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REFINEMENT OF HARDY'S INEQUALITIES INVOLVING MANY FUNCTIONS VIA SUPERQUADRATIC FUNCTIONS

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Abstract. Some new refined Hardy type integral inequalities involving n functions ($n \in \mathbb{Z}_+$) via superquadratic functions are established for $p \geq 2$ and their dual inequalities are also derived. In particular, the results obtained complement and improve some recent results of Oguntuase and Persson.

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1. Introduction

In 1920, Hardy [3] announced and proved in [4] the following integral inequality (see also [5, Theorem 330, p. 245])

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \quad p > 1, \tag{1.1}$$

where f is a nonnegative measurable function and the constant $(\frac{p}{p-1})^p$ is the best possible. The prehistory of (1.1) up to the time when Hardy finally proved (1.1) in 1925 can be found in [5], [6] and [7]. Concerning the history and development of inequality (1.1) we refer interested reader to the books [7], [8] and [10] devoted to this subject and also the recent historical article [6] and the references given therein.

A well-known simple fact is that (1.1) can equivalently (via the substitution $f(x) = h(x^{(1-\frac{1}{p})}x^{-\frac{1}{p}})$, be rewritten in the form

$$\int_0^\infty \left(\frac{1}{x}\int_0^x h(t)dt\right)^p \frac{dx}{x} \leq \int_0^\infty h^p(x)\frac{dx}{x} \tag{1.2}$$

and in this form it even holds with equality when $p = 1$. In this form we see that Hardy's inequality is a simple consequence of Jensen's inequality but this was not discovered in the dramatic period when Hardy discovered and finally proved this inequality in [4].

In a recent paper, Oguntuase and Persson [9] using mainly the notion of superquadratic and subquadratic functions proved for $p \geq 2$ the refined form of Hardy's inequality (1.1), namely

$$\begin{aligned}
&\int_0^b x^{-m}\left(\int_0^x f(t)dt\right)^p dx \\
&+ \frac{m-1}{p}\int_0^b \int_t^b \left|\frac{p}{m-1}\left(\frac{t}{x}\right)^{1-\frac{m-1}{p}} f(t) - \frac{1}{x}\int_0^x f(t)dt\right|^p \\
&\times x^{p-m-\frac{m-1}{p}} dx t^{\frac{m-1}{p}-1} dt \\
&\leq \left(\frac{p}{m-1}\right)^p \int_0^b \left[1-\left(\frac{x}{b}\right)^{\frac{m-1}{p}}\right] x^{p-m} f^p(x)dx
\end{aligned} \tag{1.3}$$

for $m > 1$, $0 < b \leq \infty$, and

$$\begin{aligned}
&\int_b^\infty x^{-m}\left(\int_x^\infty f(t)dt\right)^p dx \\
&+ \frac{1-m}{p}\int_b^\infty \int_b^t \left|\frac{p}{1-m}\left(\frac{t}{x}\right)^{\frac{1-m}{p}+1} f(t) - \frac{1}{x}\int_x^\infty f(t)dt\right|^p \\
&\times x^{\frac{1-m}{p}+p-m} dx t^{\frac{m-1}{p}-1} dt \\
&\leq \left(\frac{p}{1-m}\right)^p \int_b^\infty \left[1-\left(\frac{b}{x}\right)^{\frac{1-m}{p}}\right] x^{p-m} f^p(x)dx
\end{aligned} \tag{1.4}$$

for $m < 1$, $0 \leq b < \infty$.

Furthermore, it was shown in [9] that (1.3)-(1.4) hold in the reversed direction if $1 < p \leq 2$. In particular, this means that for $p = 2$ we have equality in (1.3)-(1.4), respectively.

In this paper we generalize and unify some of these results by proving some new refined Hardy type integral inequalities involving n functions ($n \in \mathbb{Z}_+$) via superquadratic functions. Moreover the corresponding dual inequalities are also derived. In particular, our results further complement and improve some recent results in [9].

Our main tool in the proofs is to use the notion of superquadratic functions introduced by ABRAMOVICH, JAMESON and SINNAMON in [1] (see also [2]).

