ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
MATEMATICĂ, Tomul LVIII, 2012, f.2

ULTIMATE BOUNDEDNESS OF THE SOLUTIONS OF CERTAIN THIRD ORDER NONLINEAR NON-AUTONOMOUS DIFFERENTIAL EQUATIONS*

BY

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Abstract. In this paper, we shall establish sufficient conditions for the uniform ultimate boundedness of solutions of a certain third order nonlinear non-autonomous differential equation, by using a Lyapunov function as basic tool. In doing so we extend some existing results. Examples are given to illustrate our results.

Mathematics Subject Classification 2010: 34D40, 34D20.

Key words: third order, nonlinear differential equation, non-autonomous, stability, uniform ultimate boundedness.

1. Introduction

Consider the following non-autonomous differential equation of the form

$$\dddot{x} + a(t)f(x,\dot{x},\ddot{x}) + b(t)g(x,\dot{x}) + c(t)h(x) = p(t) \tag{1.1}$$

and

$$\dddot{x} + a(t)f(x,\dot{x},\ddot{x}) + b(t)g(x,\dot{x}) + c(t)h(x) = p(t,x,\dot{x},\ddot{x}), \tag{1.2}$$

where a, b, c, p, f, g and h continuous functions depending only on the arguments shown and $g_x(x,\dot{x})$, $h'(x)$ exist and are continuous for all x and $\dot{x}$.

*This research was supported by University of Antioquia Research Grant CODI No. IN568CE.

The uniform ultimate boundedness of solutions of equations of the special forms of these equations have been discussed by different authors using Lyapunov's second method (cf. [20]). Many of these include the survey book Reissig, Sansone and Conti [25], the recent book by Haddad and Chellaboina [16] and the survey paper Andres [6]. Also other works with their references are Afuwape [1, 2, 3], Afuwape and Omeike [4], Afuwape et al. [5], Burton [9], Chukwu [11], Ezeilo [12, 13, 14], Hara [19], Qian [23, 24] and Tunç [27, 29, 30].

However, equations in the form of (1.1) and (1.2), especially when the $\ddot{x}$ term is inside the $f(x, \dot{x}, \ddot{x})$ term, not much have been considered.

Our objective in this paper is to study the uniform ultimate boundedness of solutions of equations (1.1) and (1.2). We shall use appropriate Lyapunov function and impose suitable conditions on the functions a, b, c, p, f, g, h and $p(t)$ (or $p(t, x, \dot{x}, \ddot{x})$). In the process, we shall extend earlier results of Hara [19], Tunç [29] and Ezeilo [14].

Moreover, the differential equations of third order have important applications in various areas of science, such as in the modeling of HIV infection, as recorded in Zhou [31]; the study of viscous fluid flows near bank edges as studied by Giacomelli [15]; the spacecraft flight control investigated by Bai [7]; modeling in Biopharmaceutics, Pharmacokinetics, and Pharmacodynamics as recorded by Macheras and Iliadis [21]; draining flows involving third-order differential equations as investigated by Bernis and Peletier [8]; and the transmission of rabies among foxes (three species) studied in Murray [22], among others.

Many results concerning qualitative solutions of non-autonomous nonlinear third-order ordinary differential equations use Lyapunov's second method. However the construction of such functions which are positive definite with corresponding negative definite derivatives are difficult to obtain, especially for higher order differential equations. Such functions and their time derivatives along the system under consideration must satisfy some fundamental inequalities.

We point out that Hara ([17, 18, 19]) proved that the differential equations

$$\dddot{x} + a(t)\ddot{x} + b(t)\dot{x} + c(t)x = p(t)$$
$$\dddot{x} + a(t)\ddot{x} + b(t)\dot{x} + c(t)h(x) = p(t, x, \dot{x}, \ddot{x})$$
$$\dddot{x} + a(t)f(x, \dot{x})\ddot{x} + b(t)g(x, \dot{x})\dot{x} + c(t)h(x) = p(t, x, \dot{x}, \ddot{x})$$

and $\dddot{x} + a(t)f(x,\dot{x},\ddot{x})\ddot{x} + b(t)g(x,\dot{x}) + c(t)h(x) = p(t,x,\dot{x},\ddot{x})$ have uniform ultimate bounded solutions.

Also, TUNÇ ([26, 27, 28]) proved some results on the uniformly ultimately boundedness of solution of scalar differential equation

$$\dddot{x} + a(t)f(x,\dot{x},\ddot{x})\ddot{x} + b(t)g(x,\dot{x}) + c(t)h(x) = p(t) \tag{1.3}$$

and

$$\dddot{x} + a(t)f(x,\dot{x},\ddot{x})\ddot{x} + b(t)g(x,\dot{x}) + c(t)h(x) = p(t,x,\dot{x},\ddot{x}). \tag{1.4}$$

These equations ((1.3) and (1.4)) are particular cases of the equations (1.1) and (1.2).

Motivated by results of HARA [19], we shall assume that

(i) $P(t) = e^{-t}\int_0^t e^s p(s)ds$ is such that $\sup_{t\geq 0}|P(t)| < +\infty$ in (1.1), which is a generalization of results given earlier by other researchers who imposed either boundedness on $|p(t)|$ or on the integral $\left|\int_0^t p(s)ds\right|$ (see [25]).

(ii) $p(t,x,y,z)$ in (1.2) is such that $|p(t,x,y,z)| \leq q_1(t) + \varepsilon(|x|+|y|+|z|)$ for all $t \geq 0$ and $|x|+|y|+|z| \geq K$, for $K > 0$ large enough, and $\varepsilon \geq 0$, sufficiently small, $q_1(t)$ is a non-negative function, and $\sup_{t\geq 0}\int_t^{t+1} q_1(s)ds < +\infty$.

We now give a lemma which will play an important role in the proof of our main results.

