ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
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ON THE Ψ-EXPONENTIAL ASYMPTOTIC STABILITY OF NONLINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

BY

AUREL DIAMANDESCU

Abstract. In this paper we prove sufficient conditions for Ψ-exponential asymptotic stability in variation on $\mathbb{R}_+$ of trivial solution of the system $x' = F(t,x)$ and for Ψ-exponential asymptotic stability on $\mathbb{R}_+$ of trivial solution of the system $x' = F(t,x) + G(t,x)$.

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1. Introduction

The purpose of our paper is to prove sufficient conditions for Ψ-exponential asymptotic stability of trivial solution of the system

$$y' = F(t,y) + G(t,y) \tag{1}$$

which can be seen as a perturbed system of

$$x' = F(t,x). \tag{2}$$

We investigate conditions on the fundamental matrix $\Phi(t, t_0, x_0)$ for the variational system

$$z' = \frac{\partial F}{\partial x}(t, x(t, t_0, x_0))z \tag{3}$$

and on the functions F and G under which the trivial solution of (2) is Ψ-exponentially asymptotically stable in variation on $\mathbb{R}_+$ and the trivial solution of (1) is Ψ-exponentially asymptotically stable on $\mathbb{R}_+$. Here, Ψ is a matrix function whose introduction permits us obtaining a mixed asymptotic behavior for the components of solutions.

The problem of Ψ-stability for systems of ordinary differential equations has been studied by many authors, as e.g. Akinyele [1], Constantin [3], [4], Morchalo [11]. In these papers, the function Ψ is a scalar (positive) continuous function (and monotone in [1] or nondecreasing and such that $\Psi(t) \geq 1$, for $t \geq 0$ in [4]).

In our papers [8] and [9], we have proved sufficient conditions for Ψ-uniform stability and Ψ-asymptotic stability on $\mathbb{R}_+$ of the trivial solution of the systems (1) and (2). In these papers, the function Ψ is a matrix function.

Our study is based on the nonlinear variation of constants formula of Alekseev [2] and on the theory of integral inequalities.

2. Definitions, notations and hypotheses

Let $\mathbb{R}^n$ denote the Euclidean n-space. For $x = (x_1, x_2, x_3, \ldots, x_n)^T \in \mathbb{R}^n$, let $\| x \| = \max\{| x_1 |, | x_2 |, | x_3 |, \ldots, | x_n |\}$ be the norm of x. For an $n \times n$ matrix $A = (a_{ij})$, we define the norm $| A | = \sup_{\|x\| \leq 1} \| Ax \|$.

In the system (2), we assume that the function $F : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable and $F(t, 0) = 0$, for $t \in \mathbb{R}_+ = [0, \infty)$.

Let $x(t, t_0, x_0)$ be the unique solution of (2) passing through the point $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$. Suppose that $x(t, t_0, x_0)$ is defined on $[t_0, \infty)$. The variational system of (2) associated with $x(t, t_0, x_0)$ is the linear system (3). It is well-known that the matrix $\Phi(t, t_0, x_0)$ given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0}\left(x(t, t_0, x_0)\right)$$

is the fundamental matrix of the system (3) which is the identity matrix at $t = t_0$ (see [5], Ch. I, Th. 7).

Similarly, $y(t, t_0, y_0)$ will denote a solution of (1) which passes through the point (t_0, y_0).

The nonlinear variation of constants formula can be written as

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s, t_0, y_0)) G(s, y(s, t_0, y_0)) ds \tag{4}$$

(see [2]).

We note that under the above hypotheses on F, we have

(5) $$x(t,t_0,x_0)=\left[\int_0^1 \Phi(t,t_0,sx_0)ds\right]\cdot x_0,\ \text{for t } \geq t_0.$$

Let $\Psi_i : \mathbb{R}_+ \to (0,\infty)$, $i = 1,2,\ldots,n$, be continuous functions and $\Psi = \text{diag}\,[\Psi_1,\Psi_2,\ldots,\Psi_n]$. For $p \geq 1$ and $x_0 \in \mathbb{R}_+$, we denote by $L^p_\Psi([t_0,\infty))$ the linear space of all functions $x : [t_0,\infty) \to \mathbb{R}^n$ satisfying the condition $\int_{t_0}^\infty \parallel \Psi(t)x(t) \parallel^p dt < +\infty$. Observe that for bounded matrix functions Ψ and Ψ^{-1}, we have $L^p_\Psi([t_0,\infty)) \equiv L^p_{I_n}([t_0,\infty)) \equiv L^p([t_0,\infty))$.

We now give the definitions of different kinds of Ψ-stability in terms of the behavior of solutions of (2) and of the variational system (3).

