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ON η-EINSTEIN TRANS-SASAKIAN MANIFOLDS

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Abstract. A systematic study of η-Einstein trans-Sasakian manifold is performed. We find eight necessary and sufficient conditions for the structure vector field ξ of a trans-Sasakian manifold to be an eigenvector field of the Ricci operator. We show that for a 3-dimensional almost contact metric manifold $(M, \varphi, \xi, \eta, g)$, the conditions of being normal, trans-K-contact, trans-Sasakian are all equivalent to $\nabla\xi \circ \varphi = \varphi \circ \nabla\xi$. In particular, the conditions of being quasi-Sasakian, normal with $0 = 2\beta = \mathrm{div}\xi$, trans-$K$-contact of type $(\alpha, 0)$, trans-Sasakian of type $(\alpha, 0)$, and $\mathcal{C}_6$-class are all equivalent to $\nabla\xi = -\alpha\varphi$, where $2\alpha = \mathrm{Trace}(\varphi\nabla\xi)$. In last, we give fifteen necessary and sufficient conditions for a 3-dimensional trans-Sasakian manifold to be η-Einstein.

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1. Introduction

In [54], Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension. For such a manifold, the sectional curvature of plane sections containing ξ is a constant, say c. He showed that they can be divided into three classes: (1) homogeneous normal contact Riemannian manifolds with $c > 0$, (2) global Riemannian products of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature if $c = 0$ and (3) a warped product space $R \times_f C^n$ if $c < 0$. It is known that the manifolds of class (1) are characterized by admitting a Sasakian structure, while the manifolds of class (2) are characterized by admitting a cosymplectic structure. Kenmotsu

[30] characterized the differential geometric properties of the manifolds of class (3); the structure so obtained is now known as Kenmotsu structure. In general, these structures are not Sasakian [30]. In the Gray-Hervella classification of almost Hermitian manifolds [25], there appears a class, W_4, of Hermitian manifolds which are closely related to locally conformal Kaehler manifolds [61, 24]. An almost contact metric structure on a manifold M is called a trans-Sasakian structure [46] if the product manifold $M \times R$ belongs to the class W_4. The class $C_6 \oplus C_5$ [34] coincides with the class of trans-Sasakian structures of type (α, β). We note that trans-Sasakian structures include cosymplectic [5], α-Sasakian [28], Sasakian, β-Kenmotsu [28], Kenmotsu and normal locally conformal almost cosymplectic [35] or f-Kenmotsu [43] structures. Moreover, these structures provide a large class of generalized quasi-Sasakian structures also [59, Proposition 2].

On the other hand, on the basis of the studies done in [1], [7], [15], [16], [17], [21], [23] etc., there is a strong feeling that the concepts of pseudosymmetry, quasi Einstein and η-Einstein are closely related on 3-dimensional contact, Sasakian, Kenmotsu and trans-Sasakian manifolds. Keeping this fact in mind, we present a systematic study of η-Einstein trans-Sasakian manifolds. Section 2 contains a brief review of trans-Sasakian manifolds and its particular cases. In this section we also find eight necessary and sufficient conditions for the structure vector field to be an eigenvector field of the Ricci operator (cf. Lemma 2.2). In Section 3, among others we show that for a 3-dimensional almost contact metric manifold $(M, \varphi, \xi, \eta, g)$, the conditions of being normal, trans-K-contact, trans-Sasakian are all equivalent to $\nabla\xi \circ \varphi = \varphi \circ \nabla\xi$ (cf. Theorem 3.5). In particular, the conditions of being quasi-Sasakian, normal with $0 = 2\beta = \text{div}\xi$, trans-$K$-contact of type $(\alpha, 0)$, trans-Sasakian of type $(\alpha, 0)$, and $\mathcal{C}_6$-class are all equivalent to $\nabla\xi = -\alpha\varphi$, where $2\alpha = \text{Trace}(\varphi\nabla\xi)$ (cf. Theorem 3.9). In section 4, we study η-Einstein trans-Sasakian manifolds using concepts of quasi-Einstein and pseudo symmetric manifolds. We present fifteen necessary and sufficient conditions for a 3-dimensional trans-Sasakian manifold to be η-Einstein (cf. Theorem 4.19).

2. Trans-Sasakian manifolds

Let M be a differentiable manifold of odd dimension $2n + 1$ $(n \geq 1)$. A 1-form η on M is called a *contact form* if $\eta \wedge (d\eta)^n \neq 0$ everywhere on M, and M equipped with a contact form η is a *contact manifold*. Since rank of

$d\eta$ is $2n$ on the Grassmann algebra $\bigwedge T_p^*M$ at each point $p \in M$, therefore there exists a unique global vector field ξ, called the *characteristic vector field*, such that $\eta(\xi) = 1$, $d\eta(\xi, \cdot) = 0$, and consequently $\pounds_\xi \eta = 0$, $\pounds_\xi d\eta = 0$, where $\pounds_\xi$ denotes the Lie differentiation by ξ. In 1953, Chern [14] proved that the structural group of a $(2n+1)$-dimensional contact manifold can be reduced to $\mathcal{U}(n) \times 1$. A $(2n+1)$-dimensional differentiable manifold M is called an *almost contact manifold* [26] if its structural group can be reduced to $\mathcal{U}(n) \times 1$. Equivalently, there is an *almost contact structure* (φ, ξ, η) [49] consisting of a tensor field φ of type $(1,1)$, a vector field ξ, and a 1-form η satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0. \tag{2.1}$$

First and one of the remaining three relations of (2.1) imply the other two relations of (2.1). An almost contact structure is *normal* [50] if the torsion tensor $[\varphi, \varphi] + 2d\eta \otimes \xi$, where $[\varphi, \varphi]$ is the Nijenhuis tensor of φ, vanishes identically. We say that the almost contact structure (φ, ξ, η) has rank r if and only if the 1-form η has rank r. Consequently, (φ, ξ, η) has rank $r = 2s$ if $(d\eta)^s \neq 0$ and $\eta \wedge (d\eta)^s = 0$, and has rank $r = 2s+1$ if $\eta \wedge (d\eta)^s \neq 0$ and $(d\eta)^{s+1} = 0$.