Definition 1.1 ([1, Definition 2.1])**.** A function $\varphi : [0,\infty) \to \mathbb{R}$ is superquadratic provided that for all $x \geq 0$ there exists a constant $C_x \in \mathbb{R}$ such that

$$\varphi(y) - \varphi(x) - \varphi(|y-x|) \geq C_x (y-x)$$

for all $y \geq 0$. We say that f is subquadratic if $-f$ is superquadratic.

The paper is organized as follows: In Section 2 we present some preliminaries, including some inequalities involving superquadratic and subquadratic functions involving n functions ($n \in \mathbb{Z}_+$) of independent interest. In Section 3 we present some new refined Hardy type inequalities involving n functions and their proofs. Our final Section 4 is devoted to some concluding remarks and examples.

Notations. Throughout this paper, all functions are assumed to be measurable and expressions of the form $0 \cdot \infty$, $\frac{\infty}{\infty}$, and $\frac{0}{0}$ are taken to be equal to zero. In addition, by a weight function u we mean a nonnegative measurable function on the actual interval.

2. Preliminaries

First, we state the following refinement of Jensen's inequality in [1], which is very useful in the proofs of our results.

Lemma 2.1 ([1, Theorem 2.3])**.** *Let (Ω, μ) be a probability measure space. The inequality*

$$\varphi\left(\int_\Omega f(s)d\mu(s)\right) \leq \int_\Omega \varphi(f(s))d\mu(s) - \int_\Omega \varphi\left(\left|f(s) - \int_\Omega f(s)d\mu(s)\right|\right)d\mu(s) \tag{2.1}$$

holds for all probability measures μ and all nonnegative $\mu-$integrable functions f if and only if φ is superquadratic. Moreover, (2.1) *holds in the reversed direction if and only if φ is subquadratic.*

Proposition 2.1. *Let $f_1,\ f_2, ..., f_n$ be nonnegative integrable functions and define*

$$F_k(x) := \frac{1}{x}\int_0^x f_k(t)\,dt, \quad k = 1, 2, ..., n. \tag{2.2}$$

Suppose that $0 < b \leq \infty$, let $u : (0,\infty) \to \mathbb{R}$ be a nonnegative weight function such that the function $x \to \frac{u(x)}{x^2}$ is locally integrable on $(0,\infty)$, and define the weight function v by

$$v(t) = t\int_t^b \frac{u(x)}{x^2}dx, \quad t \in (0,b).$$

If the real-valued function φ is superquadratic and nondecreasing on (a,c), $0 \leq a < c \leq \infty$, then

$$\begin{aligned}
&\int_0^b u(x)\varphi\left(\left[\prod_{k=1}^n F_k(x)\right]^{\frac{1}{n}}\right)\frac{dx}{x} \\
&+\int_0^b\int_t^b \varphi\left(\left|\frac{1}{n}\sum_{k=1}^n f_k(t) - \frac{1}{x}\int_0^x \frac{1}{n}\sum_{k=1}^n f_k(t)dt\right|\right)\frac{u(x)}{x^2}dxdt \\
&\leq \int_0^b v(x)\varphi\left(\frac{1}{n}\sum_{k=1}^n f_k(x)\right)\frac{dx}{x}
\end{aligned} \tag{2.3}$$

holds for all f_k with $a < f_k(x) < c,\ 0 < x \leq b,\ k \in \{1, 2, ..., n\}$.

Proof. First we apply the Arithmetic-Geometric Mean inequality

$$\left(\prod_{k=1}^n F_k(x)\right)^{\frac{1}{n}} \leq \frac{1}{n}\sum_{k=1}^n F_k(x), \tag{2.4}$$

to the first term on the left hand side of inequality (2.3) and thereafter, by using Lemma 2.1 and Fubini's theorem we have that

$$\int_0^b u(x)\varphi\left(\left[\prod_{k=1}^n F_k(x)\right]^{\frac{1}{n}}\right)\frac{dx}{x} \leq \int_0^b u(x)\varphi\left(\frac{1}{n}\sum_{k=1}^n F_k(x)\right)\frac{dx}{x}$$