Definition 1. The trivial solution of (2) is said to be:

a) Ψ-stable on $\mathbb{R}_+$ iff for every $\varepsilon > 0$ and every $t_0 \in \mathbb{R}_+$, there exists a $\delta = \delta(\varepsilon,t_0) > 0$ such that any solution $x(t,t_0,x_0)$ of (2) which satisfies the inequality $\parallel \Psi(t_0)x_0 \parallel < \delta$, also satisfies the inequality $\parallel \Psi(t)x(t,t_0,x_0) \parallel < \varepsilon$, for all $t \geq t_0$;

b) Ψ-uniformly stable on $\mathbb{R}_+$ iff it is Ψ-stable on $\mathbb{R}_+$ and the δ above is independent of t_0;

c) Ψ-exponentially asymptotically stable on $\mathbb{R}_+$ iff there exists a $\lambda > 0$ and, for every $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that any solution $x(t,t_0,x_0)$ of (2) which satisfies the inequality $\parallel \Psi(t_0)x_0 \parallel < \delta(\varepsilon)$ for some $t_0 \geq 0$, satisfies the inequality $\parallel \Psi(t)x(t,t_0,\ x_0)\parallel \leq \varepsilon e^{-\lambda(t-t_0)}$, for all $t \geq t_0$;

d) Ψ-stable in variation on $\mathbb{R}_+$ iff for every $t_0 \in \mathbb{R}_+$, there exist the constants $M(t_0) \geq 1$ and $\delta_0 = \delta_0(t_0) > 0$ such that $\mid \Psi(t)\Phi(t,t_0,x_0)\Psi^{-1}(t_0) \mid \leq M(t_0)$, for all $t \geq t_0$ and for all $x_0 \in \mathbb{R}^n$ with $\parallel \Psi(t_0)x_0 \parallel < \delta_0$. If, in addition, the M and δ_0 above are independent of t_0, then the trivial solution of (2) is said to be Ψ-uniformly stable in variation on $\mathbb{R}_+$.

If, in addition, $\delta_0(t_0) = +\infty$, then the trivial solution of (2) is said to be globally Ψ-(uniformly) stable in variation on $\mathbb{R}_+$.

e) Ψ-exponentially asymptotically stable in variation on $\mathbb{R}_+$ iff there exist the constants $L \geq 1$, $\lambda > 0$ and $\delta > 0$ such that $\mid \Psi(t)\Phi(t,t_0,x_0)\Psi^{-1}(t_0) \mid \leq Le^{-\lambda(t-t_0)}$, for all $t \geq t_0 \geq 0$, whenever $\parallel \Psi(t_0)x_0 \parallel < \delta$;

If, in addition, $\delta = +\infty$, then the trivial solution of (2) is said to be globally Ψ-exponentially asymptotically stable in variation on $\mathbb{R}_+$.

f) $(\Psi - L^p_\Psi)$-stable on $\mathbb{R}_+$ iff it is Ψ-stable on $\mathbb{R}_+$ and in addition, for every $t_0 \in \mathbb{R}_+$, there exists $\tilde{\delta}_0 = \tilde{\delta}_0(t_0) > 0$ such that any solution $x(t, t_0, x_0)$ of (2) which satisfies the inequality $\| \Psi(t_0)x_0 \| < \tilde{\delta}_0$, also satisfies the condition $x(t, t_0, x_0) \in L^p_\Psi([t_0, \infty))$.

Remark 1. 1. For $\Psi = I_n$ (identity $n \times n$ matrix), we obtain the different kinds of classical stability (see [5], Ch. III).

In addition, we obtain the notions of (uniform) stability in variation (see [6]).

2. For $\Psi_1 = \Psi_2 = \ldots = \Psi_n = \psi$, we obtain the different kinds of ψ-stability (see [11]).

In addition, we obtain the notion of $(\psi - L^p_\Psi)$- (asymptotic) stability on $\mathbb{R}_+$ (see [11]).

3. Main results

The first theorem gives new sufficient conditions for the Ψ-exponential asymptotic stability in variation and for the global $(\Psi - L^p_{e^{\lambda t}\Psi})$-exponential asymptotic stability of the trivial solution of (2).

Theorem 1. *Let $\Phi(t, t_0) = \Phi(t, t_0, 0)$ be a fundamental matrix for the variational system (3) in which $x_0 = 0$.*

Suppose that there exist a positive constants λ, M and δ such that

$$H(t, t_0, x_0) \equiv | e^{\lambda t}\Psi(t)\Phi(t, t_0)\Psi^{-1}(t_0) | \cdot e^{\int_{t_0}^t |\Psi(s)\Phi(s,t_0)\Psi^{-1}(t_0)|\cdot|\Psi(t_0)\Phi(t_0,s)\left(\frac{\partial F}{\partial x}(s,x(s,t_0,x_0)) - \frac{\partial F}{\partial x}(s,0)\right)\Psi^{-1}(s)|ds} \leq M,$$

for all $t \geq t_0 \geq 0$ and all $x_0 \in \mathbb{R}^n$ with $\| \Psi(t_0)x_0 \| < \delta$.