Let the solution through (t_0, x_0) of system

$$\dot{x} = F(t,x) \tag{1.5}$$

be denoted by $x(t,t_0,x_0)$, where $F(t,x) : [0,+\infty)\times\mathbb{R}^n \to \mathbb{R}^n$ is continuous.

Lemma 1.1 ([19]). *Suppose that there exists a Lyapunov function $V(t,x)$, continuously differentiable on $0 \leq t < \infty$, $\|x\| \geq K$, where K can be large, which satisfies the following conditions:*

(i) $a(\|x\|) \leq V(t,x) \leq b(\|x\|)$, where $a(r)$, $b(r)$ are continuous and $a(r) \to \infty$ as $r \to \infty$,

(ii)

$$\begin{aligned} \dot{V}|_{(1.5)}(t,x) &\equiv \lim_{h\to 0^+}\frac{1}{h}\{V(t+h, x+hF(t,x)) - V(t,x)\} \\ &\leq -\{c-\lambda_1(t)\}V(t,x) + \lambda_2(t)\sqrt{V(t,x)}, \end{aligned} \tag{1.6}$$

where $c > 0$ *is a constant and* $\lambda_i(t) \geq 0$ $(i = 1, 2)$ *are continuous functions satisfying*

$$\limsup_{(t,\nu)\to(\infty,\infty)} \frac{1}{\nu}\int_t^{t+\nu} \lambda_1(s)ds < c \tag{1.7}$$

and

$$\sup_{t\geq 0}\int_t^{t+1} \lambda_2(s)ds < +\infty \tag{1.8}$$

then the solutions of (1.5) *are uniformly ultimately bounded.*

See the sketch of the proof in [19].

We also note that this lemma actually implies Theorem 0.4.3 (e) of Burton [10].

This paper is organized as follows: after the introduction in Section 1, we give the basic assumptions and the main results in Section 2. Section 3 will be devoted to the preliminaries to the proofs of some of the results. In Section 5, we give an example that illustrates the results of Theorem 2.2. Section 6 is devoted to the completion of the proof of uniform ultimate boundedness of equation (1.2) in Theorem 2.3, while some examples are given in Sections 6 and 7. The references are given in the following section.

2. Assumptions and main results

We shall first state here some assumptions which will be used on the functions that appeared in equations (1.1) and (1.2)

(I) $a(t)$, $b(t)$ and $c(t)$ are continuously differentiable, and $p(t)$ is continuous on $[0, \infty)$, $p(t, x, y, z)$ is continuous on $[0, \infty) \times \mathbb{R}^3$.

(II) $g(x, y)$, $g_x(x, y)$ are continuous for all $(x, y) \ \epsilon\ \mathbb{R}^2$ and $h(x)$ is continuously differentiable for all $x \ \epsilon\ \mathbb{R}$.

(III) $1 \leq a_0 \leq a(t) \leq A$, $1 \leq b_0 \leq b(t) \leq B$ and $0 < c_0 \leq c(t) \leq C$ for all $t \ \epsilon [0, \infty)$.

(IV) $0 < \delta \leq \frac{h(x)}{x}$ with $x \neq 0$ and $h(0) = 0$.

(V) $0 < f_0 \leq \frac{f(x,y,z)}{z} \leq \overline{f}$, with $z \neq 0$.

(VI) $0 < g_0 \leq \frac{g(x,y)}{y} \leq \overline{g}$, with $y \neq 0$ $g_x(x,y) \leq 0$ for all $(x,y)\epsilon\ \mathbb{R}^2$.

(VII) $h'(x) \leq h_1 \leq \frac{a_0 b_0 f_0 g_0}{C}$ for all $x \in \mathbb{R}$.

(VIII)

$$\frac{2\mu_2 c_0 \delta}{a_0 b_0 f_0^2 g_0 (A - a_0) + \mu_2 [A(\overline{f} - f_0) + B(\overline{g} - g_0)] + \delta a_0 b_0 f_0 g_0} \geq \varepsilon_1 \geq 0,$$
$$\frac{2\left[b_0 g_0 (a_0 f_0 + \mu_1) - 2Ch_1 - \mu_2 A f_0\right]}{a_0 b_0 f_0^2 g_0 (A - a_0) + \mu_2 B(\overline{g} - g_0) + A(Af_0 + \mu_1)(\overline{f} - f_0) + \delta B\overline{g}} \geq \varepsilon_2 \geq 0,$$
$$\frac{2(a_0 f_0 - \mu_1 + \delta a_0 f_0)}{(\overline{f} - f_0) A(\mu_2 + Af_0 + \mu_1) + \delta(a_0 b_0 f_0 g_0 + B\overline{g})} \geq \varepsilon_3 \geq 0$$

and $b_0 g_0 \geq \frac{C\delta}{\mu_1} + \mu_2$ where μ_1 and μ_2 are arbitrarily fixed constants satisfying $\frac{Ch_1}{b_0 g_0} < \mu_1 < a_0 f_0$ and $0 < \mu_2 < \frac{a_0 b_0 f_0 g_0 - Ch_1}{Af_0}$.

(IX) $\limsup_{(t,s)\to(\infty,\infty)} \frac{1}{v} \int_t^{t+v} \left\{|a'(s)| + b'_+(s) + |c'(s)|\right\} ds < \gamma$ where γ is a small positive constant whose magnitude depends only on the constant that appeared in (III)-(VIII) and $b'_+(t) = \max(b'(t), 0)$.

(X) $\sup_{t\geq 0} \left| e^{-t} \int_0^t e^s p(s) ds \right| < +\infty$.

(XI) $|p(t,x,y,z)| \leq q_1(t) + \varepsilon(|x| + |y| + |x|)$ for $t \geq 0$ and $|x| + |y| + |z| \geq K$, where ε and K are positive constants and $q_1(t)$ is a non-negative function.

(XII) $\sup_{t\geq 0} \int_t^{t+1} q_1(s) ds < +\infty$.

