ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
MATEMATICĂ, Tomul LVIII, 2012, f.2

ON ABELIAN GROUPS WITH COMMUTATIVE CLEAN ENDOMORPHISM RINGS

BY

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Abstract. Results on some classes of Abelian groups having clean and commutative endomorphism rings are given. Some open questions are stated.

Mathematics Subject Classification 2010: 20K30, 16L30.

Key words: (strongly) clean ring, commutative endomorphism ring, strongly indecomposable, purely indecomposable.

1. Introduction

A ring with identity is called *clean* if every element decomposes as a sum of an idempotent and a unit, and *strongly clean* if in this decomposition the idempotent and the unit commute. As everybody does it nowadays for modules, an Abelian group will be called (strongly) clean if its endomorphism ring is such.

In dealing separately with clean Abelian groups, or strongly clean Abelian groups, two different difficulties occur: not every direct summand of a clean group is clean, but (finite) direct sums of clean groups are clean, respectively direct summands of strongly clean groups are strongly clean but (finite) direct sums of strongly clean groups may not be strongly clean. Moreover, clean is a Morita invariant property but strongly clean is not.

Recent research on torsion clean Abelian groups (see [6]) shows that clean p-groups exist in such abundance that they are unlikely to be classifiable by any (known or) reasonable set of invariants. On the other hand, the studies devoted to the structure of the Abelian groups having a commutative endomorphism ring were abandoned from long time due to the lack of structure theory in the torsion-free class.

Obviously, these separate difficulties may disappear, when the endomorphism ring of an Abelian group is supposed to be clean and *also* commutative. It is the purpose of this note to give some information about Abelian groups having a commutative (strongly) clean endomorphism ring.

We first mention that a ring R is strongly regular if and only if R is regular and *abelian* (all idempotents are central).

Hence, for commutative rings, strongly regular $\equiv$ unit-regular $\equiv$ regular, and all these $\Longrightarrow$ strongly π-regular $\Longrightarrow$ strongly clean $\equiv$ clean.

Example 1. Any pure subgroup G of $\prod_p \mathbf{Z}(p)$ or $\mathbf{Q} \oplus (\bigoplus_p \mathbf{Z}(p))$.

Indeed, commutativity follows from the fact that $G = \ker(f) \oplus \mathrm{Im(f)}$ for every $f \in \mathrm{End}(G)$ implies $G_p \cong \mathbf{Z}(p)$; actually these groups have strongly regular endomorphism rings.

Results on Abelian groups with commutative endomorphism ring (mostly all due to Szele, Szendrei - 1951) are known from long time. Since some of them will be used in the sequel, here is a short summary:

Theorem A. *For a torsion group G the following conditions are equivalent*:

1) $\mathrm{End}(G)$ *commutative*;

2) *all endomorphic images of G are fully invariant subgroups*;

3) $G \cong \bigoplus_{p\in S} G_p$ *where G_p is a cocyclic p-group and p runs over some set S of* (*different*) *primes.*

Theorem B. *Let $G = T \oplus A$ be a splitting mixed group. Then* $\mathrm{End}(G)$ *is commutative if and only if $T = T(G) \cong \bigoplus_{p\in S} G_p$ where S is a set of primes with G_p is a finite cocyclic p-group, and $pA = A$ for all $p \in S$, and* $\mathrm{End}(A)$ *is commutative.*

Theorem C. *If G is a mixed group and* $\mathrm{End}(G)$ *is commutative then $T(G) \cong \bigoplus_{p\in S} G_p$ where S is a set of primes with G_p is a finite cocyclic p-group and $p(G/T(G)) = G/T(G)$, for every $p \in S$.*

2. The torsion groups

Due to recent results on clean Abelian groups (see [6]), we can easily dispose of the torsion case:

Proposition 2. *Let G be a torsion group. If* $\mathrm{End}(G)$ *is commutative then* $\mathrm{End}(G)$ *is (strongly) clean, and this happens if and only if* $G \cong \bigoplus_{p\in S} G_p$ *where G_p is a cocyclic p-group and p runs over some set S of (different) primes.*

Proof. A torsion group is clean if and only if all its primary components are clean, and, a countable p-group C is clean if $C = D \oplus B$, with D divisible and B bounded (see [6]). Since (by Theorem A) for a torsion group G, $\mathrm{End}(G)$ is commutative iff $G \cong \bigoplus_{p\in S} G_p$ where G_p is a cocyclic p-group (see [14]) and cocyclic p-groups are divisible or bounded, we finally infer that for torsion groups, if $\mathrm{End}(G)$ is commutative then it is also (strongly) clean. □

3. The divisible groups

Since every divisible group is clean, we just have to find the divisible groups with commutative endomorphism ring. For this purpose we use Theorem B (see [14]): indeed, every divisible group D is splitting, so if $D = T(D) \oplus F$ is its decomposition with torsion part $T(D)$, and $F \neq 0$ then $T(D) = 0$ (as direct sum of finite cocyclic p-groups).

Hence, *a divisible group with commutative endomorphism ring is torsion, or, torsion-free* (not genuine mixed).

Torsion divisible groups with commutative endomorphism rings are, according to the preceding section, *direct sums of quasicyclic groups* $\mathbf{Z}(p^\infty)$, *at most one, for every prime number p.*

Since torsion-free divisible groups are direct sums of $\mathbf{Q}$, and, completely decomposable groups with commutative endomorphism rings are exactly the rigid ones (e.g. see [9]), *the only divisible torsion-free group with commutative endomorphism ring is* $\mathbf{Q}$.

Notice that once again, for divisible groups, $\mathrm{End}(G)$ commutative implies $\mathrm{End}(G)$ (strongly) clean.

Now, the expected usual reduction theorem, *let $G = D(G) \oplus R$ with $D(G)$ the divisible part of G. Then* $\mathrm{End}(G)$ *is (strongly) clean and commutative if and only if* $\mathrm{End}(R)$ *is (strongly) clean and commutative,* must be slightly adapted (using the remarks above).

Since *direct summands of strongly clean Abelian groups are strongly clean,* and *if* $A_1, \dots, A_n$ *are clean then* $A_1 \oplus \dots \oplus A_n$ *is clean*, no trouble with the (strongly) clean part of the above statement.

However, since a divisible group with commutative endomorphism ring is torsion, or torsion-free, only the following is possible:

- $D(G) = \mathbf{Q}$ or, $D(G) = 0$, and R is mixed reduced;
- By Theorem C (see [14]), if G is mixed, its torsion part has only finite cocyclic summands, that is, if G is mixed, $T(G)$ must be reduced; hence, only torsion groups have a torsion divisible part.

Summarizing:

Proposition 3. *For Abelian groups having* (*strongly*) *clean and commutative endomorphism rings, with respect to divisible part decomposition, only the following is possible:*

(a) *G is torsion and G is a direct sum of non-isomorphic quasicyclic groups* (*say corresponding to* $p \in S$) *and a direct sum of finite cocyclic groups corresponding to prime numbers* $p \notin S$.

(b) *G is mixed reduced, and*

(c) $G \cong Q \oplus R$ *with R mixed reduced.*

Therefore the usual reduction to reduced groups is possible.

4. The torsion-free groups

Since there are only few and rather restricted classes of torsion-free groups for which a satisfactory theory was ever developed, to some extent, we will follow the sections of chapter XIII in Fuchs's treatise ([5]). As usual, due to the immense complexity of infinite rank torsion-free groups, we restrict ourselves to finite rank torsion-free groups having commutative and (strongly) clean endomorphism rings.

Since, any torsion-free group of finite rank decomposes into the direct sum of a finite number of indecomposable groups, we first focus on such groups.

