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KÖTHE SPACES OF VECTOR FIELDS

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Abstract. The standard theory of Köthe spaces of measurable functions is extended for vector fields on a separated locally compact space. Our theory extends the theory of Orlicz spaces of vector fields. Basic results, concerning completeness and relations between different types of convergence are obtained. The paper ends with two examples, illustrating the theoretical facts previously presented.

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1. Introduction

Function spaces have been a very important object of study in Functional Analysis. Successive generalizations appeared (e.g. from Lebesgue spaces L^p to Orlicz spaces and continuing with Köthe spaces). At the same time, the generalizations involved the range of the functions in the function spaces, considering first scalar valued functions and then vector valued functions. In this respect, the idea of considering vector fields instead of vector functions appeared in connection with differential geometry and physics.

A major step in the development of the theory of vector fields was the series of papers [2]-[5] by Dinculeanu, where a theory of Orlicz spaces of vector fields was introduced. The theory in [2]-[5] is sistematically presented in the monograph [6] by the same author.

The aim of the present paper is to continue the ideas in [2]-[5], initiating a more general theory: the theory of Köthe spaces of vector fields. To this end, we have intensively used the monographs [6] ,[8] and [1]. In order to have a suitable theory of measurability, compatible with the theory of vector fields, the framework of (locally compact) topological spaces as basic spaces seems natural. It is our intention to continue the development of this theory.

2. Preliminary facts

Throughout the paper, K will be the scalar field (either $K = \mathbb{R}$ = the real numbers, or $K = \mathbb{C}$ = the complex numbers). We shall write $\mathbb{R}_+ \stackrel{def}{=} [0, \infty)$ and $\overline{\mathbb{R}}_+ \stackrel{def}{=} [0, \infty]$. As usual, $\mathbb{N} = \{0, 1, 2, \ldots\}$ = the set of natural numbers and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

2.1. Assume (S,Σ,μ) is a *measure space*, i.e. S is a non empty set, Σ is a σ - algebra of subsets of S and $\mu : \Sigma \to \overline{\mathbb{R}}_+$ is a complete non null measure. Write $M_+(\mu) = \{u : S \to \overline{\mathbb{R}}_+ | \ u \ is \ \mu - measurable\}$.

A function norm is a function $\rho : M_+(\mu) \to \overline{\mathbb{R}}_+$ having the following properties (here u,v are in $M_+(\mu)$ and a is in $\mathbb{R}_+$):

(i) $\rho(u) = 0$ iff $u(t) = 0 \mu$ - a.e.;

(ii) $u \leq v \Rightarrow \rho(u) \leq \rho(v)$;

(iii) $\rho(u + v) \leq \rho(u) + \rho(v)$;

(iv) $\rho(au) = a\rho(u)$,

with the convention $0 \cdot \infty = 0$.

One knows that $\rho(u) < \infty \Rightarrow u(t) < \infty$ μ - a.e. and $u(t) \leq v(t)$ μ - a.e $\Rightarrow \rho(u) \leq \rho(v)$.

We say that ρ has *the Riesz - Fischer property* (and write ρ $R - F$) in case

$$\rho(\sum_{n=0}^{\infty} u_n) \leq \sum_{n=0}^{\infty} \rho(u_n),$$

for any sequence $(u_n)_n$ in $M_+(\mu)$.

We say that ρ has *the Fatou property* (and write ρ F) in case

$$\rho(\sup_n u_n) = \sup_n \rho(u_n),$$

for any increasing sequence $(u_n)_n$ in $M_+(\mu)$.

It is known that $\rho\ F \Rightarrow \rho\ R - F$ (the converse implication is not true). For any $A \in \Sigma$, we denote by φ_A the characteristic (indicator) function of A. We shall write

$$\rho(A) \overset{def}{=} \rho(\varphi_A)$$

$$M(\mu) = \{f : S \to K|\ f \text{ is } \mu \text{ - measurable}\}$$

$$\mathcal{L}_\rho(\mu) = \mathcal{L}_\rho = \{f \in M(\mu)|\ \rho|f| < \infty\}$$

(where $\rho|f| \overset{def}{=} \rho(|f|)$). Then $\mathcal{L}_\rho$ is a vector subspace of $M(\mu)$, seminormed with the seminorm $f \mapsto \rho|f|$. The null space of this seminorm is

$$N(\mu) = \{f \in \mathcal{L}_\rho \mid \rho|f| = 0\} = \{f \in M(\mu) \mid f(t) = 0\ \ \mu \text{ - a.e.}\}$$

(actually, $N(\mu) = \{f : S \to K|\ f(t) = 0\mu \text{ - a.e.}\}$).

The associate normed space is $L_\rho(\mu) = L_\rho = \mathcal{L}_\rho(\mu)/N(\mu)$ (the equivalence relation being given via $f \sim g \Leftrightarrow \rho|f - g| = 0 \Leftrightarrow f(t) = g(t)\ \mu$ - a.e.) normed with the norm $\tilde{f} \mapsto \|\tilde{f}\| = \rho|f|$, for any $f \in \tilde{f}$.

The spaces L_ρ are called *Köthe spaces.* One knows that L_ρ is Banach iff $\rho\ R - F$. The spaces L_ρ generalize the Lebesgue spaces L^p (for these spaces the generating function norm $\|\ \|_p$ has the Fatou property) or the Orlicz spaces.

2.2. Now let us consider a separated locally compact topological space T. Let us consider also a family $\mathcal{E} = (E_t)_{t\in T}$ of Banach spaces. The norm in each E_t will be denoted by $\|z\|$, for $z \in E_t$. A more precise notation would have been $\|z\|_t$, but we think there is no danger of confusion.

A *vector field* (with respect to $\mathcal{E}$) is a function $x : T \to \bigcup_{t\in T} E_t$ such that $x(t) \in E_t$ for any $t \in T$. The set of all vector fields with respect to $\mathcal{E}$ will be denoted by $\mathcal{C}(\mathcal{E})$. Clearly, $\mathcal{C}(\mathcal{E})$ is a vector space with respect to the pointwise defined operations

$$(x, y) \mapsto x + y \text{ where } (x + y)(t) \overset{def}{=} x(t) + y(t)$$
$$(\alpha, x) \mapsto \alpha x \text{ where } (\alpha x)(t) \overset{def}{=} \alpha x(t),$$

for any $x, y \in \mathcal{C}(\mathcal{E})$ and any $\alpha \in K$. Actually, one has

$$\mathcal{C}(\mathcal{E}) = \prod_{t\in T} E_t.$$

For any $x \in \mathcal{C}(\mathcal{E})$, one can define the function $|x| : T \to \mathbb{R}_+$, via $|x|(t) = \|x(t)\|$.

A *fundamental family of continuous vector fields* is a vector subspace $\mathcal{A}$ of $\mathcal{C}(\mathcal{E})$, satisfying the following axioms:

$(A1)$ For any $x \in \mathcal{A}$, the function $|x|$ is continuous.

$(A2)$ For any $t \in T$, the set $\{x(t) \mid x \in \mathcal{A}\}$ is dense in E_t.

Particular case. *The unicity case* $\mathcal{C}(E)$.

Assume that $\mathcal{E}$ is such that $E_t = E$ (where E is a Banach space) for any $t \in T$. Then we say that we are in the *unicity case* $\mathcal{C}(E)$. In this case:

- A vector field is a function $x : T \to E$.

- A fundamental family of continuous vector fields is $\mathcal{A} =$ all the constant functions $x : T \to E$. We shall write $\mathcal{A} = E$ (identifying any function $x \in \mathcal{A}$ with its constant value $x(t)$ for all $t \in T$).

2.3. Let us consider a fundamental family of continuous vector fields $\mathcal{A}$. We shall say that $x \in \mathcal{C}(\mathcal{E})$ *is continuous at* $t_0 \in T$ *with respect to* $\mathcal{A}$ if, for any $\epsilon > 0$, there exist $V \in \mathcal{V}(t_0) =$ the set of all neighbourhoods of t_0 and $y \in \mathcal{A}$ such that $\|x(t) - y(t)\| < \epsilon$ for any $t \in V$.

If $\emptyset \neq A \subset T$ we say that x is *continuous on* A if x is continuous at any $t \in A$. In case $A = T$ we say that x is *continuous* (with respect to $\mathcal{A}$).

Concerning the above definition, we remark that:

- Any $x \in \mathcal{A}$ is continuous with respect to $\mathcal{A}$ (take $y = x$ in definition).