Then, we have the following conclusions:

a) the system (3) is Ψ-exponentially asymptotically stable on $\mathbb{R}_+$, for all $t_0 \geq 0$ and $x_0 \in \mathbb{R}^n$ with $\| \Psi(t_0)x_0 \| < \delta$;

b) the trivial solution of (2) is Ψ-exponentially asymptotically stable in variation on $\mathbb{R}_+$;

c) the system (3) is $e^{\lambda t}\Psi$- uniformly stable on $\mathbb{R}_+$, for all $t_0 \geq 0$ and $x_0 \in \mathbb{R}^n$ with $\| \Psi(t_0)x_0 \| < \delta$;

d) all solutions of the system (3) *belong to* $L^p_\Psi(\mathbb{R}_+)$*, for all* $p \geq 1$.

If, in addition, there exists a $\delta_1 > 0$ *such that* $H(t, t_0, x_0) \in L^p([t_0, \infty))$, $p \geq 1$, *for all* $t_0 \geq 0$ *and* $x_0 \in \mathbb{R}^n$ *with* $\| \Psi(t_0)x_0 \| < \delta_1$, *then the trivial solution of (3) is* $(e^{\lambda t}\Psi - L^p_{e^{\lambda t}\Psi})$*-uniformly stable on* $\mathbb{R}_+$.

If, in addition, $H(t, t_0, x_0) \leq M$ *for all* $t \geq t_0 \geq 0$ *and* $x_0 \in \mathbb{R}^n$ *and* $H(t, t_0, x_0) \in L^p(\mathbb{R}_+)$, $p \geq 1$, *for all* $t_0 \geq 0$ *and* $x_0 \in \mathbb{R}^n$, *then, the trivial solution of (2) is globally* $(\Psi - L^p_{e^{\lambda t}\Psi})$*-exponentially asymptotically stable in variation on* $\mathbb{R}_+$.

Proof. For $t_0 \in \mathbb{R}_+$ and $z_0 \in \mathbb{R}^n$, let $z(t, t_0, z_0)$ be the solution of (3) such that $z(t_0, t_0, z_0) = z_0$. The equation (3) can be write in the form

$$z' = \frac{\partial F}{\partial x}(t, 0)z + \left[\frac{\partial F}{\partial x}(t, x(t, t_0, x_0)) - \frac{\partial F}{\partial x}(t, 0)\right] z.$$

Therefore, by the variation of constants formula, we have that

$$\begin{aligned} z(t, t_0, z_0) =& \Phi(t, t_0)z_0 \\ &+ \int_{t_0}^{t} \Phi(t, s) \left[\frac{\partial F}{\partial x}(s, x(s, t_0, x_0)) - \frac{\partial F}{\partial x}(s, 0)\right] z(s, t_0, z_0)ds, \forall t \geq t_0. \end{aligned}$$

From this, we have

$$\begin{aligned} &e^{\lambda t}\Psi(t)z(t, t_0, z_0) = e^{\lambda t}\Psi(t)\Phi(t, t_0)\Psi^{-1}(t_0)\Psi(t_0)z_0 \\ &+ e^{\lambda t}\Psi(t)\Phi(t, t_0)\Psi^{-1}(t_0) \\ &\cdot \int_{t_0}^{t} \Psi(t_0)\Phi(t_0, s) \left[\frac{\partial F}{\partial x}(s, x(s, t_0, x_0)) - \frac{\partial F}{\partial x}(s, 0)\right] \Psi^{-1}(s)e^{-\lambda s} \\ &\cdot \left(e^{\lambda s}\Psi(s)z(s, t_0, z_0)\right) ds, t \geq t_0 \end{aligned}$$

and then,

$$\begin{aligned} &\| e^{\lambda t}\Psi(t)z(t, t_0, z_0) \| \leq | e^{\lambda t}\Psi(t)\Phi(t, t_0)\Psi^{-1}(t_0) | \| \Psi(t_0)z_0 \| \\ &+ | e^{\lambda t}\Psi(t)\Phi(t, t_0)\Psi^{-1}(t_0) | \\ &\cdot \int_{t_0}^{t} e^{-\lambda s} | \Psi(t_0)\Phi(t_0, s) \left[\frac{\partial F}{\partial x}(s, x(s, t_0, x_0)) - \frac{\partial F}{\partial x}(s, 0)\right] \Psi^{-1}(s) | \\ &\cdot \| e^{\lambda s}\Psi(s)z(s, t_0, z_0) \| ds, \forall t \geq t_0. \end{aligned}$$

