) + \mathcal{A}Z(t) + BZ(t) + \mathcal{A}_0 Z(t) + \mathcal{R}Z(t) = U(t),\ t > 0 \\ Z(0) = z_0 - z_e, \end{cases}$$

where $\mathcal{A}_0 \in L(V_B, V_B')$ is defined by relation (13), for z_e in (44).

Consider the problem

$$\begin{cases} \dfrac{dZ}{dt}(t) + (\mathcal{A} + B + \mathcal{A}_0 + \mathcal{R})Z(t) + \partial I_K(Z(t)) \\ \qquad\qquad -\mathcal{P}m\mathcal{S}P_K Z(t) \ni 0, \qquad t > 0 \\ Z(0) = z_0 - z_e, \end{cases} \tag{49}$$

where the operator $\mathcal{P} : (L^2(\Omega))^{d+1} \to H_B$ is given by

$$\mathcal{P}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} Ph_1 \\ h_2 \end{pmatrix}, \text{for all } h_1 \in (L^2(\Omega))^d, h_2 \in L^2(\Omega),$$

with $P : (L^2(\Omega))^d \to H$ the Leray projector. Given $\omega \subset\subset \Omega$, with a smooth boundary, the application

$$m : (L^2(\omega))^{d+1} \to (L^2(\Omega))^{d+1},\ (mU)(x) = \begin{cases} U(x), & x \in \omega \\ 0, & x \in \Omega \setminus \omega. \end{cases}$$

extends the functions of the subdomain $(L^2(\omega))^{d+1}$ by 0 to the whole space $(L^2(\Omega))^{d+1}$. The operator $\mathcal{S} : K \to (L^2(\omega))^{d+1}$ is a linear continuous operator.

The following local stability result holds,

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be an open, bounded set, with a smooth boundary, and let an open subset $\omega \subset\subset \Omega$, with a smooth boundary. Let $F_e \in H_B$, $z_e \in D(\mathcal{A})$ satisfying (44) and K defined by (47). Then for all $\delta > 0$ there exist a linear continuous operator $\mathcal{S} : K \to (L^2(\omega))^{d+1}$ and a constant $\rho > 0$ such that if $z_0 - z_e \in K$, with $|z_0 - z_e| < \rho$, then the problem (49) has at least one weak solution $Z : [0, \infty) \to V_B$, having the property that for all $T > 0$,*

$$Z \in L^2(0, T; V_B) \cap C_w([0, T]; H_B) \text{ and } \frac{dZ}{dt} \in \begin{cases} L^2(0, T; V_B'), & \text{if } d = 2 \\ L^{4/3}(0, T; V_B'), & \text{if } d = 3 \end{cases};$$

in addition, $Z(t) \in K$, $t \geq 0$ and $\lim_{t\to\infty} |Z(t)|e^{\delta t} = 0$. If $d = 2$, the weak solution Z is unique.

Proof. Theorem 2.1 implies that, for any linear continuous operator $\mathcal{S} : K \to (L^2(\omega))^{d+1}$, the problem (49) has for all $T > 0$ at least a weak solution $Z = \begin{pmatrix} Y \\ \Theta \end{pmatrix}$ (because $\mathcal{G} = -Pm\mathcal{S}P_K$ is a composition of linear continuous operators, hence linear continuous); if $d = 2$, the solution is unique. The condition $Z(t) \in K$ is verified on the whole existence interval of Z.

In order to get an operator $\mathcal{S}$ which would ensure stability, let us start from the problem

$$(50)\quad \begin{cases} \dfrac{dZ}{dt}(t) + (\mathcal{A} + B + \mathcal{A}_0 + \mathcal{R})Z(t) + \partial I_K(Z(t)) - Pm\bar{U}(t) \ni 0, t > 0 \\ Z(0) = z_0 - z_e. \end{cases}$$

We multiply equation (50) scalarly with $E_i = \begin{pmatrix} E_{i1} \\ E_{i2} \end{pmatrix} \in H \times L^2(\Omega)$, $i = \overline{1,r}$ and recall that $\partial I_K(\cdot) = K^\perp$ in the elements of $K = \partial K$. Then,

$$\begin{aligned} &\frac{d}{dt}(Z(t), E_i) + (Z(t), \mathcal{A}E_i) + b\,(Y(t), Y(t), E_{i1}) + \bar{b}\,(Y(t), \Theta(t), E_{i2}) \\ &+ b\,(Y(t), y_e, E_{i1}) + b\,(y_e, Y(t), E_{i1}) + \bar{b}\,(Y(t), \theta_e, E_{i2}) \\ &+ \bar{b}\,(y_e, \Theta(t), E_{i2}) - (\gamma\Theta(t)e_d, E_{i1}) = (m\bar{U}(t), E_i). \end{aligned} \tag{51}$$