- The vector field x is continuous at t_0 (with respect to $\mathcal{A}$) iff for any $y \in \mathcal{A}$ and any $\alpha \in K$, the vector field $x + \alpha y$ is continuous at t_0 (with respect to $\mathcal{A}$).

- The vector field x is continuous at t_0 with respect to $\mathcal{A}$ iff $|x - y|$ is continuous at t_0 for any $y \in \mathcal{A}$. Hence, if x is continuous at t_0 with respect to $\mathcal{A}$, it follows that $|x|$ is continuous at t_0.

Write: $\mathcal{C}_{\mathcal{A}}(\mathcal{E}) = \{x \in \mathcal{C}(\mathcal{E}) \mid x \text{ is continuous with respect to } \mathcal{A}\}$. Then $\mathcal{A} \subset \mathcal{C}_{\mathcal{A}}(\mathcal{E})$ and $\mathcal{C}_{\mathcal{A}}(\mathcal{E})$ is a fundamental family of continuous vector fields.

- In the unicity case $\mathcal{C}(\mathcal{E})$: a vector field (function) $x : T \to E$ is continuous at t_0 with respect to E iff x is continuous at t_0 in the usual sense.

2.4. From now on, we shall work within the following

Framework. Assume T is a separated locally compact topological space with Borel sets $\mathcal{B}$. Let $\mathcal{T}$ be a σ - algebra of subsets of T such that $\mathcal{B} \subset \mathcal{T}$ and let $\mu : \mathcal{T} \to \overline{\mathbb{R}_+}$ be a complete non null measure with $\mu(A) < \infty$ for any $A \in \mathcal{K} =$ the compact sets of T. We also assume that μ is regular. So, we have the measure space $(T,\mathcal{T},\mu)$. We consider also a family $\mathcal{E} = (E_t)_{t\in T}$ of Banach spaces, the vector space $\mathcal{C}(\mathcal{E})$ and a fundamental family of continuous vector fields $\mathcal{A}$.

A vector field $x \in \mathcal{C}(\mathcal{E})$ is called *μ - measurable with respect to* $\mathcal{A}$ (shortly, x is *$(\mathcal{A},\mu)$ - measurable*) if, for any $A \in \mathcal{K}$ and any $\epsilon > 0$, there exists $\mathcal{K} \ni A_\epsilon \subset A$ such that $\mu(A \setminus A_\epsilon) < \epsilon$ and x is continuous with respect to $\mathcal{A}$ on A_ϵ.

Write $\mathcal{M}(\mathcal{A},\mu) = \{x \in \mathcal{C}(\mathcal{E}) \mid x \text{ is } (\mathcal{A},\mu) \text{ - measurable}\}$ hence $\mathcal{C}_{\mathcal{A}}(\mathcal{E}) \subset \mathcal{M}(\mathcal{A},\mu)$. Notice that $\mathcal{M}(\mathcal{A},\mu)$ is a vector subspace of $\mathcal{C}(\mathcal{E})$ having the following properties:

- For any $x, y \in \mathcal{C}(\mathcal{E})$ such that $x \in \mathcal{M}(\mathcal{A},\mu)$ and $y = x$ μ - a.e., one has $y \in \mathcal{M}(\mathcal{A},\mu)$.

- For any $x \in \mathcal{M}(\mathcal{A},\mu)$ the function $|x|$ is μ - measurable.

- If $(x_n)_n$ is a sequence in $\mathcal{M}(\mathcal{A},\mu)$ and x is in $\mathcal{C}(\mathcal{E})$ such that $x_n \underset{n}{\to} x$ μ - a.e., then $x \in \mathcal{M}(\mathcal{A},\mu)$. Moreover (Egorov's theorem), it follows that, for any $A \in \mathcal{K}$ and any $\epsilon > 0$, there exists $\mathcal{K} \ni A_\epsilon \subset A$ such that $\mu(A \setminus A_\epsilon) < \epsilon$ and $(x_n)_n$ converges to x uniformly on A_ϵ.

- An element $x \in \mathcal{C}(\mathcal{E})$ is in $\mathcal{M}(\mathcal{A},\mu)$ iff $\varphi_A x$ is in $\mathcal{M}(\mathcal{A},\mu)$ for any $A \in \mathcal{K}$.

Particular case. Assume we are in the unicity case $\mathcal{C}(E)$, with $\mathcal{A} = E$. Then, one can prove (Lusin's theorem) that a vector field (i.e. a function) $x : T \to E$ is (E,μ) - measurable iff x is μ - measurable. In this case we write $\mathcal{M}(E,\mu) \overset{def}{=} M_E(\mu)$.

3. Results

3.1. In this section we introduce the object of study of the present paper, namely the Köthe spaces of vector fields. We shall work within the framework of 4., section 1, and we shall assume, supplementarily, that ρ is a function norm on $(T,\mathcal{T},\mu)$.

The *seminormed Köthe space* $\mathcal{L}_\rho(\mathcal{E},\mathcal{A})$ is defined as follows:

$$\mathcal{L}_\rho(\mathcal{E},\mathcal{A}) = \{x \in \mathcal{M}(\mathcal{A},\mu) \mid \rho|x| < \infty\}.$$

The definition is meaningful, because, as we saw, for any $x \in \mathcal{M}(\mathcal{A},\mu)$, one has $|x| \in M_+(\mu)$. Clearly, $\mathcal{L}_\rho(\mathcal{E},\mathcal{A})$ is a vector subspace of $\mathcal{M}(\mathcal{A},\mu)$, seminormed with the seminorm given via $x \mapsto \rho|x|$.

The null space of this seminorm is

$$N_\rho(\mathcal{E},\mathcal{A}) = \{x \in \mathcal{M}(\mathcal{A},\mu) |\ \rho|x| = 0\} = \{x \in \mathcal{M}(\mathcal{A},\mu) \mid x(t) = 0\ \mu - a.e.\}.$$

We get the quotient space $L_\rho(\mathcal{E},\mathcal{A}) = \mathcal{L}_\rho(\mathcal{E},\mathcal{A})/N_\rho(\mathcal{E},\mathcal{A})$ (the equivalence relation is given via $x \sim y \Leftrightarrow x(t) = y(t)\ \mu$ - a.e.) normed with the norm given via $\tilde{x} \mapsto \|\tilde{x}\| = \rho|x|$ for any $x \in \tilde{x}$.

Definition 1. The vector space $L_\rho(\mathcal{E},\mathcal{A})$, normed with the norm from above, is called *Köthe space of vector fields.*

Particular cases. 1. Assume we are in the unicity case $\mathcal{C}(E)$. We shall write $\mathcal{L}_\rho(\mathcal{E},E) \stackrel{def}{=} \mathcal{L}_\rho(E,\mu)$. It is seen that $\mathcal{L}_\rho(E,\mu) = \{x \in M_E(\mu) \mid \rho|x| < \infty\}$ seminormed with the seminorm $x \mapsto \rho|x|$. We shall write $L_\rho(\mathcal{E},E) \stackrel{def}{=} L_\rho(E,\mu)$ normed with $\tilde{x} \mapsto \|\tilde{x}\| = \rho|x|$ for any $x \in \tilde{x}$.

Of course, $\mathcal{L}_\rho(K,\mu) = \mathcal{L}_\rho$ and $L_\rho(K,\mu) = L_\rho$.

2. Assume we have, for any $t \in T$, a measure space (S_t,Σ_t,μ_t) and a function norm ρ_t on (S_t,Σ_t,μ_t). We shall consider that, for any $t \in T$, $E_t = L_{\rho_t}$.(hence all $\rho_t\ R-F$, because all E_t must be Banach spaces). In this case we shall write:

$$\mathcal{L}_\rho(\mathcal{E},\mathcal{A}) \stackrel{def}{=} \mathcal{L}_\rho((\rho_t)_t,\mathcal{A}) \text{ and } L_\rho(\mathcal{E},\mathcal{A}) \stackrel{def}{=} L_\rho((\rho_t)_t,\mathcal{A}).$$

Concerning this particular case, we add the fact that, the most "normal" situation is that one when all the measure spaces are equal, i.e. $(S_t,\Sigma_t,\mu_t) = (S,\Sigma,\mu)$ for any $t \in T$. In this case, the variability is furnished by the function norms $\rho_t, t \in T$.

Our aim in this paper is to study some properties of the spaces $L_\rho(\mathcal{E},\mathcal{A})$, (e.g. completeness), thus initiating a theory which extends the theory of the spaces L_ρ, in particular the theory of the Lebesgue spaces L^p and of the Orlicz spaces.

3.2.