Our main results are as follows:

First, we prove that it is necessary for assumptions (X) to hold, whenever the solutions of (1.1) are uniformly ultimate bounded.

Theorem 2.1. *Suppose that the solutions of equation* (1.1) *are uniformly ultimately bounded. Then,*

$$\left| e^{-t} \int_0^t e^s p(s) ds \right| < \infty.$$

Proof. Let $x(t)$ be a solution of (1.1) such that $(x(t), \dot{x}(t), \ddot{x}(t))$ are uniformly ultimately bounded. Then, $|x(t)| \leq L_1,\ |\dot{x}(t)| \leq L_2,\ |\ddot{x}(t)| \leq L_3,$ for all $t > 0$ and by the integration by parts we have

$$\begin{aligned}\left|e^{-t}\int_0^t e^s p(s)ds\right| &= \left|e^{-t}\int_0^t e^s \dddot{x}(s)ds + e^{-t}\int_0^t e^s a(s)f(x(x),\dot{x}(s),\ddot{x}(s))ds\right.\\ &\quad \left.+\, e^{-t}\int_0^t e^s b(s)g(x,\dot{x}(s))ds + e^{-t}\int_0^t e^s c(s)h(x(s))ds\right|\\ &\leq |\ddot{x}(t)| + e^{-t}\,|\ddot{x}(0)| + e^{-t}\int_0^t e^s|\ddot{x}(x)|ds\\ &\quad + A\overline{f}e^{-t}\int_0^t e^s\,|\ddot{x}(s)|\,ds + B\overline{g}e^{-t}\int_0^t e^s\,|\dot{x}(s)|\,ds\\ &\quad + Ce^{-t}\int_0^t e^s\,|h(x(s))|\,ds < \infty.\end{aligned}$$

□

Now we state the main results that it suffices to have assumption (X) for the existence of uniform ultimate boundedness of solutions on (1.1).

Theorem 2.2. *Suppose that assumptions (I) through (IX) hold. Then the solution $x(t)$ of* (1.1) *and their derivatives $\dot{x}(t)$ and $\ddot{x}(t)$ are uniformly ultimately bounded if (X) holds.*

Remark 2.1. It should be pointed out that in the special case where $a(t)f(x,y,z) = az$, $b(t)g(x,y) = by$ and $c(t)h(x) = cx$ in (1.1) and (1.2), where a, b, and c are constants, (so that all the assumptions (I), (II), (IV) (V), (VI), (VIII) and (IX) are automatically fulfilled), the assumptions (III) and (VII) reduce to $a > 0$, $b > 0$, $c > 0$, $ab - c > 0$, which is the Routh-Hurwitz criterion for the global asymptotic stability of the zero solution of the equation $\dddot{x} + a\ddot{x} + b\dot{x} + cx = 0$. see, for example, [25]

Next, for equation (1.2) we have:

Theorem 2.3. *Suppose that assumptions (I)-(IX), (XI) and (XII) hold. Then there exist a continuous $\varepsilon_0{=}\varepsilon_0(A, a_0, B, b_0, C, c_0, \delta, \overline{f}, f_0, \overline{g}, g_0, h_1)$ > 0 such that if $\varepsilon < \varepsilon_0$ then the solution $x(t)$ of* (1.2) *and their derivatives $\dot{x}(t)$ and $\ddot{x}(t)$ are uniformly ultimately bounded.*

Remark 2.2. These results are generalization of the results of Tunç [29], Hara [19] and Ezeilo [14] to equations (1.1) and (1.2).

3. Preliminaries for the proofs of the main results

We note that equation (1.1) is equivalent the following system of differential equations

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z + P(t) \\ \dot{z} &= -a(t)f(x,y,\dot{y}) - b(t)g(x,y) - c(t)h(x) + P(t) \end{aligned} \tag{3.1}$$

where $P(t) \equiv e^{-t}\int_0^t e^s p(s)ds$.

In proving Theorem 2.2, we shall use the following Lyapunov function

$$\begin{aligned} V(t,x,y,z) =& c(t)(\mu_1 + a_0 f_0)\int_0^x h(\xi)d\xi + 2c(t)h(x)y + 2b(t)\int_0^y g(x,\eta)d\eta \\ &+ \left(1 + \frac{\delta}{2}\right)z^2 + \frac{1}{2}(a^2(t)f_0^2 - \mu_2)y^2 + (a(t)f_0 + \mu_1)yz \\ &+ \frac{1}{2}\mu_2 b(t)g_0 x^2 + \mu_2 a(t)f_0 xy + \mu_2 xz + \frac{1}{2}a(t)f_0\mu_1 y^2. \end{aligned} \tag{3.2}$$

Lemma 3.1. *There exist positive constants k_1 and k_2 such that*

$$k_1(x^2 + y^2 + z^2) \le V(t,x,y,z) \le k_2(x^2 + y^2 + z^2) \tag{3.3}$$

for all $t \ge 0$ and $(x,y,z) \in \mathbb{R}^3$, where

$$2k_1 := \min\left\{ \begin{array}{l} c_0\delta\left(\frac{a_0b_0f_0g_0 - C\delta}{Bg_0}\right) + \mu_2(b_0g_0 - \mu_2), \\ \\ \mu_1(a_0f_0 - \mu_1) + b_0g_0 - \mu_2 - \frac{C\delta}{\mu_1}, \delta \end{array} \right\}$$

and

$$k_2 := \frac{1}{2}\max\left\{ \begin{array}{l} [(\mu_1 + a_0f_0 + 2)a_0b_0f_0g_0 + \mu_2 Bg_0 + \mu_2(Af_0 + 1)], \\ \\ [(Af_0 + \mu_1)(Af_0 + 1) + 2B\overline{g} + 2a_0b_0f_0g_0 + \mu_2 Af_0], \\ \\ (Af_0 + \mu_1 + \mu_2 + 2 + \delta) \end{array} \right\}.$$