Let g be a compatible Riemannian metric with (φ, ξ, η), that is,

$$g(X,Y) = g(\varphi X, \varphi Y) + \eta(X)\eta(Y), \qquad X, Y \in \mathfrak{X}(M), \tag{2.2}$$

where $\mathfrak{X}(M)$ is the Lie algebra of vector fields in M. Then, M becomes an *almost contact metric manifold* equipped with an *almost contact metric structure* (φ, ξ, η, g). The equation (2.2) is equivalent to

$$g(X, \varphi Y) = -g(\varphi X, Y) \text{ alongwith } g(X, \xi) = \eta(X), \; X, Y \in \mathfrak{X}(M). \tag{2.3}$$

Let M be an almost contact manifold equipped with an almost contact structure (φ, ξ, η). Let g be any Riemannian metric compatible with (φ, ξ, η). For a nonconstant positive function σ on M, the pseudo-conformal deformation g^σ of g, defined by

$$g^\sigma = \sigma g + (1 - \sigma)\eta \otimes \eta, \tag{2.4}$$

is also compatible with (φ, ξ, η). Note that g is also a pseudo-conformal deformation of g^σ. In particular, in dimension 3, we have the following

Proposition 2.1 ([41], [42, p. 42]). *Let M be an almost contact 3-manifold equipped with an almost contact structure (φ, ξ, η). Then two Riemannian metrics are compatible with (φ, ξ, η) if and only if one of them is a pseudo-conformal deformation of the other.*

An almost contact metric structure becomes a contact metric structure if $\Phi = d\eta$, where Φ is the fundamental 2-form given by

$$\Phi(X, Y) = g(X, \varphi Y), \qquad X, Y \in \mathfrak{X}(M).$$

In a contact metric manifold M, the $(1,1)$-tensor field h defined by $2h = \pounds_\xi \varphi$, which is the Lie derivative of φ in the characteristic direction ξ, is symmetric and satisfies

$$h\xi = 0, \quad h\varphi + \varphi h = 0, \quad \nabla\xi = -\varphi - \varphi h, \quad \mathrm{trace}(h) = \mathrm{trace}(\varphi h) = 0. \tag{2.5}$$

A contact metric manifold is called a *K-contact manifold* [53] if the characteristic vector field ξ is a Killing vector field. An almost contact metric manifold is a *K-contact manifold* if and only if $\nabla\xi = -\varphi$, where ∇ is the Levi-Civita connection. A K-contact manifold is a contact metric manifold, while the converse is true if $h = 0$. In a K-contact manifold, the curvature tensor R satisfies

$$(\nabla_X \varphi)Y = R(\xi, X)Y, \qquad X, Y \in \mathfrak{X}(M).$$

A normal contact metric manifold is a *Sasakian manifold*. An almost contact metric manifold is Sasakian if and only if [51]

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \qquad X, Y \in \mathfrak{X}(M). \tag{2.6}$$

Also, a contact metric manifold M is Sasakian if and only if the curvature tensor R satisfies the equation

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \qquad X, Y \in \mathfrak{X}(M). \tag{2.7}$$

A Sasakian manifold is always a K-contact manifold and the converse is true in the dimension three. Thus a 3-dimensional contact metric manifold is Sasakian if and only if $h = 0$.

An almost contact metric structure (φ, ξ, η, g) on M is a *trans-Sasakian structure* [46, 8] if

$$(\nabla_X \varphi) Y = \alpha (g(X, Y)\xi - \eta(Y) X) + \beta (g(\varphi X, Y)\xi - \eta(Y)\varphi X) \tag{2.8}$$

for some smooth functions α and β on M, and we say that the trans-Sasakian structure is of type (α, β). The formula (2.8) is equivalent to

$$\begin{aligned} (\nabla_X \Phi)(Y,Z) &= \alpha\left(g(X,Z)\eta(Y) - g(X,Y)\eta(Z)\right) \\ &\quad -\beta\left(g(X,\varphi Z)\eta(Y) - g(X,\varphi Y)\eta(Z)\right), \end{aligned} \tag{2.9}$$

for all $X, Y, Z \in \mathfrak{X}(M)$. Trans-Sasakian manifolds are always normal [46].

A trans-Sasakian structure of type (α, β) is reduced to be a

- *Sasakian structure* ([51], [5]) if $(\alpha, \beta) = (1, 0)$,
- *Kenmotsu structure* [30] if $(\alpha, \beta) = (0, 1)$,
- *cosymplectic structure* [5] if $(\alpha, \beta) = (0, 0)$.

More generally a trans-Sasakian manifold of type $(\alpha, 0)$ with nonzero constant α is homothetic to a Sasakian manifold and is called a *homothetic Sasakian manifold* or *α-Sasakian manifold* [28]. Analogously, a *homothetic Kenmotsu manifold* (or *β-Kenmotsu manifold*) [28] is a trans-Sasakian manifold of type $(0, \beta)$ with nonzero constant β [28].

In the classification of almost contact metric structures/manifolds, there appear two classes namely $\mathcal{C}_6$-class and $\mathcal{C}_5$-class [34]. The class $\mathcal{C}_6 \oplus \mathcal{C}_5$ [34] coincides with the class of trans-Sasakian structures/manifolds of type (α, β). A trans-Sasakian manifold of type (α, β) is of $\mathcal{C}_5$-class if $\alpha = 0$. The manifolds belonging to the class $\mathcal{C}_5$ are also known as *f-Kenmotsu manifolds* [43] or a *normal locally conformal almost cosymplectic manifolds* (cf. [35], [45]). The class $\mathcal{C}_5$ contains the class of β-Kenmotsu manifolds. On the other hand, a trans-Sasakian manifold of type (α, β) is of $\mathcal{C}_6$-class if $\beta = 0$. α-Sasakian manifolds belong to the $\mathcal{C}_6$-class. The cosymplectic manifolds are common to the $\mathcal{C}_6$-class and the $\mathcal{C}_5$-*class*.