Lemma 2. *Assume $\rho\ R-F$. Let $(x_n)_n$ be a sequence in $\mathcal{L}_\rho(\mathcal{E},\mathcal{A})$ such that $\sum_{n=0}^\infty \rho|x_n| < \infty$. Then:*

a) The series $\sum_{n=0}^\infty x_n$ converges μ - a.e. , i.e. there exists $A \in \mathcal{T}$ with $\mu(A) = 0$ such that, for any $t \in T \setminus A$, the series $\sum_{n=0}^\infty x_n(t)$ converges in E_t.

b) Define $x \in \mathcal{C}(\mathcal{E})$ *via* $x(t) = \begin{cases} \sum_{n=0}^{\infty} x_n(t), & \text{if } t \in T \setminus A \\ \text{arbitrary in } E_t, & \text{if } t \in A. \end{cases}$ *Then* $x \in \mathcal{L}_\rho(\mathcal{E}, \mathcal{A})$ *and, for any natural* n

$$\rho|x - \sum_{k=0}^{n} x_k| \leq \sum_{k=n+1}^{\infty} \rho|x_k| \underset{n}{\to} 0.$$

Proof. $a)$ Let us define $f : T \to \overline{\mathbb{R}}_+$ via $f = \sum_{n=0}^{\infty} |x_n|$. Then $f \in M_+(\mu)$ and $\rho(f) \leq \sum_{n=0}^{\infty} \rho|x_n| < \infty$ hence $f(t) < \infty$ μ - a.e. and the series $\sum_{n=0}^{\infty} x_n(t)$ converges (absolutely) μ - a.e. (i.e. on $T \setminus A$, for some $A \in \mathcal{T}$ with $\mu(A) = 0$).

$b)$ Defining x as in the enunciation, one can see that $x \in \mathcal{M}(\mathcal{A}, \mu)$ and, for any $t \in T \setminus A$:

$$\|x(t)\| = \lim_n \|\sum_{k=0}^{n} x_k(t)\| \leq \lim_n \sum_{k=0}^{n} \|x_k(t)\| = \sum_{k=0}^{\infty} \|x_k(t)\| = f(t)$$

hence $\rho|x| \leq \rho(f) < \infty$, i.e. $x \in \mathcal{L}_\rho(\mathcal{E}, \mathcal{A})$. For any $t \in T \setminus A$, using the same device:

$$\|x(t) - \sum_{k=0}^{n} x_k(t)\| = \|\sum_{k=n+1}^{\infty} x_k(t)\| \leq \sum_{k=n+1}^{\infty} \|x_k(t)\|$$

hence

$$\rho|x - \sum_{k=0}^{n} x_k| \leq \rho(\sum_{k=n+1}^{\infty} |x_k|) \leq \sum_{k=n+1}^{\infty} \rho|x_k|.$$

□

With the aid of the preceding Lemma, we can prove

Theorem 3 (Completeness of Köthe spaces of vector fields)**.** *Assume* ρ $R - F$*. Then* $L_\rho(\mathcal{E}, \mathcal{A})$ *is Banach.*

Proof. Let $(\tilde{x}_n)_{n \geq 0}$ be a Cauchy sequence in $L_\rho(\mathcal{E}, \mathcal{A})$ and take $x_n \in \tilde{x}_n$ for any n. We shall find $\tilde{x} \in L_\rho(\mathcal{E}, \mathcal{A})$ such that $\tilde{x}_n \underset{n}{\to} \tilde{x}$, or, which is the same, we shall find $x \in \mathcal{L}_\rho(\mathcal{E}, \mathcal{A})$ such that $x_n \underset{n}{\to} x$ in $\mathcal{L}_\rho(\mathcal{E}, \mathcal{A})$.

Let $\epsilon > 0$. There exists $n(\epsilon)$ such that, for any $n \geq n(\epsilon), m \geq n(\epsilon)$, one has

$$\|\tilde{x}_n - \tilde{x}_m\| = \rho|x_n - x_m| < \frac{\epsilon}{2}. \tag{1}$$

We can find a strictly increasing sequence $n_1 < n_2 < \ldots < n_k < \ldots$ such that, for any k

$$\rho|x_{n_{k+1}} - x_{n_k}| < \frac{1}{2^k} \tag{2}$$

as follows. First, we find n_1 such that, for any $n \geq n_1$, one has $\rho|x_n - x_{n_1}| < \frac{1}{2}$. Then, we find $n_2 > n_1$ such that, for any $n \geq n_2$ $\rho|x_n - x_{n_2}| < \frac{1}{2^2}$ and (of course) $\rho|x_{n_2} - x_{n_1}| < \frac{1}{2}$. Continuing, we find $n_3 > n_2$ such that, for any $n \geq n_3$ $\rho|x_n - x_{n_3}| < \frac{1}{2^3}$ and (of course) $\rho|x_{n_3} - x_{n_2}| < \frac{1}{2^2}$.

The procedure continues and we find inductively $(n_k)_k$ satisfying (2). Write, for all k, $y_k = x_{n_{k+1}} - x_{n_k}$ and we have (see (2))

$$\rho|y_k| < \frac{1}{2^k} \Rightarrow \sum_{k=1}^{\infty} \rho|y_k| < \infty.$$

Lemma 2 says that the series $\sum_{k=1}^{\infty} y_k(t)$ converges μ - a.e. and the function (defined μ - a.e.) $y = \sum_{k=1}^{\infty} y_k$ belongs to $\mathcal{L}_\rho(\mathcal{E}, \mathcal{A})$. For any $k \geq 2$, one has

$$y_1 + y_2 + \ldots + y_{k-1} = x_{n_k} - x_{n_1}. \tag{3}$$

Let us define $x = y + x_{n_1}$ hence, we have pointwise μ - a.e. (see (3))

$$x = \lim_k (y_1 + y_2 + \ldots + y_{k-1} + x_{n_1}) = \lim_k x_{n_k}.$$

It is seen that, for any $k \geq 2$, one has

$$\rho|x_n - x_{n_k}| < \frac{1}{2^{k-1}}. \tag{4}$$

Indeed, using (3) and Lemma 2:

$$\rho|x - x_{n_k}| = \rho|(y + x_{n_1}) - (y_1 + y_2 + \ldots + y_{k-1} + x_{n_1})|$$
$$= \rho|y - (y_1 + y_2 + \ldots + y_{k-1})| \leq \sum_{p=k}^{\infty} \rho|y_p| = \frac{1}{2^{k-1}}.$$

We can find k such that:

$$n_k \geq n(\epsilon) \text{ and } \frac{1}{2^{k-1}} < \frac{\epsilon}{2}. \tag{5}$$

Finally, let $n \geq n(\epsilon)$. One has, for this k:

$$\rho|x - x_n| \leq \rho|x - x_{n_k}| + \rho|x_{n_k} - x_n|. \tag{6}$$

But, using (5) we have

$$\rho|x - x_{n_k}| < \frac{1}{2^{k-1}} < \frac{\epsilon}{2} \text{ (with (4))}, \quad \rho|x_{n_k} - x_n| < \frac{\epsilon}{2} \text{ (with (1))}$$

and (6) gives: $\rho|x - x_n| < \epsilon$ which shows that $x_n \underset{n}{\to} x$, in $\mathcal{L}_\rho(\mathcal{E}, \mathcal{A})$. □

3.3. We have introduced the space $\mathcal{L}_\rho(\mathcal{E}, \mathcal{A})$ and the corresponding convergence of sequences in this space. In this section we shall introduce another type of convergence (on all of $\mathcal{M}(\mathcal{A}, \mu)$) which is weaker than the convergence of $\mathcal{L}_\rho(\mathcal{E}, \mathcal{A})$ on this space.

Notice first that in case A, A_1 are in $\mathcal{T}$ such that $A_1 \subset A$ and $\mu(A \setminus A_1) = 0$, it follows that $\rho(A) = \rho(A_1)$ (because $\varphi_A = \varphi_{A_1} + \varphi_{A \setminus A_1}$ hence $\rho(A) \leq \rho(A_1) + \rho(A \setminus A_1) = \rho(A)$). Using $\varphi_{A \cup B} \leq \varphi_A + \varphi_B$, we get $\rho(A \cup B) \leq \rho(A) + \rho(B)$ for any A, B in $\mathcal{T}$.

Definition 4. Let $(f_n)_n$ be a sequence in $\mathcal{M}(\mathcal{A}, \mu)$.