$$
\begin{aligned}
&= \int_0^b u(x)\varphi\left(\frac{1}{x}\int_0^x \left(\frac{1}{n}\sum_{k=1}^n f_k(t)\right)dt\right)\frac{dx}{x} \\
&\le \int_0^b \frac{u(x)}{x^2}\int_0^x \varphi\left(\frac{1}{n}\sum_{k=1}^n f_k(t)\right)dtdx \\
&- \int_0^b \frac{u(x)}{x^2}\int_0^x \varphi\left(\left|\frac{1}{n}\sum_{k=1}^n f_k(t) - \frac{1}{x}\int_0^x \frac{1}{n}\sum_{k=1}^n f_k(t)dt\right|\right)dtdx \\
&= \int_0^b \varphi\left(\frac{1}{n}\sum_{k=1}^n f_k(t)\right)\int_t^b \frac{u(x)}{x^2}dxdt \\
&- \int_0^b\int_t^b \varphi\left(\left|\frac{1}{n}\sum_{k=1}^n f_k(t) - \frac{1}{x}\int_0^x \frac{1}{n}\sum_{k=1}^n f_k(t)dt\right|\right)\frac{u(x)}{x^2}dxdt \\
&= \int_0^b v(t)\varphi\left(\frac{1}{n}\sum_{k=1}^n f_k(t)\right)\frac{dt}{t} \\
&- \int_0^b\int_t^b \varphi\left(\left|\frac{1}{n}\sum_{k=1}^n f_k(t) - \frac{1}{x}\int_0^x \frac{1}{n}\sum_{k=1}^n f_k(t)dt\right|\right)\frac{u(x)}{x^2}dxdt.
\end{aligned}
$$

from which (2.3) follows and the proof is complete. □

Remark 2.1. For the case $n = 1$ Proposition 2.1 coincides with Proposition 2.1 (a) in [9].

Proposition 2.2. *Let $0 \le b < \infty$, $u : (b,\infty) \to \mathbb{R}$ be a nonnegative locally integrable function on (b,∞), and define the function v by*

$$
v(t) = \frac{1}{t}\int_b^t u(x)dx, \quad t \in (b,\infty).
$$

If the real-valued function φ is superquadratic and nondecreasing on (a,c), $0 \le a < c \le \infty$, then the inequality

$$
\int_b^\infty u(x)\varphi\left(\left[\prod_{k=1}^n G_k(x)\right]^{\frac{1}{n}}\right)\frac{dx}{x}
$$

$$+\int_b^\infty \int_b^t \varphi\left(\left|\frac{1}{n}\sum_{k=1}^n f_k(t) - x\int_x^\infty \frac{1}{n}\sum_{k=1}^n f_k(t)\frac{dt}{t^2}\right|\right) u(x)dx\frac{dt}{t^2} \tag{2.5}$$
$$\leq \int_b^\infty v(x)\varphi\left(\frac{1}{n}\sum_{k=1}^n f_k(x)\right)\frac{dx}{x}$$

holds for all f_k *with* $a < f_k(x) < c,\ 0 < x \leq b,\ k \in \{1, 2, ..., n\}$ *., where*

$$G_k(x) := x\int_x^\infty f_k(t)\frac{dt}{t^2}.$$

Proof. The proof is similar to that of Proposition 2.1 so we omit the details. □

Remark 2.2. For the case $n = 1$ Proposition 2.2 coincides with Proposition 2.2 (a) in [9].

3. Refinement of Hardy-type inequalities involving many functions

Our first result reads:

Theorem 3.1. *Let* $p > 1,\ m > 1,\ 0 < b \leq \infty,\ n \in \mathbb{Z}_+$ *and let the functions* $f_k,\ k \in \{1, 2, ..., n\}$, *be nonnegative and locally integrable on* $(0, b)$ *such that*

$$0 < \int_0^b x^{p-m}\left(\sum_{k=1}^n f_k(x)\right)^p dx < \infty.$$

(i) *If* $p \geq 2$, *then*

$$\int_0^b x^{-m}\left(\prod_{k=1}^n \int_0^x f_k(t)dt\right)^{\frac{p}{n}} dx$$
$$+\frac{m-1}{p}\int_0^b \int_t^b \left|\frac{p}{m-1}\sum_{k=1}^n \frac{f_k(t)}{n}\left(\frac{t}{x}\right)^{1-\frac{m-1}{p}}\right. \tag{3.1}$$
$$\left.-\frac{1}{x}\int_0^x \sum_{k=1}^n \frac{f_k(t)}{n}dt\right|^p x^{p-m-\frac{m-1}{p}} dx t^{\frac{m-1}{p}-1}dt$$
$$\leq \left(\frac{p}{nm-n}\right)^p \int_0^b \left[1 - \left(\frac{x}{b}\right)^{\frac{m-1}{p}}\right] x^{p-m}\left(\sum_{k=1}^n f_k(x)\right)^p dx.$$