A transformation in an n-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a *concircular transformation* ([33], [64]). A concircular transformation is always a conformal transformation [33]. Thus, the geometry of concircular transformations, that is, the concircular geometry, is a generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism (see also [4]). It is interesting to note that Kirichenko [31] obtained Kenmotsu structures from cosymplectic structures (that is, $\nabla\varphi = 0$ (cf. [5]) by the canonical concircular transformations [64].

In a $(2n+1)$-dimensional trans-Sasakian manifold M, the following results are known [21]

$$R(X,Y)\xi = \left(\alpha^2 - \beta^2\right)(\eta(Y)X - \eta(X)Y) + 2\alpha\beta\,(\eta(Y)\varphi X - \eta(X)\varphi Y) \tag{2.10}$$
$$+ (Y\alpha)\,\varphi X - (X\alpha)\,\varphi Y + (Y\beta)\,\varphi^2 X - (X\beta)\,\varphi^2 Y,$$

$$S(X,\xi) = \left(2n\left(\alpha^2-\beta^2\right) - \xi\beta\right)\eta(X) - (2n-1)\,X\beta - (\varphi X)\,\alpha, \tag{2.11}$$

$$Q\xi = \left(2n\left(\alpha^2-\beta^2\right) - \xi\beta\right)\xi - (2n-1)\,\mathrm{grad}\,\beta + \varphi\,(\mathrm{grad}\,\alpha) \tag{2.12}$$

for all $X, Y \in \mathfrak{X}(M)$, where R and S are curvature and Ricci curvature tensors, while Q is the Ricci operator given by $S(X,Y) = g(QX,Y)$.

In a trans-Sasakian manifold of type (α,β), the functions α and β cannot be arbitrary and satisfy [21, Eq. (9)]

$$2\alpha\beta + \xi\alpha = 0; \tag{2.13}$$

thus a trans-Sasakian structure of type (α,β) with α a nonzero constant is always α-Sasakian. This conclusion is also obtained by a different method in [8]. In particular, in a trans-Sasakian manifold of type $(\alpha, 0)$, it follows that $d\alpha\,(\xi) = 0$. The relation (2.13) was first/also obtained by Olszak [42, Eq. (16)] for the 3-dimensional case.

From (2.11) and (2.13), it follows that

$$S(\xi,\xi) = 2n\left(\alpha^2 - \beta^2 - \xi\beta\right). \tag{2.14}$$

We close this section with the following

Lemma 2.2. *Let M be a $(2n+1)$-dimensional trans-Sasakian manifold. Then the following statements are equivalent*:
(1) *The structure vector field ξ is an eigenvector field of the Ricci operator Q.*
(2) *For each $X \in \mathfrak{X}(M)$, which is orthogonal to the structure vector field ξ,*

$$g\left((2n-1)\,\mathrm{grad}\,\beta - \varphi\,\mathrm{grad}\,\alpha, X\right) = 0. \tag{2.15}$$

(3) *The functions α and β satisfy*

$$(2n-1)\,\mathrm{grad}\,\beta - \varphi\,\mathrm{grad}\,\alpha = (2n-1)\,(\xi\beta)\,\xi. \tag{2.16}$$

(4) *For each* $X \in \mathfrak{X}(M)$,

$$(2n-1)\, X\beta + (\varphi X)\,\alpha = (2n-1)\,(\xi\beta)\,\eta(X). \tag{2.17}$$

(5) *The functions* α *and* β *satisfy*

$$(2n-1)\,\varphi\,\mathrm{grad}\,\beta + \mathrm{grad}\,\alpha = (\xi\alpha)\,\xi. \tag{2.18}$$

(6) *For each* $X \in \mathfrak{X}(M)$,

$$X\alpha - (2n-1)\,(\varphi X)\,\beta = (\xi\alpha)\,\eta(X). \tag{2.19}$$

(7) *For each* $X \in \mathfrak{X}(M)$, *which is orthogonal to the structure vector field* ξ,

$$g\left((2n-1)\,\varphi\,\mathrm{grad}\,\beta + \mathrm{grad}\,\alpha, X\right) = 0. \tag{2.20}$$

(8) *The Ricci operator satisfies*

$$Q\xi = 2n\left(\alpha^2 - \beta^2 - \xi\beta\right)\xi. \tag{2.21}$$

(9) *We have*

$$\eta\,(Q\varphi - \varphi Q) = 0. \tag{2.22}$$

Proof. Equivalence of (1) and (2) follows from (2.12). It is easy to verify the equivalence of (2), (3) and (4). Similarly, (5), (6) and (7) are equivalent. Operating φ to (2.16) we get (2.18) and operating φ to (2.18) we get (2.16); Thus (3) and (5) are equivalent. Equivalence of (2.16) and (2.21) follows from (2.12). In last, we note that (8) implies (9) and (9) implies (5). □

In [58], it is proved that a pseudo projective φ-recurrent trans-Sasakian manifold which satisfies $\varphi\mathrm{grad}\,\alpha = (2n-1)\,\mathrm{grad}\,\beta$ is an Einstein manifold. We remark that the condition $\varphi\mathrm{grad}\,\alpha = (2n-1)\,\mathrm{grad}\,\beta$ on a trans-Sasakian manifold is a very strong condition on the manifold making the structure vector field ξ an eigenvector field of the Ricci operator Q, provided that $d\beta\,(\xi) = 0$. Further it is proved that in a pseudo projective φ-recurrent trans-Sasakian manifold the characteristic vector field ξ and the vector field ρ associated to the 1-form are opposite directional. Thus in a pseudo projective φ-recurrent trans-Sasakian manifold all the equivalent statements of Lemma 2.2 are equivalent to the condition of ρ being an eigenvector field of the Ricci operator Q.