1. For $f \in \mathcal{M}(\mathcal{A}, \mu)$, we say that $(f_n)_n$ *converges to* f *in* ρ *- measure* (and write $f_n \overset{\rho}{\underset{n}{\to}} f$), if for any $a > 0$, one has

$$\lim_n \rho(|f_n - f| > a) = 0, \text{ where } \rho(|f_n - f| > a) \overset{def}{=} \rho(A_n^a)$$

and $A_n^a = \{t \in T \mid \|f_n(t) - f(t)\| > a\} \in \mathcal{T}$.

2. We say that $(f_n)_n$ is *Cauchy in* ρ *- measure* if, for any $a > 0$, one has: for any $\epsilon > 0$, there exists a natural $n(\epsilon)$ such that, for any $m \geq n(\epsilon)$, $n \geq n(\epsilon)$,

$$\rho(|f_m - f_n| > a) < \epsilon, \text{ where } \rho(|f_m - f_n| > a) \overset{def}{=} \rho(A_{m,n}^a)$$

and $A_{m,n}^a = \{t \in T| \, \|f_m(t) - f_n(t)\| > a\} \in \mathcal{T}$.

Remark. We have $A_n^a \in \mathcal{T}$ and $A_{m,n}^a \in \mathcal{T}$, because $f \in \mathcal{M}(\mathcal{A}, \mu)$ and $f_n \in \mathcal{M}(\mathcal{A}, \mu)$.

Proposition 5. *Assume $(f_n)_n$ is convergent in ρ - measure (i.e. there exists $f \in \mathcal{M}(\mathcal{A}, \mu)$, such that $f_n \overset{\rho}{\underset{n}{\to}} f$). Then $(f_n)_n$ is Cauchy in ρ - measure.*

Proof. For any ϵ and any $a > 0$, if m, n are large enough

$$A^a_{m,n} \subset A^{\frac{a}{2}}_m \cup A^{\frac{a}{2}}_n \Rightarrow \rho(A^a_{m,n}) \leq \rho(A^{\frac{a}{2}}_m) + \rho(A^{\frac{a}{2}}_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon.$$

□

Proposition 6. 1. *If* $f_n \underset{n}{\overset{\rho}{\rightarrow}} f$ *and* $f = g$ μ *- a.e., then* $f_n \underset{n}{\overset{\rho}{\rightarrow}} g$.

2. *If* $f_n \underset{n}{\overset{\rho}{\rightarrow}} f$ *and* $f_n \underset{n}{\overset{\rho}{\rightarrow}} g$, *then* $f = g$ μ *- a.e..*

Proof. 1. Notice first that $g = f$ μ - a.e. and $f \in \mathcal{M}(\mathcal{A}, \mu)$, implies $g \in \mathcal{M}(\mathcal{A}, \mu)$ and the enunciation is meaningful. Let us denote $M = \{t \in T \mid f(t) \neq g(t)\}$ hence $M \in \mathcal{T}$ and $\mu(M) = 0$. It follows that, for any natural n and any $a > 0$

$$\rho(|f_n - f| > a) = \rho(|f_n - g| > a). \tag{7}$$

Equality (7) is obtained as follows: if we write:

$$A^a_n = \{t \in T | \; \|f_n(t) - f(t)\| > a\} \text{ and } B^a_n = \{t \in T | \; \|f_n(t) - g(t)\| > a\}$$

we have $A^a_n \setminus M = B^a_n \setminus M$ and, of course $A^a_n \setminus (A^a_n \setminus M) = A^a_n \cap M$ and $B^a_n \setminus (B^a_n \setminus M) = B^a_n \cap M$, which implies (see the Remark at the beginning of the section) $\rho(A^a_n) = \rho(A^a_n \setminus M) = \rho(B^a_n \setminus M) = \rho(B^a_n)$.

2. Fix $a > 0$ and write $A_a = \{t \in T | \; \|f(t) - g(t)\| > a\}$. We get

$$A_a \subset A^{\frac{a}{2}}_n \cup B^{\frac{a}{2}}_n \Rightarrow \rho(A_a) \leq \rho(A^{\frac{a}{2}}_n) + \rho(B^{\frac{a}{2}}_n) \underset{n}{\rightarrow} 0.$$

It follows that $\rho(A_a) = 0$, hence $\mu(A_a) = 0$. The number a being arbitrary, we obtain $\rho(A) = 0 \Rightarrow \mu(A) = 0$, where

$$A = \bigcap_{n=1}^{\infty} A_{\frac{1}{n}} = \{t \in T | \; f(t) \neq g(t)\}.$$

□

The following theorem shows that convergence in $\mathcal{L}_\rho(\mathcal{E}, \mathcal{A})$ is stronger than convergence in ρ - measure.

Theorem 7. *If* $f_n \underset{n}{\rightarrow} f$ *in* $\mathcal{L}_\rho(\mathcal{E}, \mathcal{A})$, *then* $f_n \underset{n}{\overset{\rho}{\rightarrow}} f$.

Proof. Take an arbitrary $a > 0$. Then, with previously introduced notations $|f_n - f| = |f_n - f|\varphi_{A_n^a} + |f_n - f|\varphi_{T\setminus A_n^a} \geq |f_n - f|\varphi_{A_n^a}$ which implies $\rho|f_n - f| \geq \rho(|f_n - f|\varphi_{A_n^a}) \geq \rho(a\varphi_{A_n^a}) = a\rho(A_n^a)$. Hence

$$\rho(A_n^a) \leq \frac{1}{a}\rho|f_n - f| \underset{n}{\to} 0$$

and a is arbitrary, which shows that $f_n \overset{\rho}{\underset{n}{\to}} f$. □

Using Theorem 7 and Proposition 6, we get

Corollary 8. *Assume $f_n \underset{n}{\to} f$ and $f_n \underset{n}{\to} g$ in $\mathcal{L}_\rho(\mathcal{E}, \mathcal{A})$. Then $f = g$ μ - a.e.*

3.4. In this section we generalize the almost uniform (asymptotic) convergence and introduce relationships between all types of convergence.

Definition 9. Let $(f_n)_n$ be a sequence in $\mathcal{M}(\mathcal{A}, \mu)$.

1. For $f \in \mathcal{M}(\mathcal{A}, \mu)$, we say that $(f_n)_n$ *converges ρ - almost uniformly (ρ - asymptotically) to f* (and write $f_n \overset{\rho u}{\underset{n}{\to}} f$) if, for any $\epsilon > 0$, there exists $A_\epsilon \in \mathcal{T}$ such that $\rho(A_\epsilon) < \epsilon$ and $(f_n)_n$ converges uniformly to f on $T \setminus A_\epsilon$.

2. We say that $(f_n)_n$ is *ρ - almost uniformly (ρ-asymptotically) Cauchy* if, for any $\epsilon > 0$, there exists $A_\epsilon \in \mathcal{T}$ such that $\rho(A_\epsilon) < \epsilon$ and $(f_n)_n$ is uniformly Cauchy on $T \setminus A_\epsilon$ (this means that, for any $\delta > 0$, there exists a natural $n(\delta)$ having the property that, for $m \geq n(\delta)$ and $n \geq n(\delta)$ one has $\|f_m(t) - f_n(t)\| < \delta$, for any $t \in T \setminus A_\epsilon$).

Proposition 10. *Assume $f_n \overset{\rho u}{\underset{n}{\to}} f$. Then $(f_n)_n$ is almost uniformly Cauchy.*

Proof. Let $\epsilon > 0$ and take $A_\epsilon \in \mathcal{T}$ like in the definition of the fact that $f_n \overset{\rho u}{\underset{n}{\to}} f$. Pick $\delta > 0$ and find a natural $n(\delta)$ such that, for any $n \geq n(\delta)$ and any $t \in T \setminus A_\epsilon$, one has $\|f_n(t) - f(t)\| < \frac{\delta}{2}$. It follows that, for $n \geq n(\delta)$, $m \geq n(\delta)$ and $t \in T \setminus A_\epsilon$, one has $\|f_n(t) - f_m(t)\| < \delta$. □

Almost uniform convergence is stronger than almost everywhere convergence and convergence in measure, as the following theorem shows.

Theorem 11. *Let $(f_n)_n$ be a sequence in $\mathcal{M}(\mathcal{A}, \mu)$, and $f \in \mathcal{M}(\mathcal{A}, \mu)$.*

1. *If* $f_n \underset{n}{\overset{\rho u}{\rightarrow}} f$*,then* $f_n \underset{n}{\rightarrow} f$ μ *- a.e.*

2. *If* $f_n \underset{n}{\overset{\rho u}{\rightarrow}} f$*, then* $f_n \underset{n}{\overset{\rho}{\rightarrow}} f$*.*

Proof. 1. For any natural $n > 0$, pick $A_n \in \mathcal{T}$ with $\mu(A_n) < \frac{1}{n}$, such that $(f_m)_m$ converges to f uniformly on $T \setminus A_n$. Writing $A = \bigcap_{n=1}^{\infty} A_n$, we have $\rho(A) = 0$, hence $\mu(A) = 0$. For $t \in T \setminus A = \bigcup_{n=1}^{\infty}(T \setminus A_n)$ we find n such that $t \in T \setminus A_n$. It follows that $f_m(t) \underset{m}{\longrightarrow} f(t)$ a.s.o.