(ii) *If* $1 < p \leq 2$, *then inequality* (3.1) *holds in the reversed direction.*

Proof. This follows directly from the proof of Theorem 3.1 in [9] by using the substitution $f(x) = \sum_{k=1}^{n} \frac{f_k(x)}{n}$ and the arithmetic-geometric inequality (2.3). $\square$

Remark 3.1. In the special case $n = 1$ Theorem 3.1 coincides with Theorem 3.1 in [9]. For the case $p = 2$ and $n = 1$ we even have equality in (3.1).

Next, we formulate the dual version of Theorem 3.1.

Theorem 3.2. *Let $p > 1$, $m < 1$, $0 \leq b < \infty$, $n \in \mathbb{Z}_+$ and let the functions f_k, $k \in \{1, 2, ..., n\}$, be nonnegative and locally integrable on (b, ∞) such that $0 < \int_b^\infty x^{p-m} \left(\sum_{k=1}^{n} f_k(x)\right)^p dx < \infty$.*

(iii) If $p \geq 2$, then

$$\begin{aligned}
&\int_b^\infty x^{-m} \left(\prod_{k=1}^{n} \int_x^\infty f_k(t)dt \right)^{\frac{p}{n}} dx \\
&+ \frac{1-m}{p} \int_b^\infty \int_b^t \left| \frac{p}{1-m} \sum_{k=1}^{n} \frac{f_k(t)}{n} \left(\frac{t}{x}\right)^{\frac{1-m}{p}+1} \right. \\
&\left. -\frac{1}{x} \int_x^\infty \sum_{k=1}^{n} \frac{f_k(t)}{n} dt \right|^p \times x^{\frac{1-m}{p}+p-m} dx t^{\frac{m-1}{p}-1} dt \qquad (3.2) \\
&\leq \left(\frac{p}{n-nm}\right)^p \int_b^\infty \left[1 - \left(\frac{b}{x}\right)^{\frac{1-m}{p}}\right] x^{p-m} \left(\sum_{k=1}^{n} f_k(x)\right)^p dx.
\end{aligned}$$

(iv) If $1 < p \leq 2$, then inequality (3.2) holds in the reversed direction.

Proof. This follows directly from the proof of Theorem 3.2 in [9] by using the substitution $f(x) = \sum_{k=1}^{n} \frac{f_k(x)}{n}$ and the arithmetic-geometric inequality (2.3). $\square$

Remark 3.2. If we set $n = 1$, then Theorem 3.2 reduces to inequality Theorem 3.2 in [9]. For the case $p = 2$ and $n = 1$ we even have equality in (3.2).

4. Concluding remarks and examples

Next, we obtain the following generalizations of our results in Theorems 3.1 and 3.2 respectively.

Theorem 4.1. *Let $p>1, m>1, 0<b\leq\infty$, and $n\in\mathbb{Z}_+$. Let $\{\alpha_k\}_{k=1}^n$ be a positive sequence such that $\sum_{k=1}^n\alpha_k=1$ and $\{f_k\}_{k=1}^n$ be a sequence of integrable functions on $(0,b)$ such that*