3. Trans-K-contact manifolds

Definition 3.1. An almost contact metric structure (resp. manifold) is called a *trans-K-contact structure* (resp. *manifold*) *of type* (α, β) [59] if it satisfies

$$\nabla_X \xi = -\alpha\varphi X + \beta\left(X - \eta\left(X\right)\xi\right), \qquad X \in \mathfrak{X}\left(M\right) \tag{3.1}$$

for some smooth functions α, β on M. In particular, if $\alpha = 1$ and $\beta = 0$, then a trans-K-contact structure (resp. manifold) reduces to a *K-contact structure* (resp. *manifold*).

Proposition 3.2. *In a trans-K-contact manifold, integral curves* (*trajectories*) *of the structure vector field ξ are geodesics.*

Proof. From (3.1), it follows that $\nabla_\xi \xi = 0$. □

An almost contact metric manifold is a trans-K-contact manifold if and only if

$$\left(\nabla_X \eta\right)Y = -\alpha g\left(\varphi X, Y\right) + \beta g\left(\varphi X, \varphi Y\right), \qquad X, Y \in \mathfrak{X}\left(M\right). \tag{3.2}$$

It is easy to show that (3.2) is equivalent to

$$d\eta = \alpha\,\Phi \qquad \text{alongwith} \qquad \pounds_\xi g = 2\beta\left(g - \eta\otimes\eta\right). \tag{3.3}$$

In a trans-K-contact manifold, it follows that [59, Proposition 2]

$$\pounds_\xi \eta = 0 \qquad \text{and} \qquad \beta g(\varphi X, Y) = (\nabla_Y \eta)(\varphi X) + \alpha g(\varphi X, \varphi Y), \tag{3.4}$$

for all $X, Y \in \mathfrak{X}(M)$.

A trans-Sasakian structure is always a trans-K-contact structure and is normal. In particular, a Sasakian manifold is always a K-contact manifold.

Now, we need the following

Definition 3.3. An almost contact metric manifold M is *generalized quasi-Sasakian* [36] if

$$\begin{aligned} 3d\Phi(X,Y,Z) \;=\; & \eta(X)\left((\nabla_Y\eta)(\varphi Z) - (\nabla_Z\eta)(\varphi Y)\right) \\ & + \eta(Y)\left((\nabla_Z\eta)(\varphi X) - (\nabla_X\eta)(\varphi Z)\right) \\ & + \eta(Z)\left((\nabla_X\eta)(\varphi Y) - (\nabla_Y\eta)(\varphi X)\right), \end{aligned}$$

for all $X, Y, Z \in \mathfrak{X}\left(M\right)$.

A necessary and sufficient condition for an almost contact metric manifold to be trans-Sasakian is given in the following

Theorem 3.4 ([59, Theorem 3])**.** *An almost contact metric manifold is trans-Sasakian if and only if it is normal, trans-K-contact and generalized quasi-Sasakian.*

Next, we show the following

Theorem 3.5. *Let M be a 3-dimensional almost contact metric manifold. Then the following four statements are equivalent*:

(1) *M is normal.*

(2) *M satisfies $\nabla\xi \circ \varphi = \varphi \circ \nabla\xi$.*

(3) *M is trans-K-contact.*

(4) *M is trans-Sasakian of type (α, β).*

In any of these three cases

$$\alpha = \frac{1}{2}\,\mathrm{Trace}\,(\varphi\nabla\xi)\,, \qquad \beta = \frac{1}{2}\,\mathrm{div}\xi.$$

Proof. Let $(M, \varphi, \xi, \eta, g)$ be a 3-dimensional almost contact metric manifold. Then [42, Proposition 1], [29, Eq. (1.5)]

$$(\nabla_X\varphi)\,Y = g\,(\varphi\nabla_X\xi, Y)\,\xi - \eta\,(Y)\,\varphi\nabla_X\xi, \qquad X \in \mathfrak{X}\,(M)\,. \tag{3.5}$$

It is known that [42, Proposition 2] a 3-dimensional almost contact metric manifold M is normal if and only if

$$\nabla_{\varphi X}\xi = \varphi\nabla_X\xi, \qquad X \in \mathfrak{X}\,(M)\,, \tag{3.6}$$

which is equivalent to [42, Proposition 2]

$$\nabla_X\xi = -\,\alpha\varphi X + \beta\,(X - \eta\,(X)\,\xi)\,, \qquad X \in \mathfrak{X}\,(M)\,, \tag{3.7}$$

where α and β are the functions defined by

$$2\alpha = \mathrm{Trace}(\varphi\nabla\xi) = \mathrm{Trace}\,\{X \to \varphi\nabla_X\xi\}\,, \tag{3.8}$$

$$2\beta = \mathrm{div}\xi = \mathrm{Trace}\,\{X \to \nabla_X\xi\}\,. \tag{3.9}$$

Thus the statements (1), (2) and (3) are equivalent. Obviously, (4) implies (3). In view of (3.5) and (3.7), it follows that

$$(\nabla_X \varphi) Y = \alpha (g(X,Y)\xi - \eta(Y)X) + \beta (g(\varphi X, Y)\xi - \eta(Y)\varphi X), \tag{3.10}$$

for all $X, Y \in \mathfrak{X}(M)$. This concludes the proof. □

Corollary 3.6 ([5, Corollary 6.5]). *Every 3-dimensional K-contact manifold is Sasakian.*

Problem 3.7. It would be interesting to obtain an Example of a trans-K-contact manifold, which is not trans-Sasakian.

Next, we have

Definition 3.8 ([3]). An almost contact metric manifold M is *quasi-Sasakian* if it is normal and $d\Phi = 0$.