Proof. From the definition of $2V(t,x,y,z)$ in (3.2), we easily rearrange

$2V$ as

$$\begin{aligned}
2V &= 2c(t)(\mu_1 + a_0 f_0)\int_0^x h(\xi)d\xi + 4c(t)\frac{h(x)}{x}xy \\
&\quad +4b(t)\int_0^y g(x,\eta)d\eta + z^2 - \mu_2 y^2 + 2\mu_1 yz \\
&\quad +\mu_2 b(t)g_0 x^2 - \mu_2^2 x^2 + (\mu_2 x + a(t)f_0 y + z)^2 \\
&\quad +a(t)f_0\mu_1 y^2 + \delta z^2 \\
&\geq c_0\delta\mu_1\left(x + \frac{1}{\mu_1}y\right)^2 + b_0 g_0\left(y + \frac{c_0\delta}{b_0 g_0}x\right)^2 \\
&\quad +(z + \mu_1 y)^2 + (\mu_2 x + a_0 f_0 y + z)^2 + \delta z^2 \\
&\quad +\left[c_0\delta\left(\frac{a_0 b_0 f_0 g_0 - C\delta}{Bg_0}\right) + \mu_2(b_0 g_0 - \mu_2)\right]x^2 \\
&\quad +\left[\mu_1\left(a_0 f_0 - \mu_1\right) + b_0 g_0 - \mu_2 - \frac{C\delta}{\mu_1}\right]y^2 \\
&\geq 2k_1(x^2 + y^2 + z^2)
\end{aligned}$$

where

$$2k_1 := \min\left\{\begin{array}{c} c_0\delta\left(\frac{a_0 b_0 f_0 g_0 - C\delta}{Bg_0}\right) + \mu_2(b_0 g_0 - \mu_2), \\ \\ \mu_1\left(a_0 f_0 - \mu_1\right) + b_0 g_0 - \mu_2 - \frac{C\delta}{\mu_1}, \delta \end{array}\right\}.$$

On the other hand, using the assumption I-XII, and a rearranged of $2V$ in (3.2), we have

$$\begin{aligned}
2V &\leq \left[2C(\mu_1 + a_0 f_0)\frac{a_0 b_0 f_0 g_0}{2C} + \mu_2 B g_0\right]x^2 \\
&\quad + \left[A^2 f_0^2 + A f_0\mu_1 + 2B\overline{g}\right]y^2 + (2+\delta)\,z^2 \\
&\quad + \left[2C\frac{a_0 b_0 f_0 g_0}{C} + \mu_2 A f_0\right](x^2 + y^2) \\
&\quad +(Af_0 + \mu_1)(y^2 + z^2) + \mu_2(x^2 + z^2) \\
&\leq \left[(\mu_1 + a_0 f_0 + 2)a_0 b_0 f_0 g_0 + \mu_2 B g_0 + \mu_2(A f_0 + 1)\right]x^2 \\
&\quad + \left[(Af_0 + \mu_1)(Af_0 + 1) + 2B\overline{g} + 2a_0 b_0 f_0 g_0 + \mu_2 A f_0\right]y^2 \\
&\quad + \left(Af_0 + \mu_1 + \mu_2 + 2 + \delta\right)z^2 \\
&\leq k_2(x^2 + y^2 + z^2),
\end{aligned}$$

where

$$2k_2 := \max\left\{\begin{array}{l}[(\mu_1 + a_0 f_0 + 2)a_0 b_0 f_0 g_0 + \mu_2 B g_0 + \mu_2(Af_0 + 1)]\,, \\ \\ [(Af_0 + \mu_1)(Af_0 + 1) + 2B\overline{g} + 2a_0 b_0 f_0 g_0 + \mu_2 A f_0]\,, \\ \\ (Af_0 + \mu_1 + \mu_2 + 2 + \delta)\end{array}\right\}.$$

□

Next, along any solution $(x(t), y(t), z(t))$ of (3.1), we have

$$\text{(3.4)}\quad \dot{V}|_{(3.1)}(t,x,y,z) \;=\; -W(t,x,y,z) + U(t,x,y,z) + P_1(t,x,y,z),$$

where

$$\begin{aligned} U(t,x,y,z) \;=\;& c'(t)\left[(\mu_1 + a_0 f_0)\int_0^x h(\xi)d\xi + 2h(x)y\right] \\ &+b'(t)\left[2\int_0^y g(x,\eta)d\eta + \frac{1}{2}\mu_2 g_0 x^2\right] + 2b(t)y\int_0^y g_x(x,\eta)d\eta \qquad \text{(3.5)} \\ &+a'(t)\left[a(t)f_0 y^2 + f_0 yz + \mu_2 f_0 xy + \frac{1}{2}f_0\mu_1 y^2\right], \end{aligned}$$

$$\begin{aligned} W(t,x,y,z) =\;& \mu_2 c(t)\frac{h(x)}{x}x^2 \\ &+ \left[b(t)(a(t)f_0 + \mu_1)\left(\frac{g(x,y)}{y} - g_0\right) + b(t)g_0(a(t)f_0 + \mu_1)\right. \\ &\left. - 2c(t)h'(x) - \mu_2 a(t) f_0\right] y^2 \\ &+ \left[(a(t)f_0 - \mu_1) + (2+\delta)a(t)\left(\frac{f(x,y,z)}{z} - f_0\right) + \delta a(t) f_0\right] z^2 \qquad \text{(3.6)} \\ &- \left[c(t)\frac{h(x)}{x}(a(t) - a_0) + \mu_2 b(t)\left(\frac{g(x,y)}{y} - g_0\right)\right] xy \\ &+ \left(\frac{f(x,y,z)}{z} - f_0\right)(a(t)f_0 + \mu_1)yz + \delta b(t)\frac{g(x,y)}{y}yz \\ &+ \mu_2 a(t)\left(\frac{f(x,y,z)}{z} - f_0\right)xz + \delta c(t)\frac{h(x)}{x}xz \end{aligned}$$

and

$$\begin{aligned} P_1(t,x,y,z) = \big[&2\,c(t)h(x) + 2b(t)g(x,y) + (a^2(t)f_0^2 - \mu_2)y + (a(t)f_0 + \mu_1)z \\ &+\mu_2 a(t)f_0 x + a(t)f_0\mu_1 y + (2+\delta)z + (a(t)f_0+\mu_1)y + \mu_2 x \\ &- \{(2+\delta)a(t)z + a(t)(a(t)f_0+\mu_1)y + \mu_2 a(t)x\}\,\frac{f(x,y,\dot{y})}{\dot{y}}\Bigg]\,P(t). \end{aligned} \tag{3.7}$$