$$0<\int_0^b x^{p-m}\left(\sum_{k=1}^n f_k(x)\right)^p dx<\infty.$$

(v) If $p\geq 2$, then

$$\begin{aligned}
&\int_0^b x^{(1-m)n\alpha_k-1}\left(\prod_{k=1}^n\int_0^x f_k(t)dt\right)^{p\alpha_k}dx\\
&+\left(\left|\frac{m-1}{p}\right|\right)^{p+1-np\alpha_k}\int_0^b\int_t^b\left|\frac{p}{m-1}\sum_{k=1}^n\frac{f_k(t)}{n}\left(\frac{t}{x}\right)^{1-\frac{m-1}{p}}\right.\\
&\left.-\frac{1}{x}\int_0^x\sum_{k=1}^n\frac{f_k(t)}{n}dt\right|^p x^{p-m-\frac{m-1}{p}}dxt^{\frac{m-1}{p}-1}dt\\
&\leq\alpha_k^p\left(\left|\frac{p}{m-1}\right|\right)^{np\alpha_k}\int_0^b\left[1-\left(\frac{x}{b}\right)^{\frac{m-1}{p}}\right]x^{p-m}\left(\sum_{k=1}^n f_k(x)\right)^p dx.
\end{aligned}\tag{4.1}$$

(vi) If $1<p\leq 2$, then inequality (4.1) holds in the reversed direction.

Proof. Let $p\geq 2$. Then, by the more general Arithmetic-Geometric Mean inequality

$$\prod_{k=1}^n g_k^{\alpha_k}(x)\leq\sum_{k=1}^n\alpha_k g_k(x),$$

we have that

$$\left(\prod_{k=1}^n F_k^{\alpha_k}(x)\right)^p\leq\left(\sum_{k=1}^n\alpha_k F_k(x)\right)^p=\left(\frac{1}{x}\int_0^x\sum_{k=1}^n\alpha_k f_k(t)\,dt\right)^p.\tag{4.2}$$

The rest of the proof is just formal modification of the proof of Theorem 3.1 by using inequality (4.2) together with the substitution $f(x)=\sum_{k=1}^n\alpha_k f_k(x)$. So we omit the details. □

Remark 4.1. If $\alpha_k=\frac{1}{k}$, $k\in\{1,2,...,n\}$, and $m>1$, then Theorem 4.1 reduces to Theorem 3.1.

Theorem 4.2. *Let $p > 1, m < 1, 0 \leq b < \infty$, and $n \in \mathbb{Z}_+$. Let $\{\alpha_k\}_{k=1}^n$ be a positive sequence such that $\sum_{k=1}^n \alpha_k = 1$ and $\{f_k\}_{k=1}^n$ be a sequence of integrable functions on (b,∞) such that*

$$0 < \int_b^\infty x^{p-m}\left(\sum_{k=1}^n f_k(x)\right)^p dx < \infty.$$

(vii) If $p \geq 2$, then

$$\begin{aligned}
&\int_b^\infty x^{(1-m)n\alpha_k - 1}\left(\prod_{k=1}^n \int_x^\infty f_k(t)dt\right)^{p\alpha_k} dx \\
&+ \left(\frac{1-m}{p}\right)^{p+1-np\alpha_k} \int_b^\infty \int_b^t \left| \frac{p}{1-m}\sum_{k=1}^n \frac{f_k(t)}{n}\left(\frac{t}{x}\right)^{\frac{1-m}{p}+1} \right. \\
&\left. - \frac{1}{x}\int_x^\infty \sum_{k=1}^n \frac{f_k(t)}{n} dt \right|^p x^{\frac{1-m}{p}+p-m} dx t^{\frac{m-1}{p}-1} dt \qquad (4.3) \\
&\leq \alpha_k^p \left(\frac{p}{1-m}\right)^{np\alpha_k} \int_b^\infty \left[1 - \left(\frac{x}{b}\right)^{\frac{1-m}{p}}\right] x^{p-m}\left(\sum_{k=1}^n f_k(x)\right)^p dx.
\end{aligned}$$

(viii) If $1 < p \leq 2$, then inequality (4.3) holds in the reversed direction.

Proof. The proof of Theorem 4.2 is similar to the proof of Theorems 4.1 and so we omit the details. □

Remark 4.2. If $\alpha_k = \frac{1}{k}$, $k \in \{1, 2, ..., n\}$, and $m > 1$, then Theorem 4.2 reduces to Theorem 3.2.