First, recall that a ring is called (*finite*) *exchange* if ${}_RR$ has the (finite) exchange property (i.e., ${}_RM$ has the (*finite*) *exchange property* if for every module ${}_R A$ and any two decompositions $A = M' \oplus N = \bigoplus_{i \in I} A_i$ with $M' \cong M$ and (finite) set I there exist submodules $A'_i \subseteq A_i$ such that $A = M' \oplus (\bigoplus_{i \in I} A'_i)$.)

The next result settles the (strongly) clean part of our statements.

Theorem 4. *For an indecomposable torsion-free group G the following conditions are equivalent*

(i) End(G) *is local*;

(ii) End(G) (*strongly*) *clean*;

(iii) End(G) (*finite*) *exchange.*

Proof. First notice that for any ring

$$\text{local} \Longrightarrow \text{strongly clean} \Longrightarrow \text{clean} \Longrightarrow \text{exchange} \Longrightarrow \text{finite exchange.}$$

Then recall from [18], that for modules having an indecomposable decomposition, the finite exchange property implies the (full) exchange property.

Finally use two results of Warfield obtained in consecutive papers ([17] and [16]):

- *a left R-module M has the finite exchange property* (*as an R-module*) *if and only if* $\mathrm{End}_R(M)$ *is an exchange ring,* and
- *for an indecomposable group G, G has finite exchange if and only if* End(G) *is local,* (more precisely: a ring whose idempotents are only 0 and 1 is exchange if and only if it is local).

Therefore, for torsion-free indecomposables, the finite exchange property implies local, and so all properties are equivalent. □

Applications. 1) *Rank one torsion-free groups.* Clearly, **Q** has a clean and commutative endomorphism ring.

Further we deal with reduced torsion-free groups.

Proposition 5. *The only rational groups having clean and commutative endomorphism rings are the* $\mathbf{Q}^{(p)}$ *for every p prime number.*

Proof. Let R be a rational group. It is well-known that $\mathrm{End}(R)$ can be identified with a subring with identity of the field $\mathbf{Q}$, namely, the one which is generated by all the fractions $\frac{1}{p}$ with $pR = R$ (so now the commutativity is automatic).

Since the only idempotents (in $\mathbf{Q}$) are 0 and 1, we only have to find *the local subrings with unit of* $\mathbf{Q}$. But all these subrings are known: $\mathbf{Z}_p = \{\frac{m}{n} \in \mathbf{Q} | (n;p) = 1\}$. Thus these are clean (and commutative) rings and so $\mathbf{Q}^{(p)} = \{\frac{m}{n} \in \mathbf{Q} | (n;p) = 1\}$ are the only rank 1 torsion-free groups having clean and commutative endomorphism ring. □

2) *Completely decomposable torsion-free groups.* Recall (e.g., [9]) that: *for a completely decomposable group* G, $\mathrm{End}(G)$ *is commutative if and only if* G *is rigid.*

Using conclusions obtained above, we deduce

Proposition 6. *A reduced completely decomposable group has clean and commutative endomorphism group if and only if is a direct sum of* $\mathbf{Q}^{(p)}$ *for different primes* p.

Remarks. 1) These groups already appeared in [14], as examples of groups having commutative endomorphism rings.

2) This result extends to *separable (reduced) torsion-free groups.*

Indeed, for a *separable* torsion-free group $\mathrm{End}(G)$ is commutative (if and only if every endomorphic image of G is fully invariant, and) if and only if G is completely decomposable $G = \bigoplus_{i \in I} H_i$ with H_i pairwise incomparable types (see [8], Ex. 6, p. 132).

3) Once again, for reduced separable torsion-free groups, $\mathrm{End}(G)$ commutative $\Longrightarrow$ $\mathrm{End}(G)$ (strongly) clean.

Since systematic studies of torsion-free groups with commutative endomorphism rings do not exist, once again, there are better chances in studying torsion-free groups with clean **and** commutative endomorphism rings.