2. Let $a > 0$ and $\epsilon > 0$. One can find $A_\epsilon \in \mathcal{T}$ with $\rho(A_\epsilon) < \epsilon$, such that there exists $n(a)$ with the property that for any $n \geq n(a)$ and any $t \in T \setminus A_\epsilon$ one has $\|f_n(t) - f(t)\| \leq a$. Hence, we have the inclusion

$$\{t \in T \mid \|f_n(t) - f(t)\| > a\} \subset A_\epsilon \Rightarrow \rho(|f_n - f| > a) \leq \rho(A_\epsilon) < \epsilon.$$

□

Using the preceding facts, we obtain (see also the similar Proposition 6)

Proposition 12. *1. If* $f_n \underset{n}{\overset{\rho u}{\rightarrow}} f$ *and* $f = g$ μ *- a.e., then* $f_n \underset{n}{\overset{\rho u}{\rightarrow}} g$*.*

2. If $f_n \underset{n}{\overset{\rho u}{\rightarrow}} f$ *and* $f_n \underset{n}{\overset{\rho u}{\rightarrow}} g$*, then* $f = g$ μ *- a.e..*

Proof. 1. Write $M = \{t \in T | \ f(t) \neq g(t)\}$ hence $\mu(M) = 0$. Take $\epsilon > 0$ arbitrarily and find $A_\epsilon \in \mathcal{T}$ with $\rho(A_\epsilon) < \epsilon$, such that $(f_n)_n$ converges to f uniformly on $T \setminus A_\epsilon$, hence $(f_n)_n$ converges to f uniformly on $T \setminus B_\epsilon$, where $B_\epsilon = A_\epsilon \cup M$. But $\rho(B_\epsilon) = \rho(A_\epsilon) < \epsilon$ and, for $t \in T \setminus B_\epsilon$, $f(t) = g(t)$. So, $(f_n)_n$ converges to g uniformly on $T \setminus B_\epsilon$, a.s.o.

2. Assume $f_n \underset{n}{\overset{\rho u}{\rightarrow}} f$ and $f_n \underset{n}{\overset{\rho u}{\rightarrow}} g$. Using Theorem 11.1., one has $f_n \underset{n}{\rightarrow} f$ μ - a.e. and $f_n \underset{n}{\rightarrow} g$ μ - a.e. and this implies $f = g$ μ - a.e. □

The next result shows that $\mathcal{M}(\mathcal{A}, \mu)$, is "complete" with respect to ρ - almost uniform convergence.

Theorem 13. *Let* $(f_n)_n$ *be a sequence in* $\mathcal{M}(\mathcal{A}, \mu)$*. The following assertions are equivalent:*

1. $(f_n)_n$ *is* ρ *- almost uniformly Cauchy.*

2. $(f_n)_n$ *is* ρ *- almost uniformly convergent.*

(*i.e. there exists* $f \in \mathcal{M}(\mathcal{A}, \mu)$ *such that* $f_n \underset{n}{\overset{\rho u}{\rightarrow}} f$) .

Proof. The implication 2.⇒1. has been proved in Proposition 10. The implication 1.⇒2. will be proved in two steps.

Step 1. We shall find $f \in \mathcal{M}(\mathcal{A}, \mu)$ such that $f_n \underset{n}{\to} f$ μ - a.e.. The proof is similar to the proof of Theorem 11.

For any natural $k > 0$, we find $A_k \in \mathcal{T}$ with $\rho(A_k) < \frac{1}{k}$, such that, for any $\delta > 0$, there exists $n(\delta)$ having the property that, if $m \geq n(\delta)$,$n \geq n(\delta)$ and $t \in T \setminus A_k$, one has

$$\|f_m(t) - f_n(t)\| < \delta. \tag{8}$$

Writing $A = \bigcap_{k=1}^{\infty} A_k$, we have $\mu(A) = 0$. For any $t \in T \setminus A = \bigcup_{k=1}^{\infty}(T \setminus A_k)$, we find k such that $t \in T \setminus A_k$. It follows that (8) is true, hence the sequence $(f_n(t))_n$ is Cauchy in E_t, hence convergent in E_t. Writing $f(t) = \lim_n f_n(t)$ we have defined $f(t)$ for any $t \in T \setminus A$. Defining $f(t) = 0$ for $t \in A$, we got $f : T \to \bigcup_{t \in T} E_t$, with $f(t) \in E_t$ for any $t \in T$, hence we defined $f \in \mathcal{C}(\mathcal{E})$ and one can see that $f \in \mathcal{M}(\mathcal{A}, \mu)$ and $f_n \underset{n}{\to} f$ μ - a.e.

Step 2. We show that $f_n \overset{\rho u}{\underset{n}{\to}} f$.

Let $\epsilon > 0$. There exists $B_\epsilon \in \mathcal{T}$ with $\rho(B_\epsilon) < \epsilon$ and such that, for any $\delta > 0$, one can find $n(\delta)$ with the property that, if $m \geq n(\delta)$, $n \geq n(\delta)$ and $t \in T \setminus B_\epsilon$,

$$\|f_m(t) - f_n(t)\| < \delta. \tag{8'}$$

Write $A_\epsilon = A \cup B_\epsilon$, hence $\rho(A_\epsilon) = \rho(B_\epsilon) < \epsilon$. We have seen that, for any $t \in T \setminus A_\epsilon$, one has $f_n(t) \underset{n}{\to} f(t)$. At the same time, for a given $\delta > 0$, we can find $n(\delta)$ as previously and, for $m \geq n(\delta), n \geq n(\delta)$ and $t \in T \setminus A_\epsilon$, we have again (8′).

Keeping $m \geq n(\delta)$ fixed in (8′) and letting n tend to ∞, we get from (8′):

$$\|f_m(t) - f(t)\| \leq \delta$$

and this shows that $(f_n)_n$ converges to f uniformly on $T \setminus A_\epsilon$. □

Corollary 14. *Let $(f_n)_n$ be a sequence in $\mathcal{M}(\mathcal{A}, \mu)$ and $f \in \mathcal{M}(\mathcal{A}, \mu)$ such that $(f_n)_n$ is ρ - almost uniformly Cauchy and $f_n \underset{n}{\to} f$ μ - a.e. Then $f_n \overset{\rho u}{\underset{n}{\to}} f$.*

Proof. Using Theorem 13, we find $g \in \mathcal{M}(\mathcal{A},\mu)$ such that $f_n \overset{\rho u}{\underset{n}{\rightarrow}} g$. So $f_n \underset{n}{\rightarrow} g$ μ - a.e (Proposition 1.). At the same time $f_n \underset{n}{\rightarrow} f$ μ - a.e. It follows that $f = g$ μ - a.e. .

We have $f_n \overset{\rho u}{\underset{n}{\rightarrow}} g$ and $g = f$ μ - a.e.. We apply Proposition 12. and obtain that $f_n \overset{\rho u}{\underset{n}{\rightarrow}} f$. □

Before proceeding further, we make the following

Remark. Assume ρ $R-F$. Let $(A_n)_{n\geq 1}$ be a sequence of sets $A_n \in \mathcal{T}$. Then, one has

$$\rho(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \rho(A_n).$$

Indeed: $\varphi_A = \sum_{n=1}^{\infty} \varphi_{B_n}$ (pointwise), where:

$$A = \bigcup_{n=1}^{\infty} A_n;\ B_1 = A_1;\ B_2 = A_2 \setminus A_1;\ B_3 = A_3 \setminus (A_1 \cup A_2); \ldots$$

hence

$$\rho(A) \leq \sum_{n=1}^{\infty} \rho(B_n) \leq \sum_{n=1}^{\infty} \rho(A_n).$$

Theorem 15. *Assume ρ $R-F$. Let $(f_n)_n$ be a sequence in $\mathcal{M}(\mathcal{A},\mu)$ which is Cauchy in ρ - measure. Then, one can find a subsequence $(f_{n_k})_k \subset (f_n)_n$ and a function $f \in \mathcal{M}(\mathcal{A},\mu)$ such that $f_{n_k} \overset{\rho u}{\underset{k}{\rightarrow}} f$ (consequently $f_{n_k} \underset{k}{\rightarrow} f$ μ - a.e. and $f_{n_k} \overset{\rho}{\underset{k}{\rightarrow}} f$.)*

Proof. The remarks in the brackets follow from Theorem 11.

