ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII “AL.I. CUZA” DIN IAŞI (S.N.)
MATEMATICĂ, Tomul LVIII, 2012, f.2

OSBORN LOOPS AND THEIR UNIVERSALITY

BY

JAÍYÉỌLÁ TÈMÍTỌ́PẸ́ GBỌ́LÁHÀN

On the 50th Anniversary of Obafemi Awolowo University

Abstract. Kinyon's conjecture that 'every CC-quasigroup is isotopic to an Osborn loop' is shown to be true for universal (left and right universal) Osborn loops if and only if every CC-quasigroup obeys any of two gotten identities. An Osborn loop is proved to be universal if and only if some of its principal isotopes are isomorphic to some other principal isotopes of the loop, left universal if and only if some of its principal isotopes are isomorphic to some left principal isotopes of the loop and right universal if and only if any of its right principal isotopes is isomorphic to some principal isotopes of the loop. The existence of a bi-mapping in the Bryant-Schneider group (BSG) of a right universal Osborn loop is shown and the consequences of this is discussed for extra loops using some existing results in literature. It is established that there is no non-trivial: universal Osborn loop that can form a special class of left G-loop (e.g extra loops, CC-loops or VD-loops) under a tri-mapping, left universal Osborn loop that can form a special class of left G-loop under a bi-mapping. Also, it is established that there is no non-trivial: universal Osborn loop with this tri-mapping in its BSG, left universal Osborn loop with this bi-mapping in its left BSG and right universal Osborn loop with a bi-mapping in its right BSG.

Mathematics Subject Classification 2010: 20NO5, 08A05.
Key words: Osborn loops, universality, left universality, right universality.

1. Introduction and preliminaries

A loop is called an Osborn loop if it obeys any of the two identities below.

(1) $$\mathrm{OS}_3 \; : \; (x \cdot yz)x = xy \cdot [(x^{\lambda} \cdot xz) \cdot x],$$

(2) $$\mathrm{OS}_5 \; : \; (x \cdot yz)x = xy \cdot [(x \cdot x^{\rho} z) \cdot x].$$

For a comprehensive introduction to Osborn loops and its universality, and a detailed literature review on it, readers should check Jaiyéọlá and Adéníran [7]. In Kinyon [8], the author made the following conjecture: "Every CC-quasigroup is isotopic to an Osborn". And he mentioned that CC-quasigroups include CC-loops, quasigroups that are isotopic to groups and trimedial quasigroups. Trimedial quasigroups have been shown to be isotopic to commutative Moufang loops.

In this study, Kinyon's conjecture that 'every CC-quasigroup is isotopic to an Osborn loop' is shown to be true for universal(left and right universal) Osborn loops if and only if every CC-quasigroup obeys any of two gotten identities. An Osborn loop is proved to be universal if and only if some of its principal isotopes are isomorphic to some other principal isotopes of the loop, left universal if and only if some of its principal isotopes are isomorphic to some left principal isotopes of the loop and right universal if and only if any of its right principal isotopes is isomorphic to some principal isotopes of the loop. The existence of a bi-mapping in the Bryant-Schneider group (BSG) of a right universal Osborn loop is shown and the consequences of this is discussed for extra loops using some existing results in literature. It is established that there is no non-trivial: universal Osborn loop that can form a special class of left G-loop (e.g extra loops, CC-loops or VD-loops) under a tri-mapping, left universal Osborn loop that can form a special class of left G-loop under a bi-mapping. Also, it is established that there is no non-trivial: universal Osborn loop with this tri-mapping in its BSG, left universal Osborn loop with this bi-mapping in its left BSG and right universal Osborn loop with a bi-mapping in its right BSG.

In this present paper, we shall follow the style and notations used in Jaiyéọlá and Adéníran [7]. The only concepts and notions which will be introduced here are those that were not defined in Jaiyéọlá and Adéníran [7].