For $t \geq t_0$, we denote

$$a(t) = \mid e^{\lambda t}\Psi(t)\Phi(t,t_0)\Psi^{-1}(t_0) \mid,$$
$$b(t) = \mid e^{-\lambda t}\Psi(t_0)\Phi(t_0,t)\left[\frac{\partial F}{\partial x}(t,x(t,t_0,x_0)) - \frac{\partial F}{\partial x}(t,0)\right]\Psi^{-1}(t) \mid .$$

Thus, the continuous function $\parallel e^{\lambda t}\Psi(t)z(t,t_0,z_0) \parallel$ satisfies the inequality

$$\parallel e^{\lambda t}\Psi(t)z(t,t_0,z_0) \parallel \leq a(t) \parallel \Psi(t_0)z_0 \parallel$$
$$+a(t)\int_{t_0}^{t} b(s) \parallel e^{\lambda s}\Psi(s)z(s,t_0,z_0) \parallel ds, \qquad \forall t \geq t_0.$$

From a generalization of Gronwall Lemma (see [10]), we obtain that

$$\parallel e^{\lambda t}\Psi(t)z(t,t_0,z_0) \parallel \leq a(t) \parallel \Psi(t_0)z_0 \parallel$$
$$\cdot\left(1+\int_{t_0}^{t} a(s)b(s)e^{\int_s^t a(u)b(u)du}ds\right), \quad \forall t \geq t_0$$

or, integrating by parts,

$$\parallel e^{\lambda t}\Psi(t)z(t,t_0,z_0) \parallel \leq \mid e^{\lambda t}\Psi(t)\Phi(t,t_0)\Psi^{-1}(t_0) \mid\parallel \Psi(t_0)z_0 \parallel$$
$$\cdot\, e^{\int_{t_0}^{t}|\Psi(s)\Phi(s,t_0)\Psi^{-1}(t_0)|\cdot|\Psi(t_0)\Phi(t_0,s)\left(\frac{\partial F}{\partial x}(s,x(s,t_0,x_0))-\frac{\partial F}{\partial x}(s,0)\right)\Psi^{-1}(s)|ds}.$$

It follows that

$$\parallel e^{\lambda t}\Psi(t)z(t,t_0,z_0) \parallel \leq H(t,t_0,x_0) \parallel \Psi(t_0)z_0 \parallel, \tag{6}$$

for all $t \geq t_0 \geq 0$ and for all $x_0 \in \mathbb{R}^n$ with $\parallel \Psi(t_0)x_0 \parallel < \delta$. From this,

$$\parallel \Psi(t)z(t,t_0,z_0) \parallel \leq Me^{-\lambda(t-\ t_0)} \parallel \Psi(t_0)z_0 \parallel,$$

for all $t \geq t_0 \geq 0$ and for all $x_0 \in \mathbb{R}^n$ with $\parallel \Psi(t_0)x_0 \parallel < \delta$.

This shows that the linear system (3) is Ψ-exponentially asymptotically stable on $\mathbb{R}_+$ for all $t_0 \geq 0$ and for all $x_0 \in \mathbb{R}^n$ with $\parallel \Psi(t_0)x_0 \parallel < \delta$.

Now, let $u \in \mathbb{R}^n$ such that $\parallel u \parallel < 1$. If we take $z_0 = \frac{\delta}{2}\Psi^{-1}(t_0)u$, we have $\parallel \Psi(t_0)z_0 \parallel \leq \frac{\delta}{2}$ and then, $\parallel \Psi(t)z(t,t_0,z_0) \parallel \leq \frac{\delta}{2}H(t,t_0,x_0)e^{-\lambda(t-\ t_0)}$, $t \geq t_0$. Hence,

$$\parallel \Psi(t)\Phi(t,t_0,x_0)\Psi^{-1}(t_0)u \parallel = \parallel \Psi(t)\Phi(t,t_0,x_0)\frac{2}{\delta}z_0 \parallel$$
$$= \parallel \frac{2}{\delta}\Psi(t)z(t,t_0,z_0) \parallel \leq H(t,t_0,x_0)e^{-\lambda(t-\ t_0)}, t \geq t_0.$$

It follows that

$$\begin{aligned} \mid \Psi(t)\Phi(t,t_0,x_0)\Psi^{-1}(t_0) \mid &= \sup_{\|u\|<1} \| \Psi(t)\Phi(t,t_0,z_0)\Psi^{-1}(t_0)u \| \\ &\le H(t,t_0,x_0)e^{-\lambda(t-\ t_0)}, \end{aligned} \tag{7}$$

for all $t \ge t_0 \ge 0$ and for all $x_0 \in \mathbb{R}^n$ with $\| \Psi(t_0)x_0 \|< \delta$.