A. We shall show that there exists a sequence $(f_{n_k})_k \subset (f_n)_n$ which is ρ - almost uniformly Cauchy. This subsequence is constructed as follows.

Take $k = 1$. There exists $n_1 \in \mathbb{N}^*$ with the property that, for any $m,n \geq n_1$, one has $\rho(|f_m - f_n| > \frac{1}{2^1}) < \frac{1}{2^1}$. Take $k = 2$. There exists $n_2 > n_1$ such that, for any $m,n \geq n_2$, one has $\rho(|f_m - f_n| > \frac{1}{2^2}) < \frac{1}{2^2}$. Continuing in the same manner, we obtain the sequence $(n_k)_k$, $n_1 < n_2 < \ldots < n_k < n_{k+1} < \ldots$ such that, for any $m,n \geq n_k$ one has $\rho(|f_m - f_n| > \frac{1}{2^k}) < \frac{1}{2^k}$. Because $n_{k+1} > n_k$, one has for any k:

$$\rho(|f_{n_{k+1}} - f_{n_k}| > \frac{1}{2^k}) < \frac{1}{2^k}.$$

Our further goal is to show that the subsequence $(f_{n_k})_k \subset (f_n)_n$ fulfills the conditions in the enunciation (i.e. is ρ - almost uniformly Cauchy). For any $k \in \mathbb{N}^*$, let

$$E_k = \{t \in T | \, \|f_{n_{k+1}}(t) - f_{n_k}(t)\| > \frac{1}{2^k}\} \tag{9}$$

hence $\rho(E_k) < \frac{1}{2^k}$.

Let $\epsilon > 0$. There exists $n(\epsilon) \in \mathbb{N}$ such that

$$\sum_{k=n(\epsilon)}^{\infty} \frac{1}{2^k} < \epsilon.$$

Put $A_\epsilon = \bigcup_{k=n(\epsilon)}^{\infty} E_k$. It follows that (see the preceding Remark)

$$\rho(A_\epsilon) \le \sum_{k=n(\epsilon)}^{\infty} \rho(E_k) < \sum_{k=n(\epsilon)}^{\infty} \frac{1}{2^k} < \epsilon.$$

We close part A showing that, for any $\delta > 0$, there exists $m(\delta) \in \mathbb{N}$ such that, for any $i, j \ge m(\delta)$ and any $t \in T \setminus A_\epsilon$ one has $\|f_{n_i}(t) - f_{n_j}(t)\| < \delta$. Indeed, let $\delta > 0$. We can find $j_0 \in \mathbb{N}^*$ such that $\frac{1}{2^{j_0 - 1}} < \delta$. Take $m(\delta) = max(n(\epsilon), j_0)$. Let $i > j \ge m(\delta)$ and let $t \in T \setminus A_\epsilon$.

We shall show that $\|f_{n_i}(t) - f_{n_j}(t)\| < \delta$.

Indeed: because $t \in T \setminus (\bigcup_{k=n(\epsilon)}^{\infty} E_k) = \bigcap_{k=n(\epsilon)}^{\infty} (T \setminus E_k)$, it follows that, for any $k \ge n(\epsilon)$, one has $t \notin E_k$, which implies (see (9)):

$$\|f_{n_{k+1}}(t) - f_{n_k}(t)\| \le \frac{1}{2^k}.$$

So, we have $i > j \ge n(\epsilon)$, $i > j \ge j_0$:

$$\|f_{n_j}(t) - f_{n_i}(t)\| \le \|f_{n_j}(t) - f_{n_{j+1}}(t)\| + \|f_{n_{j+1}}(t) - f_{n_{j+2}}(t)\|$$

$$+ \ldots + \|f_{n_{i-1}}(t) - f_{n_i}(t)\| \le \frac{1}{2^j} + \frac{1}{2^{j+1}} + \ldots + \frac{1}{2^{i-1}} < \sum_{k=j_o}^{\infty} \frac{1}{2^k} = \frac{1}{2^{j_0-1}} < \delta.$$

B. Because $(f_{n_k})_k$ is ρ - almost uniformly Cauchy, we apply Theorem 13 and find $f \in \mathcal{M}(\mathcal{A}, \mu)$ such that $f_{n_k} \overset{\rho u}{\underset{k}{\to}} f$. $\square$

Theorem 15 is very important. Here are some of its consequences. We lay stress upon the fact that ρ $R - F$. The first consequence is Theorem

16, stating that $\mathcal{M}(\mathcal{A},\mu)$, equipped with the convergence in ρ - measure, is "complete" (analogue of Slutsky's theorem). The proof follows (directly) from Theorem 15 and Proposition 5.

Theorem 16. *Assume ρ $R-F$. Let $(f_n)_n$ be a sequence in $\mathcal{M}(\mathcal{A},\mu)$. The following assertions are equivalent*:

1. *The sequence $(f_n)_n$ is convergent in ρ - measure (i.e. there exists $f \in \mathcal{M}(\mathcal{A},\mu)$ such that $f_n \underset{n}{\overset{\rho}{\rightarrow}} f$).*
2. *The sequence $(f_n)_n$ is Cauchy in ρ - measure.*

Proof. 1. $\Rightarrow$ 2. Follows from Proposition 5.

2. $\Rightarrow$ 1. Theorem 15 implies the existence of a measurable vector field $f \in \mathcal{M}(\mathcal{A},\mu)$ and of a subsequence $(f_{n_k})_k \subset (f_n)_n$, such that $f_{n_k} \underset{k}{\overset{\rho}{\rightarrow}} f$. Using a standard device, we shall prove that $f_n \underset{n}{\overset{\rho}{\rightarrow}} f$.

To this end, let $\epsilon > 0$ and $a > 0$. There exists $n(\epsilon)$ such that, for any $m \geq n(\epsilon)$,$n \geq n(\epsilon)$, one has

$$\rho(|f_m - f_n| > \frac{a}{2}) < \frac{\epsilon}{2}. \tag{10}$$

There exists $k(\epsilon)$ such that $n_{k(\epsilon)} \geq n(\epsilon)$ and for any $k \geq k(\epsilon)$

$$\rho(|f - f_{n_k}| > \frac{a}{2}) < \frac{\epsilon}{2}. \tag{11}$$

Take $n \geq n(\epsilon)$. Then we have, using (10) and (11) and previous and devices

$$\rho(|f - f_n| > a) \leq \rho(|f - f_{n_{k(\epsilon)}}| > \frac{a}{2}) + \rho(|f_{n_{k(\epsilon)}} - f_n| > \frac{a}{2}) < \epsilon.$$

□

The second consequence of Theorem 15 is Theorem 17, which relates convergence in the seminormed Köthe space $\mathcal{L}(\mathcal{E},\mathcal{A})$ to the other types of convergence.

Theorem 17. *Assume ρ $R-F$. Let $(f_n)_n$ be a sequence in $\mathcal{L}_\rho(\mathcal{E},\mathcal{A})$ and $f \in \mathcal{L}_\rho(\mathcal{E},\mathcal{A})$ such that $f_n \underset{n}{\rightarrow} f$ in $\mathcal{L}_\rho(\mathcal{E},\mathcal{A})$.*

Then, there exists a subsequence $(f_{n_k})_k \subset (f_n)_n$ such that $f_{n_k} \underset{k}{\overset{\rho u}{\rightarrow}} f$ (hence $f_{n_k} \underset{k}{\rightarrow} f$ μ - a.e. and $f_{n_k} \underset{k}{\overset{\rho}{\rightarrow}} f$).

Proof. We have seen (Theorem 11) that the remarks in the brackets are true. Using Theorem 7, we see that $f_n \underset{n}{\overset{\rho}{\to}} f$. It follows (Proposition 5) that $(f_n)_n$ is Cauchy in ρ - measure. Using Theorem 15 we find $F \in \mathcal{M}(\mathcal{A}, \mu)$ and a subsequence $(f_{n_k})_k \subset (f_n)_n$ such that $f_{n_k} \underset{k}{\overset{\rho u}{\to}} F$. Consequently, $f_{n_k} \underset{k}{\overset{\rho}{\to}} F$ (again Theorem 11). We have also $f_{n_k} \underset{k}{\overset{\rho}{\to}} f$ (Theorem 7). Then $f = F$ μ - a.e. (Proposition 6).