Denote $(Z(t), E_i) = Z_i(t),\ i = \overline{1,r}$. Then the components of $Z(t)$ are $Y(t) = \sum_{i=1}^r Z_i(t)E_{i1}$, $\Theta(t) = \sum_{i=1}^r Z_i(t)E_{i2}$ and we may rewrite (51) as

$$\begin{cases} Z_i'(t) + \chi_i Z_i(t) + \displaystyle\sum_{j,k=1}^{r} c_{ijk} Z_j(t) Z_k(t) + \sum_{j=1}^{r} g_{ij} Z_j(t) \\ \qquad = (m\bar{U}(t), E_i),\ t > 0,\ i = \overline{1,r} \\ Z_i(0) = Z_0^i = (z_0 - z_e, E_i), \end{cases} \tag{52}$$

where

$$\begin{aligned} c_{ijk} &= b\,(E_{j1}, E_{k1}, E_{i1}) + \bar{b}\,(E_{j1}, E_{k2}, E_{i2}) \in \mathbb{R},\ i,j,k = \overline{1,r}; \\ g_{ij} &= b\,(E_{j1}, y_e, E_{i1}) + b\,(y_e, E_{j1}, E_{i1}) + \bar{b}\,(E_{j1}, \theta_e, E_{i2}) \\ &\quad + \bar{b}\,(y_e, E_{j2}, E_{i2}) - \gamma(E_{j2}e_d, E_{i1}) \in \mathbb{R},\ i,j = \overline{1,r}. \end{aligned}$$

In (52), we have a system of r nonlinear ordinary differential equations with the unknowns $Z_i(t),\ i = \overline{1,r}$.

From here on the proof follows closely the proof of Theorem 3.3, [6]. Let us resketch it. We will identify the linear space K with $\mathbb{R}^r$ by means of the basis $(E_i)_{i=\overline{1,r}}$. We introduce the linear operators $\mathcal{L} : \mathbb{R}^r \to \mathbb{R}^r$ and $\mathcal{H} : (L^2(\omega))^{d+1} \to \mathbb{R}^r$ and the quadratic form $\mathcal{B} : \mathbb{R}^r \to \mathbb{R}^r$ by:

$$\mathcal{L}v = \left(\chi_i v_i + \sum_{j=1}^{r} g_{ij} v_j \right)_{i=\overline{1,r}},\quad \mathcal{B}v = \left(\sum_{j,k=1}^{r} c_{ijk} v_j v_k \right)_{i=\overline{1,r}},$$

for $v = (v_i)_{i=\overline{1,r}} \in \mathbb{R}^r$, while $\mathcal{H}U = \big((mU, E_i)\big)_{i=\overline{1,r}}$, for $U \in (L^2(\omega))^{d+1}$. Consequently, the *finite dimensional* differential system (52) writes in vectorial form as

$$\begin{cases} W'(t) + \mathcal{L}W(t) + \mathcal{B}W(t) = \mathcal{H}\bar{U}(t),\ t > 0 \\ W(0) = W_0, \end{cases} \tag{53}$$

where $W(t) = (Z_i(t))_{i=\overline{1,r}} \in \mathbb{R}^r$, $W_0 = (Z_0^i)_{i=\overline{1,r}} \in \mathbb{R}^r$.

We linearize the system (53) in 0,

$$\begin{cases} W'(t) + \mathcal{L}W(t) = \mathcal{H}\bar{U}(t),\ t > 0 \\ W(0) = W_0, \end{cases} \tag{54}$$

and consider the adjoint system,

$$p'(t) - \mathcal{L}^* p(t) = 0,\ t > 0, \tag{55}$$

where $\mathcal{L}^*$ is the adjoint of $\mathcal{L}$.