In general, neither $\mathrm{End}(G)$ commutative implies G indecomposable, nor, G indecomposable implies $\mathrm{End}(G)$ commutative, so we must restrict ourselves to some special classes of indecomposable torsion-free groups.

4.1. The strongly indecomposables

G is called *strongly indecomposable* if it is not quasi-decomposable (we denote by $Q\mathrm{End}(G)$ the quasiendomorphism ring of G). Results on strongly indecomposable groups having commutative endomorphism ring can be found in [15], (e.g., if G is strongly indecomposable of rank 2, then $\mathrm{End}(G)$ is commutative).

Moreover, some classes of rank 2 strongly indecomposables are characterized by $Q\mathrm{End}(G) \cong \mathbf{Q}$ (e.g., those whose typeset has cardinal at least 3 - see [2]), so by an argument below, $\mathrm{End}(G)$ is not necessarily clean.

The best result obtained in [15], is for irreducible groups (a group is *irreducible* [12] if it has no proper pure fully invariant subgroups) of prime or square-free rank: in this case $\mathrm{End}(G)$ commutative is equivalent to G strongly indecomposable.

Since it is well-known ([12], Corollary 4.3) that for finite rank torsion-free groups, G is strongly indecomposable is equivalent to $Q\mathrm{End}(G)$ local, a first attempt to cover all the strongly indecomposables would be to prove that $Q\mathrm{End}(G)$ local implies $\mathrm{End}(G)$ clean. Unfortunately this fails: for $Q\mathrm{End}(G) \cong \mathbf{Q}$, $\mathrm{End}(G)$ can be any subring (with 1) of $\mathbf{Q}$ (and we already saw before which rank 1 groups are clean $\equiv$ local). Thus, not all strongly indecomposables are clean and we can ask the following

Question: do irreducible strongly indecomposable of prime (or square-free) rank have (strongly) clean (that is, local) endomorphism rings? In the affirmative, is maybe only less sufficient?

[A possible hint: we should use

Theorem 15.2 (Reid, [2])**.** If G is strongly indecomposable, it is irreducible if and only if $Q\mathrm{End}(G)$ is a division algebra with $\dim_{\mathbf{Q}}(Q\mathrm{End}(G)) = \mathrm{rk}(G)$.]

4.2. The purely indecomposables

A (torsion-free) group G is *purely indecomposable* if every pure subgroup of G is indecomposable. It was proved by Griffith (see [7]) that any reduced purely indecomposable group is isomorphic to a pure subgroup of the group $J = \prod_{p\in\{2,3,5,\dots\}} J_p$ (the converse fails; Griffith characterizes the pure subgroups of J which are purely indecomposable).

All the nonzero pure subgroups of this direct product (and so all the purely indecomposables) have commutative endomorphism ring. Indeed, these are the groups such that for every p with $pG \neq G$ the p-basis subgroup B_p of G is cyclic. This happens exactly when $G/pG \cong \mathbf{Z}(p)$.

There are classes of (reduced) purely indecomposables, which have commutative and clean (i.e., local) endomorphism ring.

We first mention from CAMILLO ET AL (2006), [4]: *Every flat cotorsion module, and every pure-injective module is clean.*

Since a torsion-free group (is flat and) is algebraically compact (equivalent to pure-injective, for Abelian groups) if and only if it is cotorsion (see 54.5, [5]), *torsion-free algebraically compact groups are clean* (e.g., J_p). Since the only indecomposable algebraically compact groups are: J_p, $\mathbf{Q}$ and the subgroups of $\mathbf{Z}(p^\infty)$, the only torsion-free reduced is J_p.