In [42, Remark 2], Olszak characterized a 3-dimensional quasi-Sasakian manifolds. He stated that an almost contact metric 3-manifold M is quasi-Sasakian if and only if $\nabla\xi = -\alpha\varphi$ for some function α on M. We can rephrase this characterization as the following

Theorem 3.9. *Let M be a 3-dimensional almost contact metric manifold equipped with an almost contact metric structure (φ, ξ, η, g). Then the following statement are equivalent*:

(1) *M is quasi-Sasakian.*

(2) *$\nabla\xi = -\alpha\varphi$, where $2\alpha = \mathrm{Trace}(\varphi\nabla\xi)$.*

(3) *M is normal with $0 = 2\beta = \mathrm{div}\xi$.*

(4) *M is a trans-K-contact manifold of type $(\alpha, 0)$.*

(5) *M is a trans-Sasakian manifold of type $(\alpha, 0)$.*

(6) *M belongs to $\mathcal{C}_6$-class.*

Remark 3.10. Note that in a quasi-Sasakian 3-manifold, it follows that $d\alpha(\xi) = 0$. In particular, if α is constant then a quasi-Sasakian 3-manifold is α-Sasakian. Moreover, if $\alpha = 1$, it becomes Sasakian.

In [44], Olszak also obtained necessary and sufficient conditions for a 3-dimensional quasi-Sasakian manifold to be conformally flat.

In [34], Marrero showed the nonexistence of proper trans-Sasakian manifolds of dimension greater than 3. More precisely, he proved that a trans-Sasakian manifold of dimension ≥ 5 is either α-Sasakian, or of $\mathcal{C}_5$-class (f-Kenmotsu) or cosymplectic ([34, Theorem 4.8],[62, Theorem 4.8]). On the other hand, he showed the following method for construction of proper trans-Sasakian 3-manifolds (see also [41]).

Proposition 3.11 ([34, Proposition 4.2], [41])**.** *Let M be a Sasakian 3-manifold and σ a nonconstant positive function on M. Then a pseudo-conformal deformation g^{σ} induces a trans-Sasakian structure $(\varphi, \xi, \eta, g^{\sigma})$ of type $(1/\sigma, (1/2)\,\xi\,(\ln \sigma))$ on M. Thus, if $d\sigma\,(\xi) \neq 0$, then g^{σ} induces a proper trans-Sasakian structure $(\varphi, \xi, \eta, g^{\sigma})$, which is neither of $\mathcal{C}_6$-class nor of $\mathcal{C}_5$-class. If $d\sigma\,(\xi) = 0$, then $(M, \varphi, \xi, \eta, g^{\sigma})$ is quasi-Sasakian. Conversely, every quasi-Sasakian 3-manifold can be obtained in this way* [41].

4. η-Einstein trans-Sasakian manifolds

We begin this section with the following

Definition 4.1 ([40],[5, p. 105])**.** An almost contact metric manifold M is said to be *η-Einstein* if the Ricci tensor S satisfies

$$S = ag + b\eta \otimes \eta, \tag{4.1}$$

where a and b are some smooth functions on the manifold. In particular, if $b = 0$ then M is an *Einstein manifold*.

Definition 4.2 ([56])**.** The *k-nullity distribution* $N(k)$ of a Riemannian manifold M is defined by

$$N(k) : p \to N_p(k) = \{U \in T_pM \mid R(X,Y)U = k(g\,(Y,U)\,X - g\,(X,U)\,Y)\},$$

for all $X, Y \in \mathfrak{X}(M)$, where k is some smooth function. A contact metric manifold is said to be an $N(k)$-*contact metric manifold* [6] if the structure vector field ξ belongs to the *k-nullity distribution* $N(k)$.

In 1990, Blair, Koufogiorgos and Sharma [7] proved that on a 3-dimensional contact metric manifold M equipped with a contact metric structure (φ, ξ, η, g) the following conditions are equivalent:

(1) M is η-Einstein,

(2) $Q\varphi = \varphi Q$, where Q is the Ricci operator and

(3) M is an $N(k)$-contact metric manifold.

More precisely, in a 3-dimensional $N(k)$-contact metric manifold, we have

$$Q = \left(\frac{r}{2} - k\right) I + \left(3k - \frac{r}{2}\right) \eta \otimes \xi. \tag{4.2}$$

A 3-dimensional contact metric manifold M satisfying one of the conditions (1), (2) or (3) is either Sasakian or flat or of constant ξ-sectional curvature $k < 1$ and constant φ-sectional curvature $-k$ or locally isometric to a left-invariant metric on the Lie group $SU(2)$ or $SL(2,\mathbb{R})$ (cf. [7], [5, p. 105]).

It is known that a K-contact manifold is Sasakian if either its dimension is three [5, Corollary 6.5], or it is compact Einstein [10, Theorem A] or compact η-Einstein with $a > -2$ ([38], [10, Theorem 7.2]). The conclusions of Theorems A and 7.2 of [10] are still true if the condition of compactness is weakened to completeness [52, Proposition 1].

For Sasakian manifolds, we have the following

Proposition 4.3 ([65, Proposition 5.3])**.** *Every Sasakian space form is η-Einstein.*

Proposition 4.4. *In an η-Einstein K-contact* (*in particular, Sasakian*) *manifold of dimension* > 3, *a and b are constants* ([55], [65, Proposition 5.4], [48]).

A necessary and sufficient condition for a K-contact manifold to be η-Einstein is given in the following:

Proposition 4.5 ([55, Proposition 2.2])**.** *A K-contact manifold is η-Einstein if and only if $(R(X,\xi)\cdot S)(U,V) = b\{\eta(U)g(V,X)+\eta(V)g(U,X)-2\eta(U)\eta(V)\eta(X)\}$, $X,U,V \in \mathfrak{X}(M)$ for some function b on M.*

For Kenmotsu manifolds, we have the following

Proposition 4.6 ([30, Proposition 8, Corollary 9])**.** *Let M be a $(2n+1)$-dimensional Kenmotsu manifold. If M is η-Einstein, we have*

$$a+b=-2n. \tag{4.3}$$

Moreover, if $n>1$, then

$$X(b)+2b\eta(X)=0, \qquad X\in\mathfrak{X}(M). \tag{4.4}$$

Consequently, in an η-Einstein Kenmotsu manifold M of dimension greater than 3, if one of a and b is constant, then M reduces to be an Einstein manifold.