The finite dimensional linear system (54) is exactly controllable if and only if (see [12]) the following unique continuation property takes place: for the solution p of the adjoint system (55),

$$\mathcal{H}^* p(t) = 0,\ t > 0 \text{ implies } p(t) = 0,\ t > 0, \tag{56}$$

where $\mathcal{H}^*$ is the adjoint of $\mathcal{H}$. In other words, if $p = (p_i)_{i=\overline{1,r}}$, we have to prove that

$$\sum_{i=1}^{r} p_i(t) E_{i|_\omega} = 0 \text{ in } (L^2(\omega))^{d+1}, t > 0$$
$$\text{implies } p_1(t) = p_2(t) = \ldots = p_r(t) = 0,\ t > 0,$$

relation that results (see [3]) from the fact that $(E_i)_{i\in\mathbb{N}^*}$ is an orthonormate system in $(L^2(\Omega))^{d+1} \supset (L^2(\omega))^{d+1}$. Thus, the linearized system (54) is controllable, which amounts to say ([12]) that it is completely stabilizable. Hence for every $\xi \in \mathbb{R}$ (and let us fix a $\xi > 0$) there exist a linear continuous feedback $\mathcal{S} : \mathbb{R}^r \to (L^2(\omega))^{d+1}$ and a constant $C_1 > 0$ such that the operator norm

$$\|e^{-(\mathcal{L}-\mathcal{H}\mathcal{S})t}\| \leq C_1 e^{-\xi t},\ t \geq 0. \tag{57}$$

For the controller $\bar{U} = \mathcal{S}W$, the linearized system (54) is asimptotically stable. The feedback $\bar{U} = \mathcal{S}W$ also stabilizes the nonlinear system (53). Indeed, the solution of the system

$$\begin{cases} W'(t) + (\mathcal{L} - \mathcal{H}\mathcal{S})W(t) + \mathcal{B}W(t) = 0,\ t > 0 \\ W(0) = W_0, \end{cases} \tag{58}$$

is given by the variation of constants formula,

$$W(t) = e^{-(\mathcal{L}-\mathcal{HS})t}W_0 - \int_0^t e^{-(\mathcal{L}-\mathcal{HS})(t-s)}\mathcal{B}W(s)ds,\ t \geq 0.$$

We have to prove that W is defined on $[0,\infty)$ and that $\lim_{t\to\infty}|W(t)| = 0$ exponentially. There exists a constant $\gamma_0 > 0$ such that $|\mathcal{B}v| \leq \gamma_0|v|^2,\ \forall v \in \mathbb{R}^r$ ($\mathcal{B}$ is a quadratic form); by $|\cdot|$ we have here denoted the euclidian norm of $\mathbb{R}^r$. This implies, by means of (57), that

$$|W(t)| \leq C_1e^{-\xi t}|W_0| + C_1\gamma_0\int_0^t e^{-\xi(t-s)}|W(s)|^2ds;$$

multiplying by $e^{\xi t}$ and applying Gronwall's lemma, we get

$$|W(t)| \leq C_1|W_0|e^{-\xi t+C_1\gamma_0\int_0^t|W(s)|ds},\ t > 0. \tag{59}$$

Take $\rho_1 \in (0, \frac{\xi}{C_1\gamma_0})$ and W_0 with $|W_0| < \frac{\rho_1}{C_1}$. Assume, without loss of generality, that $\frac{\rho_1}{C_1} \leq \rho_1$ (which amounts to say that $C_1 \geq 1$). We prove that $|W(t)| < \rho_1,\ \forall t \geq 0$. Suppose that this is false. Because W is a continuous function, there exists a $T_1 > 0$ such that $|W(t)| < \rho_1$ for $t \in [0,T_1)$ and $|W(T_1)| = \rho_1$. From (59), we get that, for $t = T_1$, $|W(T_1)| \leq C_1|W_0|e^{(-\xi+C_1\gamma_0\rho_1)T_1} < \rho_1$, contradiction. Consequently, W is defined on $[0,\infty)$ and

$$|W(t)| < \rho_1,\ \forall t \in [0,\infty). \tag{60}$$

Relations (59) and (60) imply that

$$|W(t)| \leq C_1|W_0|e^{(-\xi+C_1\gamma_0\rho_1)t} = C_1|W_0|e^{-\delta_1 t},\ \forall t \geq 0,$$

where $\delta_1 = \xi - C_1\gamma_0\rho_1 > 0$. Then $\lim_{t\to\infty}|W(t)|e^{\delta t} = 0$ for any $\delta \in (0,\delta_1)$. But δ is arbitrary in $(0,\xi)$, if ρ_1 is small enough. Because ξ is also arbitrary, we obtain that δ is arbitrary in $(0,\infty)$.

Hence, for the feedback $\bar{U} = \mathcal{S}W$ the solution W of problem (53) is defined on $[0,\infty)$ and $\lim_{t\to\infty}|W(t)|e^{\delta t} = 0$. From $Z(t) \in K,\ t \geq 0$ it results that for the controller $\bar{U} = \mathcal{S}P_KZ$, the solution Z of problem (50) is, also, defined on $[0,\infty)$ and $\lim_{t\to\infty}|Z(t)|e^{\delta t} = 0$. □

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