For a pure subgroup $G \neq 0$ *of* J_p, $\mathrm{End}(G)$ *is local* (*and so* (*strongly*) *clean*) *if and only if* $G = PSoc(G)$, the pseudo-socle (i.e., the subgroup generated by the collection of non-zero minimal pure fully invariant subgroups of G). Hence (ORSATTI, [10]):

every *finite rank pure subgroup* G *of* J_p has $\mathrm{End}(G)$ local ($\equiv$ clean), and, if G *is pure in* J_p *and* $\mathrm{rank}(J_p/G) < \infty$ then $\mathrm{End}(G)$ is local ($\equiv$ clean).

Thus, all these have commutative and (strongly) clean endomorphism rings.

Remark. Not every purely indecomposable group has commutative and (strongly) clean endomorphism ring. It suffices to mention an early example due to ORSATTI, [11]: a pure subring of the ring of p-adic integers generated by 1 and an invertible element that is not an algebraic number.

Open questions: 1) Characterize all the purely indecomposables having local (and so (strongly) clean) endomorphism rings, with a special case: the *cohesive Dubois groups* (a torsion-free group G such that G/P is divisible for every $0 \neq P$ pure subgroup of G).

2) A group with commutative endomorphism ring has fully invariant direct summands (and kernels and images of endomorphisms). The converse fails. Counterexamples were given by Orsatti, Reid and Lawver.

Are there indecomposable groups among these examples? (i.e., does this converse fail, also for indecomposables?)

3) Since the class of all the Abelian groups with commutative and (strongly) clean endomorphism ring is, with respect to the (strongly) clean property, closed under direct summands and finite direct sums, the only thing left is, if $G = \bigoplus_{i=1}^n H_i$ is an indecomposable decomposition, under what conditions, $\mathrm{End}(G)$ commutative implies all $\mathrm{End}(H_i)$'s are commutative, and conversely?

Possible hints: ([12]) $G = PSoc(G)$ if and only if $Q\mathrm{End}(G)$ is semisimple, or/and, ([15]) if G is torsion-free of finite rank and $G = PSoc(G)$ then $\mathrm{End}(G)$ is commutative if and only if $G \overset{\bullet}{=} \bigoplus_{i\in I} H_i$ with H_i strongly indecomposable, $\mathrm{Hom}(H_i, H_j) = 0$ for $i \neq j$ and all $\mathrm{End}(H_i)$ are commutative.

5. The mixed groups

In the splitting case we would expect to have: *the endomorphism ring of a splitting mixed group* $G = T(G) \oplus F$ *is (strongly) clean and commutative if and only if both* $\mathrm{End}(T(G))$ *and* $\mathrm{End}(F)$ *are (strongly) clean and commutative.*

As we saw previously, (since this is true for (strongly) clean) this reduces to the commutativity statement, which just needs a little precaution with respect to the relevant primes (Theorem C):

Let $G = T \oplus A$ *be a splitting mixed group. Then* $\mathrm{End}(G)$ *commutative if and only if* $T = T(G) \cong \bigoplus_{p\in S} G_p$ *where* S *is a set of primes with* G_p *is a finite cocyclic p-group,* $pA = A$ *for all* $p \in S$*, and* $\mathrm{End}(A)$ *is commutative.*

Therefore everything reduces to reduced torsion-free groups.

Not every mixed group having commutative and (strongly) clean endomorphism ring is splitting.

Example: $\prod_p \mathbf{Z}(p)$.

However, this is true in the following case

Proposition 7. *For a finite rank group* G*,* $\mathrm{End}(G)$ *clean implies* G *splitting.*

Proof. A ring is called *I-finite* if it contains no infinite set of orthogonal idempotents. If $\mathrm{End}(G)$ is I-finite then clean $\equiv$ semiperfect, which implies semilocal. Hence G is splitting (see [3]). Since $\mathrm{End}(G)$ is I-finite is equivalent to G has finite Goldie dimension (i.e., finite rank), the proof is complete. □

In the general case, an improvement due to Schultz (see [13]), to Szele-Szendrei conditions, namely, necessary and sufficient conditions for a mixed group in order to have commutative endomorphism ring, could be a start for more results in this research.

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Received: 9.IX.2011

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