A conjugacy closed quasigroup(CC-quasigroup) is a quasigroup that obeys the identities $x \cdot (yz) = \{[x \cdot (y \cdot (x\backslash x))]/x\} \cdot (xz)$ and $(zy) \cdot x = (zx) \cdot \{x\backslash[((x/x) \cdot y) \cdot x]\}$. A loop is called a right G-loop(G_ρ-loop) if and only if it is isomorphic to all its right loop isotopes. A loop is called a left G-loop (G_λ-loop) if and only if it is isomorphic to all its left loop isotopes. A loop is a G-loop if and only if it is a G_ρ-loop and a G_λ-loop. Kunen [12] demonstrated the use of G_ρ-loops and G_λ-loops. We shall treat the G-loops and G_ρ-loops of some universal and right universal Osborn loops (respectively) in the following manner.

Definition 1.1. Let $(L,\cdot,\backslash,/)$ be an Osborn loop with a mapping $\Theta\in SYM(L,\cdot)$. Suppose Θ is an element of the multiplication group $\mathcal{M}\mathrm{ult}(L)$ of L such that $\Theta(x,y,z)$, i.e Θ is the product of right, left translation mappings $R_{\alpha(x,y,z)},L_{\beta(x,y,z)}$ and their inverses $\mathbb{R}_{\alpha(x,y,z)},\mathbb{L}_{\beta(x,y,z)}$ such that $\alpha(x,y,z)$ and $\beta(x,y,z)$ are words in L in terms of arbitrary elements $x,y,z\in L$ with a minimum of length one. Then Θ is called a tri-mapping of L.

1. L is called a G(Θ_3)-loop if it is a G-loop such that there exists a tri-mapping Θ which is the isomorphism from L to all its principal isotopes.

2. L is called a G$_\lambda$(Θ_2)-loop if it is a G$_\rho$-loop such that there exists a bi-mapping Θ which is the isomorphism from L to all its principal isotopes.

Remark 1.1. Some popular examples of bi-mappings are the right and left inner mappings $R(x,y)$ and $L(x,y)$ respectively. The middle inner mapping $T(x)$ is a familiar mono-mapping. Tri-mappings, tetra-mappings e.t.c can be obtained by multiplying bi-mappings and mono-mappings. A demonstration of this can be seen in Bruck and Paige [2] and Kinyon et. al. [10]. In fact, according to Kinyon et. al. [9], in a CC-loop, $R(x,y)$ and $L(u,v)$ all commute with each other. So, it is sensible to consider tetra-mappings in some universal Osborn loops.

Theorem 1.1 (Chiboka and Solarin [5], Kunen [11]). *Let $(G,\cdot)$ be a loop.*

1. *G is called a G_ρ-loop if and only if there exists $\theta\in SYM(G,\cdot)$ such that $(\theta,\theta L_y^{-1},\theta)\in AUT(G,\cdot),\forall\ y\in G$.*

2. *G is called a G_λ-loop if and only if there exists $\theta\in SYM(G,\cdot)$ such that $(\theta R_x^{-1},\theta,\theta)\in AUT(G,\cdot),\forall\ x\in G$.*

3. *G is called a G-loop if and only if there exists $\theta\in SYM(G,\cdot)$ such that $(\theta R_x^{-1},\theta L_y^{-1},\theta)\in AUT(G,\cdot),\forall\ x,y\in G$.*

Definition 1.2 (Robinson [13]). Let $(G,\cdot)$ be a loop.

1. A mapping $\theta\in SYM(G,\cdot)$ is a right special map for G means that there exist $f\in G$ so that $(\theta,\theta L_f^{-1},\theta)\in AUT(G,\cdot)$.

2. A mapping $\theta \in SYM(G,\cdot)$ is a left special map for G means that there exist $g \in G$ so that $(\theta R_g^{-1}, \theta, \theta) \in AUT(G,\cdot)$.

3. A mapping $\theta \in SYM(G,\cdot)$ is a special map for G means that there exist $f, g \in G$ so that $(\theta R_g^{-1}, \theta L_f^{-1}, \theta) \in AUT(G,\cdot)$.

From Definition 1.2, it can be observed that θ is a left or right special map for a loop $(G,\cdot)$ with identity element e if and only if θ is an isomorphism of $(G,\cdot)$ onto some e,g- or f,e- principal isotope $(G,\circ)$ of $(G,\cdot)$. Moreso, θ is a special map for a loop $(G,\cdot)$ if and only if θ is an isomorphism of $(G,\cdot)$ onto some f,g-principal isotope $(G,\circ)$ of $(G,\cdot)$.