The following are immediate consequences of the basic assumptions with some form of rearrangements of the terms:

Lemma 3.2. *There exists a positive constant* k_3 *such that for all* $t \geq 0$, *and* $(x,y,z) \in \mathbb{R}^3$,

$$U(t,x,y,z) \quad \leq \quad k_3(|a'(t)| + b'_+(t) + |c'(t)|)(x^2+y^2+z^2). \tag{3.8}$$

where $k_3 = \max\{l_1, l_2, l_3\}$ *with*

$$l_1 = \max\left\{(\mu_1 + a_0 f_0 + 2)\frac{a_0 b_0 f_0 g_0}{2C}, \frac{a_0 b_0 f_0 g_0}{C}\right\},\; l_2 = \max\left\{\frac{1}{2}\mu_2 g_0, \overline{g}, 1\right\}$$

$$l_3 = \frac{1}{2}\max\{\mu_2 f_0, f_0\,(2A + \mu_1 + \mu_2 + 1)\,, f_0\}\,.$$

Proof. Using the schwaltz inequality $|uv| \leq \frac{1}{2}(u^2+v^2)$ for any reals $u, v \in \mathbb{R}$, we rearrange

$$\begin{aligned} U(t,x,y,z) = c'(t)&\left[(\mu_1 + a_0 f_0)\int_0^x \frac{h(\xi)}{\xi}\xi\;d\xi + 2\frac{h(x)}{x}xy\right] \\ &+ b'(t)\left[2\int_0^y \frac{g(x,\eta)}{\eta}\eta\;d\eta + \frac{1}{2}\mu_2 g_0 x^2\right] + 2b(t)y\int_0^y g_x(x,\eta)d\eta \\ &+ a'(t)\left[\frac{1}{2}f_0(2a(t)+\mu_1)y^2 + f_0 yz + \mu_2 f_0 xy\right] \\ &\leq c'(t)\left[(\mu_1 + a_0 f_0)\frac{a_0 b_0 f_0 g_0}{2C}x^2 + 2\frac{a_0 b_0 f_0 g_0}{C}xy\right] \\ &+ b'(t)\left[\overline{g}y^2 + \frac{1}{2}\mu_2 g_0 x^2\right] \\ &+ a'(t)\left[\frac{1}{2}f_0(2A+\mu_1)y^2 + f_0 yz + \mu_2 f_0 xy\right] \\ &\leq |c'(t)|\left[(\mu_1 + a_0 f_0 + 2)\frac{a_0 b_0 f_0 g_0}{2C}x^2 + \frac{a_0 b_0 f_0 g_0}{C}y^2\right] \end{aligned}$$

$$+ b'(t)\left[\overline{g}y^2 + \frac{1}{2}\mu_2 g_0 x^2\right]$$
$$+ |a'(t)|\left[\frac{1}{2}\mu_2 f_0 x^2 + \frac{1}{2} f_0 (2A + \mu_1 + \mu_2 + 1)y^2 + \frac{1}{2} f_0 z^2\right]$$
$$\leq k_3(|a'(t)| + b'_+(t) + |c'(t)|)\left(x^2 + y^2 + z^2\right),$$

where $k_3 = \max\{l_1, l_2, l_3\}$ with

$$l_1 = \max\left\{(\mu_1 + a_0 f_0 + 2)\frac{a_0 b_0 f_0 g_0}{2C}, \frac{a_0 b_0 f_0 g_0}{C}\right\},\ l_2 = \max\left\{\frac{1}{2}\mu_2 g_0, \overline{g}, 1\right\}$$
$$l_3 = \frac{1}{2}\max\{\mu_2 f_0, f_0(2A + \mu_1 + \mu_2 + 1), f_0\}.$$

□

Lemma 3.3. *There exists a positive constant k_4 such that for all $t \geq 0$, and $(x, y, z) \in \mathbb{R}^3$,*

$$W(t, x, y, z) \quad \geq \quad k_4(x^2 + y^2 + z^2), \tag{3.9}$$

where $k_4 = \max\{l_4, l_5, l_6\}$ with

$$l_4 = \Big[\mu_2 c_0 \delta - \frac{\varepsilon_1}{2}\left\{a_0 b_0 f_0 g_0 (A - a_0) + \mu_2\left[A\left(\overline{f} - f_0\right) + B\left(\overline{g} - g_0\right)\right]\right.$$
$$\left. + \delta a_0 b_0 f_0 g_0\right\}\Big],$$
$$l_5 = \Big[\{\mu_1 b_0 g_0 - Ch_1\} + \{a_0 b_0 f_0 g_0 - Ch_1 - \mu_2 A f_0\} - \frac{\varepsilon_2}{2}\left\{a_0 b_0 f_0^2 g_0 (A - a_0)\right.$$
$$\left. + \mu_2 B(\overline{g} - g_0) + A(Af_0 + \mu_1)(\overline{f} - f_0) + \delta B\overline{g}\right\}\Big]$$
$$l_6 = \Big[a_0 f_0 - \mu_1 + \delta a_0 f_0 - \frac{\varepsilon_3}{2}\left\{A(Af_0 + \mu_1)(\overline{f} - f_0) + \delta B\overline{g}\right.$$
$$\left. + \mu_2 A(\overline{f} - f_0) + \delta a_0 b_0 f_0 g_0\right\}\Big].$$