Finally, we have $f_{n_k} \underset{k}{\overset{\rho u}{\to}} F$ and $F = f$ μ - a.e. and we use Proposition 12 to conclude that $f_{n_k} \underset{k}{\overset{\rho u}{\longrightarrow}} f$. □

3.5. This last section or our paper is dedicated to some examples.

Example 18 (Trivial vector fields)**.** We consider a non empty set T and the locally compact (separated) space will be T equipped with the discrete topology. The σ - algebra $\mathcal{T}$ will be $\mathcal{P}(T) =$ the set of all subsets of T and the measure $\mu : \mathcal{P}(T) \to \overline{\mathcal{R}}_+$ will be the counting measure ($\mu(A) =$ the number of elements in A, in case A is finite and $\mu(A) = \infty$, in case A is infinite). Because the only negligible set is $\emptyset$, it is clear that $f(t) = g(t)$ μ - a.e. means $f = g$.

Let also $\mathcal{E} = (E_t)_{t \in T}$ be a family of Banach spaces. We shall take as fundamental family of continuous vector fields $\mathcal{A} = \mathcal{C}(\mathcal{E})$. This is possible, because any function defined on T is continuous and, for any $t \in T$, we have the equality $\{x(t) |\ x \in \mathcal{C}(\mathcal{E})\} = E_t$. Any $x \in \mathcal{C}(\mathcal{E})$ is continuous (with respect to $\mathcal{A} = \mathcal{C}(\mathcal{E})$ - take $y = x$ in the definition of continuity), hence we have the equalities $\mathcal{C}(\mathcal{E}) = \mathcal{C}_{\mathcal{A}}(\mathcal{E}) = \mathcal{M}(\mathcal{A}, \mu)$. Because $M_+(\mu) = \{u : T \to \overline{\mathbb{R}}_+\}$, we have for any function norm $\rho : M_+(\mu) \to \overline{\mathbb{R}}_+$ the equality $\mathcal{L}_\rho(\mathcal{E}, \mathcal{A}) = \{x \in \mathcal{C}(\mathcal{E}) \mid \rho|x| < \infty\}$ and also, identifying classes with functions $L_\rho(\mathcal{E}, \mathcal{A}) \equiv \mathcal{L}_\rho(\mathcal{E}, \mathcal{A}) \overset{def}{=} L_\rho$ normed with the norm $x \mapsto \|x\| = \rho|x|$.

We shall work in the concrete case $T = \mathbb{N}^*$. To this end, recall first the definition of the Banach spaces of sequences l^p, for $1 \le p \le \infty$:

- for $1 \le p < \infty$:

$$l^p = \{x = (x_n)_{n \ge 1} \mid x_n \in K, \sum_{n=1}^{\infty} |x_n|^p < \infty\}$$

equipped with the norm

$$\|x\|_p = (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}};$$

- for $p = \infty$:

$$l^\infty = \{x = (x_n)_{n\geq 1} \mid x_n \in K, (x_n)_n \text{ is bounded }\}$$

equipped with norm

$$\|x\|_\infty = \sup_n |x_n|.$$

So, let us take $T = \mathbb{N}^*$ and let, for any $n \in \mathbb{N}^*, E_n = l^n$. For any function norm $\rho : M_+(\mu) \to \overline{\mathbb{R}}_+$, an element $x \in L_\rho$ is identified with an infinite matrix $(x_{mn})_{m\geq 1, n\geq 1}$ such that, for any $n \in \mathbb{N}^*$,

$$x(n) = (x_{mn})_{m\geq 1} \in l^n, |x(n)| = \|x(n)\|_n = (\sum_{m=1}^{\infty} |x_{mn}|^n)^{\frac{1}{n}}$$

and $|x| = (|x(n)|)_{n\geq 1}, \|x\| = \rho|x|$. In case $\rho = \| \|_p$, for $1 \leq p \leq \infty$, we shall have:

- for $1 \leq p < \infty$:

$$\|x\| = (\sum_{n=1}^{\infty} |x(n)|^p)^{\frac{1}{p}} = (\sum_{n=1}^{\infty} (\sum_{m=1}^{\infty} |x_{mn}|^n)^{\frac{p}{n}})^{\frac{1}{p}}$$

- for $p = \infty$:

$$\|x\| = \sup_n |x(n)| = \sup_n (\sum_{m=1}^{\infty} |x_{mn}|^n)^{\frac{1}{n}}.$$

We shall exhibit an element $x \in L_\rho$ for any $\rho = \| \|_p$, $1 \leq p \leq \infty$. Take $\alpha > 1$, $\beta > 1$ and let $x \equiv (x_{mn})_{m\geq 1, n\geq 1}$ where $x_{mn} = \frac{1}{m^\alpha n^\beta}$. First, it is seen that $x \in L_\rho$, for $\rho = \| \|_1$. Namely, in this case

$$\|x\| = \sum_{n=1}^{\infty} (\sum_{m=1}^{\infty} (\frac{1}{m^\alpha n^\beta})^n)^{\frac{1}{n}} = \sum_{n=1}^{\infty} \frac{1}{n^\beta} (\sum_{m=1}^{\infty} \frac{1}{m^{\alpha n}})^{\frac{1}{n}} = \sum_{n=1}^{\infty} \frac{1}{n^\beta} A_n^{\frac{1}{n}},$$

where $A_n = \sum_{m=1}^{\infty} \frac{1}{m^{\alpha n}}$. It is seen that the sequence $(A_n)_n$ is decreasing, with $1 < A_n \leq A_1 < \infty$. Hence $1 < A_n^{\frac{1}{n}} \leq A_1^{\frac{1}{n}} \leq M$, where M is a constant (depending upon α), hence $\|x\| < \infty$.

We proved the fact that $x \in L_\rho$, for $\rho = \| \ \|_1$, which means that $(|x(n)|)_{n\leq 1} \in l^1$. But $l^1 \subset l^p$, for any $1 \leq p \leq \infty$, hence $(|x(n)|)_{n\geq 1} \in l^p$, $1 \leq p \leq \infty$, which proves the fact that $x \in L_\rho$ for $\rho = \| \ \|_p$, $1 \leq p \leq \infty$.

Let us notice also that, in case $\rho = \| \ \|_p$, $1 \leq p \leq \infty$, one has $\rho\ F$, hence $\rho\ R{-}F$ (see [1]). So, in this case, L_ρ is Banach (Theorem 3) and convergence in L_ρ implies pointwise convergence for a subsequence (Theorem 17).

Example 19. The (locally) compact separated space will be $T = [0,1]$, with natural topology. The σ - algebra $\mathcal{T}$ will be the set of Lebesgue measurable sets of T and $\mu : \mathcal{T} \to \mathbb{R}_+$ will be the Lebesgue measure. We shall consider the family of Banach spaces $\mathcal{E} = (E_t)_{t\in T}$, where, for any $t \in T = [0,1]$, we take $E_t = L^{1/t}$ (with $E_0 = L^\infty$), writing $L^p \overset{def}{=} L^p(\mu)$, for $1 \leq p < \infty$.

So, for any $t \in T = [0,1]$, we have the measure space (S_t, Σ_t, μ_t),where $S_t = T, \Sigma_t = \mathcal{T}, \mu_t = \mu$. We also have, for any $t \in T$, the function norm $\rho_t = \| \ \|_{1/t}$, giving the Köthe space $E_t = L_{\rho_t} (= L^{1/t})$. From now on, $1/0 \overset{def}{=} \infty$.

In order to complete our schema, we shall construct a fundamental family $\mathcal{A}$ of continuous vector fields for $\mathcal{C}(\mathcal{E})$. To this end, write

$$\mathcal{E}(\mathcal{T}) = \{f : T \to K \mid f \quad \text{is} \quad \mathcal{T}\text{ - simple }\} \text{ and } E(\mathcal{T}) = \{\tilde{f} \mid f \in \mathcal{E}(\mathcal{T})\}$$

(here $\tilde{f} = \{g : T \to K \mid g(t) = f(t)\ \mu\text{ - a.e. }\}$).

One knows that $E(\mathcal{T})$ is dense in all $L^p, 1 \leq p \leq \infty$. This makes possible the construction, for any $\tilde{f} \in E(\mathcal{T})$, of the element $x(\tilde{f}) \in \mathcal{C}(\mathcal{E})$ acting via $x(\tilde{f})(t) = \tilde{f} \in L^{1/t}$, for any $t \in T$.