Robinson [13] went further to show that if $BS(G,\cdot) = \{\theta \in SYM(G,\cdot) : \exists f, g \in G \ni (\theta R_g^{-1}, \theta L_f^{-1}, \theta) \in AUT(G,\cdot)\}$ i.e the set of all special maps in a loop, then $BS(G,\cdot) \leq SYM(G,\cdot)$ called the Bryant-Schneider group of the loop $(G,\cdot)$ because its importance and motivation stem from the work of Bryant and Schneider [3]. Since the advent of the Bryant-Schneider group, some studies by Adeniran [1] and Chiboka [4] have been done on it relative to CC-loops and extra loops. Let $BS_\lambda(G,\cdot) = \{\theta \in SYM(G,\cdot) : \exists\, g \in G \ni (\theta R_g^{-1}, \theta, \theta) \in AUT(G,\cdot)\}$ i.e the set of all left special maps in a loop, then $BS_\lambda(G,\cdot) \leq BS(G,\cdot)$ called the left Bryant-Schneider group of the loop $(G,\cdot)$ and $BS_\rho(G,\cdot) = \{\theta \in SYM(G,\cdot) : \exists\, f \in G \ni (\theta, \theta L_f^{-1}, \theta) \in AUT(G,\cdot)\}$ i.e the set of all right special maps in a loop, then $BS_\rho(G,\cdot) \leq BS(G,\cdot)$ called the right Bryant-Schneider group of the loop $(G,\cdot)$. We shall make a judicious use of these three groups as earlier predicted by Robinson [13].

Theorem 1.2 (Jaiyéọlá and Adéníran [7]). *A loop $(Q,\cdot,\backslash,/)$ is a universal Osborn loop if and only if it obeys the identity*

$$x\cdot u\backslash\{(yz)/v\cdot[u\backslash(xv)]\} \\ \underbrace{= (x\cdot u\backslash\{[y(u\backslash([(uv)/(u\backslash(xv))]v))]/v\cdot[u\backslash(xv)]\})/v\cdot u\backslash[((uz)/v)(u\backslash(xv))]}_{OS_0'}$$

or

$$x\cdot u\backslash\{(yz)/v\cdot[u\backslash(xv)]\} \\ \underbrace{= \{x\cdot u\backslash\{[y(u\backslash(xv))]/v\cdot[x\backslash(uv)]\}\}/v\cdot u\backslash[((uz)/v)(u\backslash(xv))]}_{OS_1'}.$$

2. Main results

2.1. Universality of Osborn loops

Theorem 2.1. *A loop $(Q,\cdot,\backslash,/)$ is a universal Osborn loop if and only if it obeys the identity*

$$[x\cdot u\backslash(yz)]/v\cdot[u\backslash(xv)]$$
$$\underbrace{=[x\cdot u\backslash(yv)]/v\cdot u\backslash\{\{x\cdot u\backslash[(u[x\backslash(uv)])/v\cdot z]\}/v\cdot u\backslash(xv)\}}_{OS_5'}$$

or

$$[x\cdot u\backslash(yz)]/v\cdot u\backslash(xv)$$
$$\underbrace{=[x\cdot u\backslash(yv)]/v\cdot u\backslash\{\{[(uv)/(u\backslash(xv))]\cdot u\backslash(xz)\}/v\cdot u\backslash(xv)\}}_{OS_3'}.$$

Proof. Let $\mathcal{Q}=(Q,\cdot,\backslash,/)$ be an Osborn loop with any arbitrary principal isotope $\mathfrak{Q}=(Q,\blacktriangle,\nwarrow,\nearrow)$ such that $x\blacktriangle y=xR_v^{-1}\cdot yL_u^{-1}=(x/v)\cdot(u\backslash y)\ \forall\ u,v\in Q$. The proof of this theorem is achieved by using identities OS_5 and OS_3 the way identities OS_0 and OS_1 were used to prove Theorem 1.2. □