From hypothesis and (7), it follows that the trivial solution of (2) is Ψ-exponentially asymptotically stable in variation on $\mathbb{R}_+$.

Now, let $t_0 \ge 0$ and $x_0 \in \mathbb{R}^n$ such that $\| e^{\lambda t_0}\Psi(t_0)x_0 \|< \delta$. Then, $\| \Psi(t_0)x_0 \|= e^{-\lambda t_0} \| e^{\lambda t_0}\Psi(t_0)x_0 \|< \delta$. From (6) it follows that the system (3) is $e^{\lambda t}\Psi$-uniformly stable on $\mathbb{R}_+$ for all $t_0 \ge 0$ and for all $x_0 \in \mathbb{R}^n$ with $\| \Psi(t_0)x_0 \|< \delta$.

If, in addition, there exists a $\delta_1 > 0$ such that $H(t,t_0,x_0) \in L^p([t_0,\infty))$, $p \ge 1$, for all $t_0 \ge 0$ and $x_0 \in \mathbb{R}^n$ with $\| \Psi(t_0)x_0 \|< \delta_1$, then, for $\| \Psi(t_0)x_0 \|< \min\{\delta,\delta_1\}$, we have (6). It follows that $z(t,t_0,z_0) \in L^p_{e^{\lambda t}\Psi}([t_0,\infty))$ (for $t_0 \ge 0$ and $z_0 \in \mathbb{R}^n$). Thus, the trivial solution of (3) is $(e^{\lambda t}\Psi - L^p_{e^{\lambda t}\Psi})$-uniformly stable on $\mathbb{R}_+$. For every solution $z(t,t_0,z_0)$ of (3), we have (6). Then, $\| \Psi(t)z(t,t_0,z_0) \|^p \le \text{const.}\cdot e^{-p\lambda t}$, $t \ge t_0$. It follows that $z(t,t_0,z_0) \in L^p_\Psi(\mathbb{R}_+)$, for all $p \ge 1$.

Finally, suppose that $H(t,t_0,x_0) \le M$, for all $t \ge t_0 \ge 0$ and $x_0 \in \mathbb{R}^n$ and $H(t,t_0,x_0) \in L^p(\mathbb{R}_+)$, $p \ge 1$, for all $t_0 \ge 0$ and $x_0 \in \mathbb{R}^n$. From (5) and (7), we have

$$\begin{aligned} &\| e^{\lambda t}\Psi(t)x(t,t_0,x_0) \| \\ &= \left\| \left[\int_0^1 e^{\lambda t}\Psi(t)\Phi(t,t_0,sx_0)\Psi^{-1}(t_0)e^{-\lambda t_0}ds\right] \cdot e^{\lambda t_0}\Psi(t_0)x_0 \right\| \\ &\le \int_0^1 \mid e^{\lambda t}\Psi(t)\Phi(t,t_0,sx_0)\Psi^{-1}(t_0)e^{-\lambda t_0} \mid ds \cdot \| e^{\lambda t_0}\Psi(t_0)x_0 \| \\ &\le H(t,t_0,x_0) \| e^{\lambda t_0}\Psi(t_0)x_0 \|, \text{ for } t \ge t_0 \ge 0 \text{ and } x_0 \in \mathbb{R}^n. \end{aligned}$$

It follows that the trivial solution of (2) is globally $(\Psi - L^p_{e^{\lambda t}\Psi})$-exponentially asymptotically stable in variation on $\mathbb{R}_+$.

The proof is complete. □

Remark 2. In particular case $F(t,x) = A(t)x$, we have $\frac{\partial F}{\partial x}(t,x) = A(t)$. Therefore, the system (3) becomes the system

$$x' = A(t)x. \tag{8}$$

If $Y(t)$ is a fundamental matrix for this system, then $\Phi(t, t_0) = Y(t)Y^{-1}(t_0)$ and the condition from Theorem becomes $|\Psi(t)Y(t)Y^{-1}(t_0)\Psi^{-1}(t_0)| \leq Me^{-\lambda(t- t_0)}$, for all $t \geq t_0 \geq 0$.

This is a necessary and sufficient condition for Ψ-exponential asymptotic stability of the linear differential system (8), given in Theorem 1, [7].

Thus, Theorem extends Theorem 1, [7] to a nonlinear system of differential equations (2).

The next theorem gives sufficient conditions for the transfer of Ψ- exponential asymptotic stability of the system (2) to the perturbed system (1).