Proof. Again using the schwaltz inequality $|uv| \leq \frac{1}{2}(u^2 + v^2)$ for any reals $u, v \in \mathbb{R}$, with some rearrangements, we have

$$W(t, x, y, z) = \mu_2 c(t)\frac{h(x)}{x}x^2$$
$$+ \Big[b(t)(a(t)f_0 + \mu_1)\left(\frac{g(x, y)}{y} - g_0\right) + (b(t)g_0\mu_1 - c(t)h'(x))$$
$$+ (a(t)b(t)f_0 g_0 - c(t)h'(x) - \mu_2 a(t)f_0)\Big]y^2$$

$$
\begin{aligned}
&+\left[a(t)f_0-\mu_1+\delta f_0 a(t)+(2+\delta)a(t)\left(\frac{f(x,y,\dot{y})}{\dot{y}}-f_0\right)\right]z^2\\
&+\left[f_0c(t)\frac{h(x)}{x}(a(t)-a_0)+\mu_2 b(t)\left(\frac{g(x,y)}{y}-g_0\right)\right]xy\\
&+\left[\left(\frac{f(x,y,\dot{y})}{\dot{y}}-f_0\right)a(t)(a(t)f_0+\mu_1)+\delta b(t)\frac{g(x,y)}{y}\right]yz\\
&+\left[\mu_2 a(t)\left(\frac{f(x,y,\dot{y})}{\dot{y}}-f_0\right)+\delta c(t)\frac{h(x)}{x}\right]xz\\
&=s_1x^2+s_2y^2+s_3z^2+l_1xy+l_2yz+l_3xz\\
&\geq s_1x^2+s_2y^2+s_3z^2-\frac{1}{2}l_1(x^2+y^2)\\
&-\frac{1}{2}l_2(y^2+z^2)-\frac{1}{2}l_3(x^2+z^2)\\
&\geq\left[s_1-\frac{\varepsilon_1}{2}(l_1+l_3)\right]x^2+\left[s_2-\frac{\varepsilon_2}{2}(l_1+l_2)\right]y^2\\
&+\left[s_3-\frac{\varepsilon_3}{2}(l_2+l_3)\right]z^2,\\
&\geq k_4(x^2+y^2+z^2)
\end{aligned}
$$

where $k_4=\max\{l_4,l_5,l_6\}$ with

$$
\begin{aligned}
s_1&=\mu_2c(t)\delta,\\
s_2&=\left[b(t)(a(t)f_0+\mu_1)\left(\frac{g(x,y)}{y}-g_0\right)+(b(t)g_0\mu_1-c(t)h_1)\right.\\
&\quad\left.+\,(a(t)b(t)f_0g_0-c(t)h_1-\mu_2a(t)f_0)\right],\\
s_3&=(a(t)f_0-\mu_1)+\delta f_0a(t)+(2+\delta)a(t)\left(\frac{f(x,y,\dot{y})}{\dot{y}}-f_0\right),\\
t_1&=f_0c(t)\frac{a_0b_0f_0g_0}{C}(a(t)-a_0)+\mu_2b(t)\left(\frac{g(x,y)}{y}-g_0\right),\\
t_2&=\left(\frac{f(x,y,\dot{y})}{\dot{y}}-f_0\right)a(t)(a(t)f_0+\mu_1)+\delta b(t)\overline{g}\\
t_3&=\mu_2a(t)\left(\frac{f(x,y,\dot{y})}{\dot{y}}-f_0\right)+\delta c(t)\frac{a_0b_0f_0g_0}{C}
\end{aligned}
$$

with $\varepsilon_1>0$, $\varepsilon_2>0$, $\varepsilon_3>0$ such that,

$l_4=s_1-\frac{\varepsilon_1}{2}(l_1+l_3)\geq 0,\ l_5=s_2-\frac{\varepsilon_2}{2}(l_1+l_2)\geq 0,\ l_6=s_3-\frac{\varepsilon_3}{2}(l_2+l_3)\geq 0.\square$

Lemma 3.4. *There exists a positive constant* k_5 *such that for all* $t \geq 0$, *and* $(x, y, z) \in \mathbb{R}^3$,

$$P_1(t, x, y, z) \quad \leq \quad \sqrt{3}k_5(x^2 + y^2 + z^2)^{1/2}|P(t)|, \tag{3.10}$$

where $P(t) = e^{-t}\int_0^t e^s p(s)ds$, *and*

$$k_5 \quad = \max \ \{2a_0b_0f_0g_0 + \mu_2 Af_0 + \mu_2, (1 + Af_0)(\mu_1 + Af_0) + 2B\overline{g}, \\ \mu_1 + Af_0 + 2 + \delta\}\,.$$

Proof. Following a similar method, we rearrange

$$\begin{aligned} P_1(t, x, y, z) = &\left[\left(2c(t)\frac{h(x)}{x} + \mu_2 a(t)f_0 + \mu_2 - \mu_2 a(t)\frac{f(x, y, \dot{y})}{\dot{y}}\right) x\right. \\ &+ \left(a(t)f_0 + \mu_1 + 2 + \delta - a(t)(a(t)f_0 + \mu_1)\frac{f(x, y, \dot{y})}{\dot{y}}\right) z \\ &+ \left(2b(t)\frac{g(x, y)}{y} + a^2(t)f_0^2 - \mu_2 + a(t)f_0\mu_1 + a(t)f_0\right. \\ &\left.\left. +\mu_1 - (2 + \delta)a(t)\frac{f(x, y, \dot{y})}{\dot{y}}\right) y\right] P(t) \\ &\leq [(2a_0b_0f_0g_0 + \mu_2 f_0 A + \mu_2)\,|x| \ + (Af_0 + \mu_1 + 2 + \delta)|z| \\ &+ (2B\overline{g} + (Af_0 + 1)(Af_0 + \mu_1))\,|y|\ ]\,P(t) \\ &\leq k_5(|x| + |y| + |z|)|P(t)| \leq \sqrt{3}k_5(x^2 + y^2 + z^2)^{1/2}|P(t)|, \end{aligned}$$

where

$$k_5 \quad = \max \ \Big\{2a_0b_0f_0g_0 + \mu_2 Af_0 + \mu_2, (1 + Af_0)(\mu_1 + Af_0) + 2B\overline{g}, \\ \mu_1 + Af_0 + 2 + \delta\ \Big\}\,.$$