Example 4.7. From (4.2), we see that every 3-dimensional Sasakian manifold is η-Einstein and its Ricci tensor is given by

$$S=\left(\frac{r}{2}-1\right)g+\left(3-\frac{r}{2}\right)\eta\otimes\eta,$$

where r is the scalar curvature of the manifold.

Example 4.8 ([20])**.** Every 3-dimensional Kenmotsu manifold is η-Einstein and its Ricci tensor S is given by

$$S=\left(\frac{r}{2}+1\right)g-\left(3+\frac{r}{2}\right)(r+6)\,\eta\otimes\eta,$$

where r is the scalar curvature of the manifold.

Proposition 4.9. *Let (M,φ,ξ,η,g) be a $(2n+1)$-dimensional η-Einstein trans-Sasakian manifold. Then*

$$S=\left(\frac{r}{2n}-(\alpha^2-\beta^2-\xi\beta)\right)g-\left(\frac{r}{2n}-(2n+1)(\alpha^2-\beta^2-\xi\beta)\right)\eta\otimes\eta. \tag{4.5}$$

Proof. In view of (4.1) and (2.14) we have

$$a+b=2n\left(\alpha^2-\beta^2-\xi\beta\right). \tag{4.6}$$

Also from (4.1), it follows that

$$r=(2n+1)\,a+b. \tag{4.7}$$

From (4.6) and (4.7) we get

$$a=\frac{r}{2n}-\left(\alpha^2-\beta^2-\xi\beta\right), \qquad b=-\frac{r}{2n}+(2n+1)\left(\alpha^2-\beta^2-\xi\beta\right);$$

thus (4.1) becomes (4.5). □

Remark 4.10. In a $(2n+1)$-dimensional η-Einstein trans-Sasakian manifold, we see that $Q\xi = 2n\left(\alpha^2 - \beta^2 - \xi\beta\right)\xi$. Consequently, in this case, all nine statements of Lemma 2.2 are true.

Example 4.11 ([8])**.** Let (x, y, z) be Cartesian coordinates in $\mathbb{R}^3$, then (φ, ξ, η, g) given by

$$\xi = \partial/\partial z\,, \qquad \eta = dz - ydx,$$

$$\varphi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -y & 0 \end{pmatrix}, \qquad g = \begin{pmatrix} e^z + y^2 & 0 & -y \\ 0 & e^z & 0 \\ -y & 0 & 1 \end{pmatrix}$$

is a trans-Sasakian structure of type $(\alpha, \beta) = (-1/(2e^z), 1/2)$ in $\mathbb{R}^3$.

Example 4.12. The Example 4.11 gives a 3-dimensional η-Einstein trans-Sasakian manifold.

Now, we consider a well known generalization of the concept of an η-Einstein almost contact metric manifold in the following

Definition 4.13 ([12])**.** A non-flat n-dimensional Riemannian manifold (M, g) is said to be a *quasi Einstein manifold* if its Ricci tensor S satisfies

$$S = ag + b\eta \otimes \eta \tag{4.8}$$

or equivalently, its Ricci operator Q satisfies

$$Q = aI + b\eta \otimes \xi \tag{4.9}$$

for some smooth functions a and b, where η is a nonzero 1-form such that

$$g(X, \xi) = \eta(X), \qquad g(\xi, \xi) = \eta(\xi) = 1 \tag{4.10}$$

for the associated vector field ξ. The 1-form η is called the associated 1-form and the unit vector field ξ is called the generator of the quasi Einstein manifold.

Chen and Yano [13] defined a Riemannian manifold (M, g) to be of *quasi-constant curvature* if it is conformally flat manifold and its Riemann-Christoffel curvature tensor R of type $(0, 4)$ satisfies the condition

$$\begin{aligned} R(X,Y,Z,W) = {} & a\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \\ & + b\{g(X,W)T(Y)T(Z) - g(X,Z)T(Y)T(W) \\ & + g(Y,Z)T(X)T(W) - g(Y,W)T(X)T(Z)\}, \end{aligned} \tag{4.11}$$

for all $X, Y, Z, W \in \mathfrak{X}(M)$, where a, b are some smooth functions and T is a non-zero 1-form defined by $g(X, \rho) = T(X)$, $X \in \mathfrak{X}(M)$ for a unit vector field ρ. On the other hand, VRĂNCEANU [63] defined a Riemannian manifold (M, g) to be of *almost constant curvature* if M satisfies (4.11). Later on, it was pointed out by MOCANU [37] that the manifold introduced by Chen and Yano and the manifold introduced by Vrănceanu were identical, as it can be verified that if the curvature tensor R is of the form (4.11), then the manifold is conformally flat. Thus, a Riemannian manifold is said to be of *quasi-constant curvature* if the curvature tensor R satisfies (4.11). If $b = 0$, then the manifold reduces to a manifold of constant curvature.

Example 4.14. A manifold of quasi-constant curvature is a quasi Einstein manifold [18, Example 1]. Conversely, a conformally flat quasi Einstein manifold of dimension n $(n > 3)$ is a manifold of quasi-constant curvature [19, Theorem 4].

Let (M, g) be a semi-Riemannian manifold with its Levi-Civita connection ∇. A tensor field F of type $(1, 3)$ is known to be *curvature-like* provided that F satisfies the symmetric properties of the curvature tensor R. For example, the tensor R_g given by

$$R_g\,(X, Y)\,Z \equiv (X \wedge_g Y)\,Z = g(Y, Z)X - g(X, Z)Y,\ X, Y \in \mathfrak{X}(M)\,, \tag{4.12}$$

is a trivial example of a curvature like tensor. Sometimes, the symbol R_g seems to be much more convenient than the symbol $(X \wedge_g Y)\,Z$. For example, a semi-Riemannian manifold (M, g) is of constant curvature c if and only if $R = cR_g$.