Theorem 2. *Suppose that:*

1) *the trivial solution of* (2) *is globally* Ψ*-exponentially asymptotically stable in variation on* $\mathbb{R}_+$;

2) *the continuous function* $G : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ *satisfies the condition*

$$\| \Psi(t)G(t, x) \| \leq \lambda(t) \| \Psi(t)x \|, t \in \mathbb{R}_+, x \in \mathbb{R}^n,$$

where $\lambda : \mathbb{R}_+ \to (0, \infty)$ *is continuous and satisfies one of the following conditions:*

a) $\Lambda = \int_0^\infty \lambda(t)dt < +\infty$;

b) $M = \sup_{t\geq 0} \lambda(t)$ *is a sufficiently small number;*

c) $\lim_{t\to\infty} \lambda(t) = 0$.

Then, the trivial solution of (1) *is* Ψ*-exponentially asymptotically stable on* $\mathbb{R}_+$.

If, in addition:

i) there exists a constant μ *such that the solution* $x(t, t_0, y_0)$ *of* (2) *belongs to the space* $L^p_{e^{\mu t}\Psi}([t_0, \infty))$, $t_0 \geq 0$, $y_0 \in \mathbb{R}^n$, $p \in (1, \infty)$,

ii) the function

$$v(t) = \left(\int_{t_0}^t \left(e^{(\mu-\lambda)(t-z)} \lambda(z) \right)^{\frac{p}{p-1}} dz \right)^{p-1}, t \geq t_0,$$

belongs to the space $L^1([t_0, \infty))$, *for* $t_0 \geq 0$, *then, the solution* $y(t, t_0, y_0)$ *of* (1) *belongs to the space* $L^p_{e^{\mu t}\Psi}([t_0, \infty))$.

Proof. From the hypothesis (1), it follows that there exist the constants $\lambda > 0$ and $L \geq 1$ such that $| \Psi(t)\Phi(t,s,\xi)\Psi^{-1}(s) | \leq Le^{-\lambda(t-s)}$, for all $t \geq s \geq 0$ and for all $\xi \in \mathbb{R}^n$.

Let $y = y(t, t_0, y_0)$ be an arbitrary solution of (1) for $t \geq t_0$ and $y_0 \in \mathbb{R}^n$. Suppose that $[t_0, t_1)$ is the existence interval of solution y.

Let $x = x(t, t_0, x_0)$ be an arbitrary solution of (2) for $t \geq t_0$ and $x_0 \in \mathbb{R}^n$.

Therefore, by the nonlinear variation of constants formula (4), we have that

$$y(t,t_0,y_0) = x(t,t_0,y_0) + \int_{t_0}^{t} \Phi(t,z,y(z,t_0,y_0))G(z,y(z,t_0,y_0))dz,$$

for $t_0 \leq t < t_1$ and $y_0 \in \mathbb{R}^n$. From (5) we have

$$x(t,t_0,y_0) = \left[\int_0^1 \Phi(t,t_0,sy_0)ds\right] y_0,$$

and then

$$\| \Psi(t)x(t,t_0,y_0) \| \leq \| \left[\int_0^1 \Psi(t)\Phi(t,t_0,sy_0)\Psi^{-1}(t_0)ds\right] \Psi(t_0)y_0 \|$$

$$\leq \int_0^1 | \Psi(t)\Phi(t,t_0,sy_0)\Psi^{-1}(t_0) | \, ds\cdot \| \Psi(t_0)y_0 \|$$

$$\leq \int_0^1 Le^{-\lambda(t-\ t_0)}ds\cdot \| \Psi(t_0)y_0 \| = Le^{-\lambda(t-\ t_0)} \| \Psi(t_0)y_0 \|, t \geq t_0.$$

It follows that

$$\| \Psi(t)y(t,t_0,y_0) \| \leq Le^{-\lambda(t-\ t_0)} \| \Psi(t_0)y_0 \|$$

$$+ \int_{t_0}^{t} | \Psi(t)\Phi(t,z,y(z,t_0,y_0))\Psi^{-1}(z) | \| \Psi(z)G(z,y(z,t_0,y_0)) \| \, dz$$

$$\leq Le^{-\lambda(t-\ t_0)} \| \Psi(t_0)y_0 \| + \int_{t_0}^{t} Le^{-\lambda(t-\ z)}\lambda(z) \| \Psi(z)y(z,t_0,y_0) \| \, dz,$$

for $t_0 \leq t < t_1$. Therefore, the positive continuous function

$$r(t) = e^{\lambda(t-t_0)}\|\Psi(t)y(t,t_0,y_0)\|$$

satisfies the inequality

$$r(t) \leq L \| \Psi(t_0)y_0 \| + L\int_{t_0}^{t} \lambda(z)r(z)dz, t_0 \leq t < t_1.$$