Example 4.1. In Proposition 2.1, by putting $u(x) \equiv 1$ we obtain the weight function v to be equal to

$$v(x) = \begin{cases} 1 - \dfrac{x}{b}, & b < \infty \\ 1, & b = \infty \end{cases}.$$

Hence for $b < \infty$, inequality (2.3) takes the form

$$\int_0^b \varphi\left(\left[\prod_{k=1}^n F_k(x)\right]^{\frac{1}{n}}\right)\frac{dx}{x}$$

$$+\int_0^b\int_t^b \varphi\left(\left|\frac{1}{n}\sum_{k=1}^n f_k(t)-\frac{1}{x}\int_0^x \frac{1}{n}\sum_{k=1}^n f_k(t)dt\right|\right)\frac{dx}{x^2}dt$$
$$\leq \int_0^b \left(1-\frac{x}{b}\right)\varphi\left(\frac{1}{n}\sum_{k=1}^n f_k(t)\right)\frac{dx}{x}$$

and

$$\int_0^\infty \varphi\left(\left[\prod_{k=1}^n F_k(x)\right]^{\frac{1}{n}}\right)\frac{dx}{x}$$
$$+\int_0^\infty\int_t^\infty \varphi\left(\left|\frac{1}{n}\sum_{k=1}^n f_k(t)-\frac{1}{x}\int_0^x \frac{1}{n}\sum_{k=1}^n f_k(t)dt\right|\right)\frac{dx}{x^2}dt$$
$$\leq \int_0^\infty \varphi\left(\frac{1}{n}\sum_{k=1}^n f_k(t)\right)\frac{dx}{x}$$

when $b=\infty$.

Example 4.2. Also by putting $u(x)\equiv 1$ in Proposition 2.2 yields

$$v(x)=\begin{cases}1-\dfrac{b}{x}, & b>0\\ 1, & b=0\end{cases}$$

and so for the case $b>0$, inequality (2.5) becomes

$$\int_b^\infty \varphi\left(\left[\prod_{k=1}^n G_k(x)\right]^{\frac{1}{n}}\right)\frac{dx}{x}$$
$$+\int_b^\infty\int_b^t \varphi\left(\left|\frac{1}{n}\sum_{k=1}^n f_k(t)-x\int_x^\infty \frac{1}{n}\sum_{k=1}^n f_k(t)\frac{dt}{t^2}\right|\right)dx\frac{dt}{t^2}$$
$$\leq \int_b^\infty \left(1-\frac{b}{x}\right)\varphi\left(\frac{1}{n}\sum_{k=1}^n f_k(x)\right)\frac{dx}{x}$$

while for $b=0$ it reads

$$\int_0^\infty \varphi\left(\left[\prod_{k=1}^n G_k(x)\right]^{\frac{1}{n}}\right)\frac{dx}{x}$$

$$+\int_0^\infty \int_0^t \varphi\left(\left|\frac{1}{n}\sum_{k=1}^n f_k(t) - x\int_x^\infty \frac{1}{n}\sum_{k=1}^n f_k(t)\frac{dt}{t^2}\right|\right) dx\frac{dt}{t^2}$$
$$\leq \int_0^\infty \varphi\left(\frac{1}{n}\sum_{k=1}^n f_k(x)\right)\frac{dx}{x}.$$

Remark 4.3. For the case $n = 1$ Examples 4.1 and 4.2 coincide with Examples 4.1 and 4.2 in [9].

By using Theorem 3.1 with $m = p$ we obtain:

Example 4.3. Let $0 < b \leq \infty$ and let the function f_k, $k \in \{1, 2, ..., n\}$ be locally integrable on $(0, b)$ such that $0 < \int_0^b x^{p-m}\left(\sum_{k=1}^n f_k(x)\right)^p dx < \infty$. If $p \geq 2$, then

$$\int_0^b \left(\frac{1}{x^n}\prod_{k=1}^n \int_0^x f_k(t)dt\right)^{\frac{p}{n}} dx$$
$$+\frac{p-1}{p}\int_0^b\int_t^b \left|\frac{p}{p-1}\frac{1}{n}\sum_{k=1}^n f_k(t)\left(\frac{t}{x}\right)^{\frac{1}{p}}\right. \tag{4.4}$$
$$\left. -\frac{1}{x}\int_0^x \frac{1}{n}\sum_{k=1}^n f_k(t)dt\right|^p x^{-\frac{p-1}{p}} dx t^{\frac{-1}{p}} dt$$
$$\leq \left(\frac{p}{np-n}\right)^p \int_0^b \left[1 - \left(\frac{x}{b}\right)^{\frac{p-1}{p}}\right]\left(\sum_{k=1}^n f_k(x)\right)^p dx.$$