□

4. Completion of the Proof of Theorem 2.2

We note that along any solution $(x(t), y(t), z(t))$ of (3.1), we have for all $t \geq 0$ and $(x, y, z) \in \ \mathbb{R}^3$, using Lemma 3.1,

$$\begin{aligned} \dot{V}|_{(3.1)}(t, x, y, z) &= -W(t, x, y, z) + U(t, x, y, z) + P_1(t, x, y, z), \\ &\leq -(k_4 - k_3(|a'(t)| + b'_+(t) + |c'(t)|))(x^2 + y^2 + z^2) \\ &+ \sqrt{3}k_5\,|P(t)|\,(x^2 + y^2 + z^2)^{1/2} \end{aligned}$$

$$\le -\left(\frac{k_4}{k_2} - \frac{k_3}{k_1}(|a'(t)| + b'_+(t) + |c'(t)|)\right) V(t,x,y,z)$$

$$+ \frac{\sqrt{3}k_5}{\sqrt{k_1}} |P(t)| V(t,x,y,z)^{1/2}. \tag{4.1}$$

We can now conclude the proof of Theorem 2.2 by choosing in Lemma 1.1,

$$c = \frac{k_4}{k_2}, \lambda_1(t) = \frac{k_3}{k_1}\left(|a'(t)| + b'_+(t) + |c'(t)|\right)$$

and

$$\lambda_2(t) = \frac{\sqrt{3}k_5}{\sqrt{k_1}}|P(t)|.$$

We note that by assumptions (IX) and (X),

$$\limsup_{(t,\nu)\to(\infty,\infty)} \frac{1}{\nu}\int_t^{t+\nu} \lambda_1(s)ds$$

$$= \frac{k_3}{k_1} \limsup_{(t,\nu)\to(\infty,\infty)} \frac{1}{\nu}\int_t^{t+\nu} \left(|a'(s)| + b'_+(s) + |c'(s)|\right) ds < \frac{k_3}{k_1}\gamma,$$

where

$$\gamma = \frac{k_1 k_4}{2k_2 k_3},$$

and

$$\sup_{t>0}\int_t^{t+1} \lambda_2(s)ds = \frac{\sqrt{3}k_5}{\sqrt{k_1}} \sup_{t>0}\int_t^{t+1} |P(s)|ds < \infty.$$

Hence the conclusions of Theorem 2.2 follow from Lemma 1.1. Thus, the proof of Theorem 2.2 is complete.

5. Completion of the proof of Theorem 2.3

Proof. (Proof of Theorem 2.3) Equation (1.2) is equivalent to the following system of differential equations

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= -a(t)f(x,y,z) - b(t)g(x,y) - c(t)h(x) + p(t,x,y,z). \end{aligned} \tag{5.1}$$

Consider the Lyapunov function $V(t,x,y,z)$ defined by (3.2). Then inequalities (3.3) hold for all $t \geq 0$ and $(x,y,z) \in \mathbb{R}^3$.

Along any solution $(x(t),y(t),z(t))$ of (5.1), we have

$$\dot{V}_{(5.1)} = -W(t,x,y,z) + U(t,x,y,z) + P_2(t,x,y,z)$$

where $W(t,x,y,z)$ and $U(t,x,y,z)$ are the functions defined by equation (3.5) and (3.6), and

$$P_2(t,x,y,z) = \{(2+\delta)z + (a(t)f_0 + \mu_1)y + \mu_2 x\}\, p(t,x,y,z).$$

Clearly, from assumption (XI), $|p(t,x,y,z)| \leq q_1(t) + \varepsilon(|x|+|y|+|z|)$, for all $t \geq 0$, sufficiently small $\varepsilon \geq 0$ and K large enough such that $|x|+|y|+|z| \geq K$. Thus, there exist a constant $k_6 > 0$, such that $P_2(t,x,y,z)$ satisfies

$$|P_2(t,x,y,z)| \;\leq\; 3\varepsilon k_6(x^2+y^2+z^2) + \sqrt{3}k_6 q_1(t)(x^2+y^2+z^2)^{1/2}.$$

We note that along any solution $(x(t),y(t),z(t))$ of (5.1), we have for all $t \geq 0$ and $(x,y,z) \in \mathbb{R}^3$, using Lemma 3.1,

$$\begin{aligned}
\dot{V}|_{(5.1)}(t,x,y,z) &= -W(t,x,y,z) + U(t,x,y,z) + P_2(t,x,y,z),\\
&\leq -(k_4 - 3\varepsilon k_6 - k_3(|a'(t)| + b'_+(t) + |c'(t)|))(x^2+y^2+z^2)\\
&+ \sqrt{3}k_6\,|q_1(t)|\,(x^2+y^2+z^2)^{1/2} \qquad (5.2)\\
&\leq -\left\{\frac{k_4}{k_2} - \frac{3k_6\varepsilon}{k_1} - \frac{k_3}{k_1}(|a'(t)| + b'_+(t) + |c'(t)|)\right\} V(t,x,y,z)\\
&+ \frac{\sqrt{3}k_6}{\sqrt{k_1}}\,|q_1(t)|\, V(t,x,y,z)^{1/2}.
\end{aligned}$$