It is well known that every curvature-like tensor field F acts on the algebra $\mathfrak{T}^1_s(M)$ of all tensor fields on M of type $(1, s)$ as a derivation [39, p. 44]:

$$\begin{aligned}(F \cdot P)\,(X_1, \dots, X_s; Y, X) &= F(X, Y)\,\{P\,(X_1, \dots, X_s)\}\\ &\quad - \sum_{j=1}^{s} P(X_1, \dots, F\,(X, Y)\,X_j, \dots, X_s),\end{aligned}$$

for all $X_1, \dots, X_s \in \mathfrak{X}(M)$, $P \in \mathfrak{T}^1_s(M)$. The derivative $F \cdot P$ of P by F is a tensor field of type $(1, s+2)$. A semi-Riemannian manifold (M, g) is said to be *semi-symmetric* if $R \cdot R = 0$. Obviously, locally symmetric spaces $(\nabla R = 0)$ are semi-symmetric. More generally, a semi-Riemannian manifold (M, g) is said to be *pseudo-symmetric* (in the sense of R. Deszcz) [22] if $R \cdot R$ and $R_g \cdot R$ in M are linearly dependent, that is, if there exists a real valued smooth function $L : M \to \mathbb{R}$ such that $R \cdot R = L\, R_g \cdot R$ is true on the set $U = \{x \in M : R \neq \frac{r}{n(n-1)} R_g \text{ at } x\}$. In particular, if L is constant, M is called a *pseudo-symmetric space of constant type* [32]. A pseudo-symmetric space is said to be *proper* if it is not semi-symmetric. For details we refer to [9, 1].

In the literature, there is also another notion of pseudo-symmetry. A semi-Riemannian manifold (M, g) is said to be *pseudo-symmetric* in the sense of Chaki [11] if

$$\begin{aligned}(\nabla R)(X_1, X_2, X_3, X_4; X) &= 2\omega(X)R(X_1, X_2, X_3, X_4)\\ &+ \omega(X_1)R((X, X_2, X_3, X_4) + \omega(X_2)R((X_1, X, X_3, X_4)\\ &+ \omega(X_3)R((X_1, X_2, X, X_4) + \omega(X_4)R((X_1, X_2, X_3, X),\end{aligned}$$

for all $X_1, X_2, X_3, X_4; X \in \mathfrak{X}(M)$, where ω is a 1-form on (M, g). Of course, both the definitions of pseudo-symmetry for a semi-Riemannian manifold are not equivalent. For example, in contact geometry, every Sasakian space form is pseudo-symmetric in the sense of Deszcz [2, Theorem 2.3], but a Sasakian manifold cannot be pseudo-symmetric in the sense of Chaki [57, Theorem 1]. We assume the pseudo-symmetry always in the sense of Deszcz, unless specifically stated otherwise.

For 3-dimensional Riemannian manifolds, the following characterizations of pseudosymmetry are known (cf. [32, 17]).

Proposition 4.15. *A 3-dimensional Riemannian manifold (M, g) is pseudo-symmetric if and only if it is quasi-Einstein, that is, if and only if there exists a 1-form η such that the Ricci tensor field S satisfies $S = ag + b\eta \otimes \eta$ for some smooth functions a and b.*

Proposition 4.16. *A 3-dimensional Riemannian manifold (M, g) is a pseudo-symmetric space of constant type if and only if there exists a 1-form η such that the Ricci tensor field S is expressed as $S = ag + b\eta \otimes \eta$, where a is a smooth function and $a + b\,\|\eta\|^2$ is a constant (provided that $\eta \neq 0$).*

Remark 4.17 ([17])**.** The preceding Proposition can be rephrased as follows: A 3-dimensional Riemannian manifold is a pseudo-symmetric space of constant type with $R \cdot R = L\, R_g \cdot R$ if and only if the principal Ricci curvatures (eigenvalues of the Ricci tensor) locally satisfy the following relations (up to numeration): $\rho_1 = \rho_2$, $\rho_3 = 2L$.

Corollary 4.18. *Every 3-dimensional η-Einstein almost contact metric manifold is always pseudo-symmetric. In particular, each 3-dimensional Sasakian manifold is pseudo-symmetric.*

Hashimoto and Sekizawa [27] investigated 3-dimensional conformally flat (irreducible) pseudo-symmetric spaces of constant type. Their (local) classification says such spaces are warped products with 1-dimensional base and constant curvature fiber. One can see that [17] every 3-dimensional warped product with 1-dimensional base and 2-dimensional fiber admits a trans-Sasakian structure of type (α, β) with $\alpha = 0$.

Now, we give a comprehensive characterization of 3-dimensional trans-Sasakian manifolds to be η-Einstein.

Theorem 4.19. *Let M be a 3-dimensional almost contact metric manifold. If M satisfies any one of the following statements:*

(a) *M is normal,*

(b) *M satisfies $\nabla\xi \circ \varphi = \varphi \circ \nabla\xi$,*

(c) *M is trans-K-contact of type (α, β),*

(d) *M is trans-Sasakian of type (α, β);*

then the following statements are equivalent:
(1) *M is η-Einstein.*
(2) *The Ricci tensor S is given by*

$$S = \left(\frac{r}{2} - \left(\alpha^2 - \beta^2 - \xi\beta\right)\right) g - \left(\frac{r}{2} - 3\left(\alpha^2 - \beta^2 - \xi\beta\right)\right) \eta \otimes \eta. \tag{4.13}$$

(3) *The Ricci operator Q is given by*

$$Q = \left(\frac{r}{2} - \left(\alpha^2 - \beta^2 - \xi\beta\right)\right) I - \left(\frac{r}{2} - 3\left(\alpha^2 - \beta^2 - \xi\beta\right)\right) \eta \otimes \xi. \tag{4.14}$$