Remark 4.4. For $p = 2$ and $b = \infty$ we obtain the following identity for all $f_k \in L_2(0, \infty)$, $k \in \{1, 2, ..., n\}$:

$$\int_0^\infty \left(\frac{1}{x^n}\prod_{k=1}^n \int_0^x f_k(t)dt\right)^{\frac{2}{n}} dx$$
$$+\frac{1}{2}\int_0^\infty\int_t^b \left(2\sqrt{\frac{t}{x}}\frac{1}{n}\sum_{k=1}^n f_k(t) - \frac{1}{x}\int_0^x \frac{1}{n}\sum_{k=1}^n f_k(t)dt\right)^2 \frac{dx}{\sqrt{x}}\frac{dt}{\sqrt{t}} \tag{4.5}$$
$$= \left(\frac{2}{n}\right)^2 \int_0^\infty \left(\sum_{k=1}^n f_k(x)\right)^2 dx.$$

In particular, if $n = 1$ inequality (4.5) coincides with Remark 4.2 in [9].

Remark 4.5. For the case $p = n$ in Theorem 3.1, inequality (3.1) reads:

$$\begin{aligned}
&\int_0^b x^{-m}\left(\prod_{k=1}^n \int_0^x f_k(t)dt\right)dx \\
&+\frac{m-1}{n}\int_0^b\int_t^b\left|\frac{1}{m-1}\sum_{k=1}^n f_k(t)\left(\frac{t}{x}\right)^{1-\frac{m-1}{n}}\right. \\
&\left.-\frac{1}{x}\int_0^x \frac{1}{n}\sum_{k=1}^n f_k(t)dt\right|^n \times x^{n-m-\frac{m-1}{n}}dxt^{\frac{m-1}{n}-1}dt \\
&\le \left(\frac{1}{m-1}\right)^n \int_0^b \left[1-\left(\frac{x}{b}\right)^{\frac{m-1}{n}}\right] x^{n-m}\left(\sum_{k=1}^n f_k(x)\right)^n dx,
\end{aligned} \tag{4.6}$$

$m > 1$, $n = 2, 3, 4, ...$

Example 4.4. For the case $n = 1$ and $b = 0$ in Theorem 3.2 we obtain that

$$\begin{aligned}
&\int_0^\infty x^{-m}\left(\int_x^\infty f(t)dt\right)^p dx + \frac{1-m}{p}\int_0^\infty\int_0^t\left|\frac{p}{1-m}f(t)\left(\frac{t}{x}\right)^{\frac{1-m}{p}+1}\right. \\
&\left.-\frac{1}{x}\int_x^\infty f(t)dt\right|^p x^{\frac{1-m}{p}+p-m}dxt^{\frac{m-1}{p}-1}dt \\
&\le \left(\frac{p}{1-m}\right)^p \int_0^\infty x^{p-m}f^p(x)dx.
\end{aligned} \tag{4.7}$$

Remark 4.6. For the case $n = p$ in Theorem 3.2 the inequality (3.2) reads:

$$\begin{aligned}
&\int_b^\infty x^{-m}\left(\prod_{k=1}^n \int_x^\infty f_k(t)dt\right)dx \\
&+\frac{1-m}{n}\int_b^\infty\int_b^t\left|\frac{1}{1-m}\sum_{k=1}^n f_k(t)\left(\frac{t}{x}\right)^{\frac{1-m}{n}+1}-\frac{1}{x}\int_x^\infty \frac{1}{n}\sum_{k=1}^n f_k(t)dt\right|^n \\
&\times x^{\frac{1-m}{n}+n-m}dxt^{\frac{m-1}{n}-1}dt \\
&\le \left(\frac{1}{1-m}\right)^n \int_b^\infty \left[1-\left(\frac{b}{x}\right)^{\frac{1-m}{n}}\right] x^{n-m}\left(\sum_{k=1}^n f_k(x)\right)^n dx,
\end{aligned} \tag{4.8}$$

$m < 1$, $n = 2, 3, 4, ...$

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