From here, applying the Gronwall inequality, we obtain

$$r(t) \leq L \parallel \Psi(t_0)y_0 \parallel e^{L\int_{t_0}^{t}\lambda(z)dz}, t_0 \leq t < t_1. \tag{9}$$

In case a), we have

$$\int_{t_0}^{t} \lambda(z)dz \leq \Lambda, \quad \text{for } t_0 \leq t < t_1,$$

and then $r(t) \leq Le^{L\Lambda} \parallel \Psi(t_0)y_0 \parallel, t_0 \leq t < t_1$ or

$$\parallel \Psi(t)y(t,t_0,y_0) \parallel \leq Le^{L\Lambda}e^{-\lambda(t-\ t_0)} \parallel \Psi(t_0)y_0 \parallel, t_0 \leq t < t_1. \tag{10}$$

In case b), we have

$$r(t) \leq Le^{LM(t-\ t_0)} \parallel \Psi(t_0)y_0 \parallel, t_0 \leq t < t_1$$

and then,

$$\parallel \Psi(t)y(t,t_0,y_0) \parallel \leq Le^{-(\lambda-LM)(t-\ t_0)} \parallel \Psi(t_0)y_0 \parallel, t_0 \leq t < t_1.$$

Suppose that $M < \frac{\lambda}{L}$. Thus, there exists a positive constant $\gamma = \lambda - LM$ such that

$$\parallel \Psi(t)y(t,t_0,y_0) \parallel \leq Le^{-\gamma(t-\ t_0)} \parallel \Psi(t_0)y_0 \parallel, t_0 \leq t < t_1. \tag{11}$$

In case c), from (9) we obtain

$$\parallel \Psi(t)y(t,t_0,y_0) \parallel \leq L \parallel \Psi(t_0)y_0 \parallel e^{-\lambda(t-\ t_0)+L\int_{t_0}^{t}\lambda(z)dz}, t_0 \leq t < t_1. \tag{12}$$

Let $M > 0$ such that $\lambda(t) < M$ for $t \geq 0$. Since $\lim_{t\to\infty} \lambda(t) = 0$, there exists $T_0 > 0$ such that $\lambda(t) < \frac{\lambda}{2L}$, for all $t \geq T_0$.

(I) in case $T_0 \leq t_0$, for $t \in [t_0, t_1)$ we have

$$-\lambda(t-\ t_0) + L\int_{t_0}^{t}\lambda(z)dz \leq -\lambda(t-\ t_0) + L\frac{\lambda}{2L}(t-\ t_0)$$
$$= -\frac{\lambda}{2}(t-\ t_0) < -\frac{\lambda}{2}(t-\ t_0) + LMT_0.$$

(II) in case $t_0 < T_0 < t_1$, for $t \in [t_0, t_1)$ we have the following two cases further:

i) for $t \in [t_0, T_0]$, we have

$$- \lambda(t - t_0) + L\int_{t_0}^{t} \lambda(z)dz \le -\lambda(t - t_0) + LM(t - t_0)$$
$$\le -\frac{\lambda}{2}(t - t_0) + LM(T_0 - t_0) \le -\frac{\lambda}{2}(t - t_0) + LMT_0.$$

ii) for $t \in (T_0, t_1)$, we have

$$- \lambda(t - t_0) + L\int_{t_0}^{t} \lambda(z)dz = - \lambda(t - t_0) + L\int_{t_0}^{T_0} \lambda(z)dz$$
$$+ L\int_{T_0}^{t} \lambda(z)dz \le -\lambda(t - t_0) + LM(T_0 - t_0) + L\frac{\lambda}{2L}(t - t_0)$$
$$\le -\frac{\lambda}{2}(t - t_0) + LM(T_0 - t_0) < -\frac{\lambda}{2}(t - t_0) + LMT_0.$$

(III) in case $T_0 \ge t_1$, for $t \in [t_0, t_1)$ we have

$$-\lambda(t - t_0) + L\int_{t_0}^{t} \lambda(z)dz \le -\frac{\lambda}{2}(t - t_0) + LMT_0$$

(as in the above case II i).

Thus, from (12), (I), (II), (III) we obtain

$$\| \Psi(t)y(t, t_0, y_0) \| \le Le^{LMT_0}e^{-\frac{\lambda}{2}(t - t_0)} \| \Psi(t_0)y_0 \|, t_0 \le t < t_1.$$

We conclude that in all cases, there exist the constants $K > 0$ and $\gamma > 0$ such that $\| \Psi(t)y(t, t_0, y_0) \| \le Ke^{-\gamma(t - t_0)} \| \Psi(t_0)y_0 \|, t_0 \le t < t_1$. Thus, we have $\| y(t, t_0, y_0) \| \le K \| \Psi(t_0)y_0 \| \cdot | \Psi^{-1}(t) |, t_0 \le t < t_1$.