We can now conclude the proof of Theorem 2.2 by choosing in Lemma 1.1

$$c = \frac{k_4}{k_2},\;\; \lambda_1(t) = \frac{k_3}{k_1}\big(|a'(t)| + b'_+(t) + |c'(t)|\big) + \frac{3k_6\varepsilon}{k_1},\;\; \lambda_2(t) = \frac{\sqrt{3}k_6}{\sqrt{k_1}}|q_1(t)|.$$

Also from assumptions (IX) and (X),

$$\begin{aligned}
&\limsup_{(t,\nu)\to(\infty,\infty)} \frac{1}{\nu}\int_t^{t+\nu} \lambda_1(s)ds\\
&= \frac{k_3}{k_1}\limsup_{(t,\nu)\to(\infty,\infty)} \frac{1}{\nu}\int_t^{t+\nu} \big(|a'(s)| + b'_+(s) + |c'(s)|\big)\,ds + \frac{3k_6\varepsilon}{k_1} < \frac{k_3}{k_1}\gamma + \frac{3k_6\varepsilon}{k_1}
\end{aligned}$$

and

$$\sup_{t>0}\int_t^{t+1}\lambda_2(s)ds = \frac{\sqrt{3}k_6}{\sqrt{k_1}}\sup_{t>0}\int_t^{t+1}|q_1(s)|ds < \infty.$$

hold. Hence the conclusions of Theorem 2.3 follow from Lemma 1.1 and thus, the proof of Theorem 2.3 is complete. □

6. Example 1

Example 6.1. We consider the following third order non-autonomous differential equation

$$\begin{aligned}
&\dddot{x} + (\cos(t)+2)(\cos(x+\dot{x})\sin\ddot{x} + 11\ddot{x}) \\
&+\left(\sin\left(\frac{t}{2}\right)+2\right)\left(-\tan^{-1}(x\dot{x})\frac{\sin^2\dot{x}}{\pi\dot{x}} + \frac{5\dot{x}}{2}\right) \\
&+(\sin(t)+2)\left(4x + \frac{x}{1+x^2}\right) = \frac{1}{1+t^2}
\end{aligned}$$

whose associated system is

$$\begin{aligned}
&\dot{x} = y, \quad \dot{y} = z \\
&\dot{z} = -(\cos(t)+2)(\cos(x+y)\sin z + 11z) \\
&\quad -\left(\sin\left(\frac{t}{2}\right)+2\right)\left(-\tan^{-1}(xy)\frac{\sin^2 y}{\pi y} + \frac{5y}{2}\right) \\
&\quad -(\sin(t)+2)\left(4x + \frac{x}{1+x^2}\right) + \frac{1}{1+t^2}.
\end{aligned}$$

Now, it is easy see

$$\begin{aligned}
1 &\le a(t) = (\cos(t)+2) \le 3; 1 \le b(t) = \left(\sin\left(\frac{t}{2}\right)+2\right) \le 3; \\
1 &\le c(t) = (\sin(t)+2) \le 3; \\
10 &\le \frac{f(x,y,z)}{z} = \cos(x+y)\frac{\sin z}{z} + 11 \le 12; \\
2 &\le \frac{g(x,y)}{y} = -\tan^{-1}(xy)\frac{\sin^2 y}{\pi y^2} + \frac{5}{2} \le 3; \\
4 &\le \frac{h(x)}{x} = 4 + \frac{1}{1+x^2} \qquad \text{and } h'(x) \le 5; \quad \text{and} \\
& g_x(x,y) = -\frac{1}{\pi(1+(xy)^2)}\sin^2 y \le 0.
\end{aligned}$$

All the assumptions (I) through (X) are satisfied and with $p(t) = \frac{1}{1+t^2}$, we can conclude using Theorem 2.2 that the solutions of equation (6.1) are uniformly ultimately bounded together with its first and second derivatives.

7. Example 2

Example 7.1. We consider the following third order non-autonomous differential equation.

$$\dddot{x} + (\cos(t) + 2)(\cos(x + \dot{x})\sin\ddot{x} + 11\ddot{x})$$
$$+ \left(\sin\left(\frac{t}{2}\right) + 2\right)\left(-\tan^{-1}(x\dot{x})\frac{\sin^2\dot{x}}{\pi\dot{x}} + \frac{5\dot{x}}{2}\right)$$
$$+(\sin(t) + 2)\left(4x + \frac{x}{1+x^2}\right) = \frac{1}{1+t^2+x^2+\dot{x}^2+\ddot{x}^2}.$$

Now, it is easy see that

$$\begin{aligned}
1 &\le a(t) = (\cos(t) + 2) \le 3; 1 \le b(t) = \left(\sin\left(\frac{t}{2}\right) + 2\right) \le 3;\\
1 &\le c(t) = (\sin(t) + 2) \le 3;\\
10 &\le \frac{f(x,y,z)}{z} = \cos(x+y)\frac{\sin z}{z} + 11 \le 12;\\
2 &\le \frac{g(x,y)}{y} = -\tan^{-1}(xy)\frac{\sin^2 y}{\pi y^2} + \frac{5}{2} \le 3;\\
4 &\le \frac{h(x)}{x} = 4 + \frac{1}{1+x^2} \quad \text{and} \quad h'(x) \le 5;
\end{aligned}$$

and

$$\begin{aligned}
g_x(x,y) &= -\frac{1}{\pi(1+(xy)^2)}\sin^2 y \le 0,\\
p(t,x,y,z) &= \frac{1}{1+t^2+x^2+y^2+z^2} \le \frac{1}{1+t^2} + \varepsilon(x^2+y^2+z^2)
\end{aligned}$$

where $\varepsilon \ge 0$ and

$$\sup_{t>0}\int_t^{t+1}\frac{1}{1+s^2}ds < \infty.$$

All the assumptions (I) through (XI) are satisfied and with

$$q_1(t) = \frac{1}{1+t^2},$$

satisfying assumption (XII), we can conclude using Theorem 2.3 that the solutions of equation (7.1) are uniformly ultimately bounded together with its first and second derivatives.

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Received: 1.II.2010

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