Write $\mathcal{A} = \{x(\tilde{f}) \mid \tilde{f} \in E(\mathcal{T})\}$ and let us prove that $\mathcal{A}$ is a fundamental family of continuous vector fields for $\mathcal{C}(\mathcal{E})$. Indeed:

a) $\mathcal{A}$ is a vector subspace of $\mathcal{C}(\mathcal{E})$ (we have $x(\tilde{f}) + x(\tilde{g}) = x(\tilde{f} + \tilde{g})$ and $\alpha x(\tilde{f}) = x(\alpha\tilde{f})$ with obvious notations).

b) Axiom (A_1) is fulfilled: for any $\tilde{f} \in E(\mathcal{T})$, the function $\varphi : [0,1] \to \mathbb{R}_+$, $\varphi(t) = |x(\tilde{f})|(t) = \|\tilde{f}\|_{1/t}$ (with $1/0 \overset{def}{=} \infty$) is continuous. This is mathematical folklore. (see, e.g., [7], pp. 2).

c) Axiom (A_2) is fulfilled: for any $t \in T$, the set $\{x(t) \mid x \in \mathcal{A}\}$ is dense in $E_t = L^{\frac{1}{t}}$.

Indeed, the last set is precisely $\{x(\tilde{f})(t) \mid \tilde{f} \in E(\mathcal{T})\} = E(\mathcal{T})$.

We completed our schema, so it is possible to speak about the Köthe space of vector fields $L_\rho((\rho_t)_t, \mathcal{A})$.

In the sequel, we shall present an example of measurable vector field with respect to $\mathcal{A}$. To this end, we shall consider a bounded Lebesgue measurable function $f : [0,1] \to K$. For any $t \in [0,1]$, let us write $x(t) = \widetilde{f\varphi_{[0,t]}} \in L^{1/t}$.

We defined $x \in \mathcal{C}(\mathcal{E})$ and we shall prove that x is continuous with respect to $\mathcal{A}$, at any $t_0 \in (0,1]$. Before proceeding to the proof, let us remark that for any bounded measurable function $g : [0,1] \to K$ and for any $0 < t \leq 1$, one has, if $A \in \mathcal{T}$:

$$\|g\varphi_A\|_{1/t} \leq \|g\|_\infty \mu(A)^t. \tag{12}$$

To prove the continuity of x at $t_0 \in T \setminus \{0\}$ means to prove the following fact: for any $\epsilon > 0$, there exists a (basic) neighbourhood V of t_0 in $[0,1]$ and an element $\widetilde{h_1} \in E(\mathcal{T})$ such that, for any $t \in V$ one has

$$d(t) = \|x(t) - x(\widetilde{h_1})(t)\|_{1/t} = \|f\varphi_{[0,t]} - h_1\|_{1/t} < \epsilon. \tag{13}$$

Let $1 > \epsilon > 0$. We shall find $\widetilde{h_1} \in E(\mathcal{T})$ and $V \in \mathcal{V}(t_0)$ which will make (13) to be true. Notice first the existence of $h \in \mathcal{E}(\mathcal{T})$ such that

$$\|h - f\|_\infty < \frac{\epsilon}{2}. \tag{14}$$

We shall see that one can take

$$h_1 = h\varphi_{[0,t_0]} \in \mathcal{E}(\mathcal{T}). \tag{15a}$$

It remains to find V. Namely, we shall take V of the form

$$V = (t_0 - \delta, t_0 + \delta) \cap [0,1], \tag{15b}$$

where $t_0 - \delta > 0$ (so $0 \notin V$) and δ must satisfy some supplementary conditions which will be exhibited in the sequel. Because $\lim_{\delta \to 0} \delta^{t_0 - \delta} = 0$ there exists $\delta_1 > 0$ such that, if $0 < \delta < \delta_1$, one has

$$(\|f\|_\infty)\delta^{t_0-\delta} < \frac{\epsilon}{2}. \tag{16}$$

Because $\lim_{\delta \to 0} \delta^{t_0} = 0$ there exists $\delta_2 > 0$ such that, for any $0 < \delta < \delta_2$, one has

$$\|f\|_\infty \cdot \delta^{t_0} < \frac{\epsilon}{2}. \tag{17}$$

The neighbouhood V will be constructed using $\delta > 0$ such that (supplementary condition) $\delta < \min(\delta_1, \delta_2)$ and this implies that (16) and (17) are true.

We finish the proof of the continuity at t_0 by showing that h_1 chosen according to (15a) and V chosen according to (15b) satisfy (13). So let $t \in V$.

In case $0 < t \leq t_0$, we can write

$$d(t) = \| f\varphi_{[0,t]} - h\varphi_{[0,t]} - h\varphi_{[t,t_0]} \|_{1/t} \leq \|(f-h)\varphi_{[0,t]} \|_{1/t} + \| h\varphi_{[t,t_0]} \|_{1/t}.$$

From (14) we get $\|h\|_\infty < \|f\|_\infty + 1$ and, using (12) and (14):

$$\begin{aligned} d(t) &\leq \frac{\epsilon}{2}(\mu([0,t]))^t + (\|f\|_\infty + 1)(t - t_0)^t < \frac{\epsilon}{2}t^t + (\|f\|_\infty + 1)\delta^t \\ &< \frac{\epsilon}{2} + (\|f\|_\infty + 1)\delta^{t_0 - \delta} < \epsilon \end{aligned}$$

(according to (16)).

In case $t_0 \leq t \leq 1$, we can write

$$d(t) = \| f\varphi_{[0,t_0]} + f\varphi_{[t_0,t]} - h\varphi_{[0,t_0]} \|_{1/t} \leq \|(f-h)\varphi_{[0,t_0]}\|_{1/t} + \| f\varphi_{[t_0,t]}\|_{1/t}.$$

Again (12) and (14) lead us to:

$$\begin{aligned} d(t) &\leq \frac{\epsilon}{2}(\mu([0,t_0]))^t + \|f\|_\infty (t - t_0)^t = \frac{\epsilon}{2}t_0^t + \|f\|_\infty (t - t_0)^t \\ &\leq \frac{\epsilon}{2} + \|f\|_\infty (t - t_0)^{t_0} \leq \frac{\epsilon}{2} + \|f\|_\infty \delta^{t_0} < \epsilon \end{aligned}$$

(according to (17)).

In all cases (13) is proved. So x is continuous on $T \setminus \{0\} = (0,1]$ with respect to $\mathcal{A}$. The reader might ask what happens for $t_0 = 0$. In this case it is possible for x to be discontinuous at t_0. For instance, let us take $f \equiv 1$. We shall see that is this case the vector field x is discontinuous at 0 (with respect to $\mathcal{A}$). Indeed, the continuity of x at $t_0 = 0$ would imply the continuity of $|x|$ at $t_0 = 0$.

We have $|x|(0) = 0$. On the other hand, for any $0 < t \leq 1$, one has $x(t) = \widetilde{\varphi_{[0,t]}}$ hence $|x(t)| = \|\widetilde{\varphi_{[0,t]}}\|_{1/t} = t^t$ and $\lim_{t\to 0} |x(t)| = 1 \neq |x|(0)$. On can prove (taking $y = 0$) that, in case $\lim_{t\to 0} f(t) = 0$, the vector field x is continuous at $t_0 = 0$. So, generally speaking, x is continuous on $(0,1]$ and (possibly) discontinuous at $t = 0$. It follows immediately that x is $(\mathcal{A}, \mu)$ - measurable. If ρ is a function norm on $(T, \mathcal{T}, \mu)$ it is possible to compute $\rho|x|$. Let us take, e.g., $f \equiv 1$ and $\rho = \| \ \|_p$,$1 \leq p \leq \infty$. We shall see that $\rho|x| < \infty$ for all such ρ, consequently we have $x \in \mathcal{L}_p((\rho_t)_t, \mathcal{A})$ (and $\tilde{x} \in L_\rho((\rho_t)_t, \mathcal{A})$). As we have seen, $x(t) = \widetilde{\varphi_{[0,t]}}$ for any $t \in [0, 1]$. Consequently: $|x(t)| = t^t$, if $t > 0$ and $|x(0)| = 0$. For $\rho = \| \ \|_p$,$1 \leq p < \infty$, we have

$$\rho|x| = (\int |x|^p d\mu)^{1/p} = (\int_{0+0}^{1} t^{pt} dt)^{1/p} < \infty$$

(because $t^{pt} \leq 1$) For $\rho = \| \ \|_\infty$, we have $\rho|x| = \sup_{t\in[0,1]} t^t = 1$ (with the convention $0^0 = 1$).

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