This shows that $t_1 = +\infty$ and the solution y is defined on $[t_0, \infty)$. Now, from the inequality

$$\| \Psi(t)y(t, t_0, y_0) \| \le Ke^{-\gamma(t - t_0)} \| \Psi(t_0)y_0 \|, t \ge t_0, y_0 \in \mathbb{R}^n,$$

it follows that the trivial solution of (1) is Ψ-exponentially asymptotically stable on $\mathbb{R}_+$.

Now, suppose that the supplementary hypotheses i) and ii) hold. From (4), we have that

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, z, y(z, t_0, y_0))G(z, y(z, t_0, y_0))dz,$$

for $t \geq t_0$. By similar arguments used in above proof, we have

$$\| e^{\mu t}\Psi(t)y(t,t_0,y_0) \| \leq \| e^{\mu t}\Psi(t)x(t,t_0,y_0) \|$$
$$+L\int_{t_0}^{t} e^{(\mu-\lambda)(t-z)}\lambda(z) \| e^{\mu z}\Psi(z)y(z,t_0,y_0) \| dz,$$

for $t \geq t_0$.

If we denote $r(t) = \| e^{\mu t}\Psi(t)y(t,t_0,y_0) \|$ and $w(t) = \|e^{\mu t}\Psi(t)x(t,t_0,y_0)\|$, we have

$$r(t) \leq w(t) + L\int_{t_0}^{t} e^{(\mu-\lambda)(t-z)}\lambda(z)r(z)dz, t \geq t_0$$

and then

$$(r(t))^p \leq 2^{p-1}\left[(w(t))^p + L^p\left(\int_{t_0}^{t} e^{(\mu-\lambda)(t-z)}\lambda(z)r(z)dz\right)^p\right]. \tag{13}$$

Using the Hölder inequality, we obtain

$$\int_{t_0}^{t} e^{(\mu-\lambda)(t-z)}\lambda(z)r(z)dz \tag{14}$$
$$\leq [\int_{t_0}^{t} (e^{(\mu-\lambda)(t-z)}\lambda(z))^q dz]^{\frac{1}{q}}[\int_{t_0}^{t} (r(z))^p dz]^{\frac{1}{p}},$$

where $q = \frac{p}{p-1}$. From (13) and (14), we have

$$(r(t))^p \leq 2^{p-1}(w(t))^p + 2^{p-1}L^p v(t)\int_{t_0}^{t} (r(z))^p dz, \quad t \geq t_0.$$

Integrating from t_0 to τ, $\tau \geq t_0$, we obtain the inequality

$$\int_{t_0}^{\tau} (r(t))^p dt \leq 2^{p-1}\int_{t_0}^{\tau} (w(t))^p dt + 2^{p-1}L^p\int_{t_0}^{\tau} v(t)\left(\int_{t_0}^{t} (r(z))^p dz\right)dt,$$

for $\tau \geq t_0$.

This implies by the Gronwall inequality that

$$\int_{t_0}^{\tau} (r(t))^p dt \leq 2^{p-1}W \cdot e^{2^{p-1}L^p\int_{t_0}^{\tau} v(t)dt}, \tau \geq t_0,$$

where $W = \int_{t_0}^{\infty}(w(t))^p dt < +\infty$. Therefore, the solution $y(t,t_0,y_0)$ of (1) belongs to the space $L^p_{e^{\mu t}\Psi}([t_0,\infty))$.

The proof is now complete. □

Corollary 1. *In the conditions of Theorem* 2*, if the trivial solution of* (2) *is globally* ($\Psi - L^p_{e^{\lambda t}\Psi}$)*-exponentially asymptotically stable in variation on* $\mathbb{R}_+$ *and the function* λ *satisfies the condition*

$$\int_0^\infty \left(\int_0^t (\lambda(z))^{\frac{p}{p-1}} \, dz \right)^{p-1} dt < +\infty,$$

then, the trivial solution of (1) *is* ($\Psi - L^p_{e^{\lambda t}\Psi}$)*-exponentially asymptotically stable on* $\mathbb{R}_+$*.*

Proof. Indeed, if in Theorem 2 we put $\mu = \lambda$, then the conclusion of the Corollary follows. □

Remark 3. Theorem 2 extends Theorem 3, [7] to a nonlinear system of differential equations (1).

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Department of Applied Mathematics,
University of Craiova,
13, "Al. I. Cuza" st., 200585 Craiova,
ROMANIA
adiamandescu@central.ucv.ro