(4) *The curvature tensor R is given by*

$$\begin{aligned} R(X,Y)Z = &\left(\frac{r}{2} - 2\left(\alpha^2 - \beta^2 - \xi\beta\right)\right)\left(g(Y,Z)X - g(X,Z)Y\right) \\ &- \left(\frac{r}{2} - 3\left(\alpha^2 - \beta^2 - \xi\beta\right)\right)\{(g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi) \\ &+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\}, \end{aligned} \tag{4.15}$$

for all $X, Y, Z \in \mathfrak{X}(M)$.
(5) *The structure tensor* φ *and the Ricci operator* Q *commute, that is,*

$$\varphi Q = Q\varphi. \tag{4.16}$$

(6) *We have*

$$\eta(\varphi Q - Q\varphi) = 0. \tag{4.17}$$

(7) *We have*

$$\operatorname{grad}\alpha + \varphi(\operatorname{grad}\beta) = 0. \tag{4.18}$$

(8) *For each* $X \in \mathfrak{X}(M)$,

$$X\alpha - (\varphi X)\beta = (\xi\alpha)\eta(X). \tag{4.19}$$

(9) *For each* $X \in \mathfrak{X}(M)$, *which is orthogonal to the structure vector field* ξ,

$$g(\operatorname{grad}\alpha + \varphi\operatorname{grad}\beta, X) = 0. \tag{4.20}$$

(10) *We have*

$$\varphi(\operatorname{grad}\alpha) - \operatorname{grad}\beta = -(\xi\beta)\xi. \tag{4.21}$$

(11) *We have*

$$X\beta + (\varphi X)\alpha = (\xi\beta)\eta(X), \tag{4.22}$$

for every $X \in \mathfrak{X}(M)$.
(12) *We have*

$$g(\operatorname{grad}\beta - \varphi\operatorname{grad}\alpha, X) = 0, \tag{4.23}$$

for every $X \in \mathfrak{X}(M)$ *orthogonal to the structure vector field* ξ.
(13) *The Ricci operator satisfies*

$$Q\xi = 2\left(\alpha^2 - \beta^2 - \xi\beta\right)\xi. \tag{4.24}$$

(14) *The structure vector field* ξ *is an eigenvector field of the Ricci operator* Q.
(15) M *is pseudo-symmetric.*
(16) *The structure vector field* ξ *is a harmonic section of the unit tangent sphere bundle* T_1M.

Proof. From Theorem 3.5 the statements (a), (b), (c), and (d) are equivalent. The equivalence of the statements (1), (2) and (3) follows by putting $n = 1$ in (4.5). The equivalence of the statements (1), (2) and (3) also follows from [21, Theorem 5.1]. Since the conformal curvature tensor vanishes in a 3-dimensional Riemannian manifold, therefore we get [7, Eq.(3.3)]

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ &\quad - \frac{r}{2}\left(g(Y,Z)X - g(X,Z)Y\right), \end{aligned}$$

where r is the scalar curvature. Using (2) and (3) in the above equation, we get the statement (4). The statement (4) implies (2).

The statements (5), (6), and (7) are equivalent by [17, Proposition 3.4]. The statements (7)–(14) are equivalent by Lemma 2.2. In last, the equivalence of statements (14), (15) and (16) is true by [17, Theorem 3.1]. □

Remark 4.20. For different combinations of α and β, from Theorem 4.19, we can find several results for 3-dimensional K-contact, (or Sasakian), Kenmotsu, f-Kenmotsu and quasi-Sasakian manifolds. For example, for a 3-dimensional K-contact (and hence Sasakian) manifold, the following equivalent statements are true:
(1) M is η-Einstein.
(2) The Ricci tensor S is given by

$$S = \left(\frac{r}{2} - 1\right) g - \left(\frac{r}{2} - 3\right) \eta \otimes \eta. \tag{4.25}$$

(3) The Ricci operator Q is given by

$$Q = \left(\frac{r}{2} - 1\right) I - \left(\frac{r}{2} - 3\right) \eta \otimes \xi. \tag{4.26}$$

(4) The curvature tensor R is given by

$$\begin{aligned} R(X,Y)Z &= \left(\frac{r}{2} - 2\right)\left(g(Y,Z)X - g(X,Z)Y\right) \\ &\quad - \left(\frac{r}{2} - 3\right)\{(g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi) \\ &\quad + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\}, \end{aligned} \tag{4.27}$$

for all vector fields X, Y and Z in M.

(5) The structure tensor φ and the Ricci operator Q commute, that is,

$$\varphi Q = Q\varphi. \tag{4.28}$$

(6) The Ricci operator satisfies

$$Q\xi = 2\xi. \tag{4.29}$$

(7) The structure vector field ξ is an eigenvector field of the Ricci operator Q.
(8) M is pseudo-symmetric.
(9) The structure vector field ξ is a harmonic section of the unit tangent sphere bundle T_1M.

In a quasi Einstein manifold M if the generator ξ belongs to some k-nullity distribution $N(k)$, then M is said to be an $N(k)$-*quasi Einstein manifold* [60]. In an $N(k)$-quasi Einstein manifold, k is not arbitrary and is always given by [47, Lemma 2.1]

$$k = \frac{a+b}{n-1}. \tag{4.30}$$

If in an η-Einstein almost contact metric manifold, the structure vector field ξ belongs to the k-nullity distribution $N(k)$, then we say that M is an $N(k)$-η-*Einstein manifold.*

Now, we state a Corollary of the Theorem 4.19.

Corollary 4.21. *A trans-Sasakian* 3*-manifold* M*, satisfying any of the statements* (1)–(16) *of Theorem* 4.19*, is an* $N(k)$-η*-Einstein manifold with* $k = \alpha^2 - \beta^2 - \xi\beta$.

Proof. From (4.15) we get $R(X,Y)\xi = (\alpha^2-\beta^2-\xi\beta)(\eta(Y)X-\eta(X)Y)$, $X, Y \in \mathfrak{X}(M)$, which entails the proof. □

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