Lemma 2.1. *A quasigroup is isotopic to a universal Osborn loop if and only if it obeys the identity OS_3' or OS_5'.*

Proof. Let Q be a quasigroup that is isotopic to a universal Osborn loop L i.e every isotope G of L is an Osborn loop. Then, the isotopisms $Q\longrightarrow L$ and $L\longrightarrow G$ implies the isotopism $Q\longrightarrow G$. Let H be any isotope of Q, then $H\longrightarrow G$ is an isotopism and so $H\longrightarrow L$ is an isotopism, hence, H is an Osborn loop. Let $H=(Q,\blacktriangle)$ be a principal isotope of $(Q,\cdot)$ such that $x\blacktriangle y=xR_v^{-1}\cdot yL_u^{-1}=(x/v)\cdot(u\backslash y)\ \forall\ u,v\in Q$. Then, thinking in line with the proof of Theorem 2.1, H obeys identity OS_3 or OS_5 if and only if Q obeys identity OS_3' or OS_5'.

The proof of the conversely is as follows. If Q obeys identity OS_3' or OS_5', then every principal isotope of Q is an Osborn loop, hence, all isotopes of Q are Osborn loops. Let L be an isotope of Q with arbitrary loop isotope L'. So L' is an isotope of Q, hence L' is an Osborn loop. Therefore, Q is isotopic to a universal Osborn loop. □

Corollary 2.1. *A quasigroup is isotopic to a Moufang loop or CC-loop or VD-loop or universal WIPL if and only if it obeys the identity* OS_3' *or* OS_5'.

Remark 2.1. Not all CC-quasigroups are isotopic to groups or Moufang loops or VD-loops.

Lemma 2.2. *Let* Q *be a loop with multiplication group* $\mathcal{M}ult(Q)$. Q *is a universal Osborn loop if and only if the triple* $(\alpha_1(x,u,v), \beta_1(x,u,v), \gamma_1(x,u,v)) \in AUT(Q)$ *or the triple*

$$(\alpha_1(x,u,v), L_x\mathbb{L}_u L_{[(uv)/(u\backslash(xv))]}\mathbb{L}_x L_u\gamma_1(x,u,v)\mathbb{L}_u, \gamma_1(x,u,v)) \in AUT(Q),$$

for all $x,u,v \in Q$, *where*

$$\alpha_1(x,u,v) = R_v\mathbb{L}_u L_x\mathbb{R}_v,\ \ \beta_1(x,u,v) = L_{[(u[x\backslash(uv)])/v]}\mathbb{L}_u L_x\mathbb{R}_v R_{[u\backslash(xv)]}\mathbb{L}_u$$

and $\gamma_1(x,u,v) = \mathbb{L}_u L_x\mathbb{R}_v R_{[u\backslash(xv)]}$ *are elements of* $\mathcal{M}ult(Q)$.

Proof. This is gotten from Theorem 2.1 by just writing identity OS_5' or OS_3' in autotopic form. □

Theorem 2.2. *Let* Q *be a loop with multiplication group* $\mathcal{M}ult(Q)$. *If* Q *is a universal Osborn loop, then the triple* $(\alpha_1(x,u,v), \gamma_1(x,u,v)\mathbb{L}_{[x\cdot u\backslash v]/v}, \gamma_1(x,u,v)) \in AUT(Q)$, $\forall x,u,v \in Q$, *where* $\alpha_1(x,u,v) = R_v\mathbb{L}_u L_x\mathbb{R}_v$ *and* $\gamma_1(x,u,v) = \mathbb{L}_u L_x\mathbb{R}_v R_{[u\backslash(xv)]}$ *are elements of* $\mathcal{M}ult(Q)$.

Proof. Theorem 2.1 will be employed. Let $y = e$ in identity OS_5', then $[x\cdot u\backslash z]/v\cdot[u\backslash(xv)] = [x\cdot u\backslash v]/v\cdot u\backslash\{\{x\cdot u\backslash[(u[x\backslash(uv)])/v\cdot z]\}/v\cdot u\backslash(xv)\}$. So, identity OS_5' can now be written as

$$[x\cdot u\backslash(yz)]/v\cdot[u\backslash(xv)] = [x\cdot u\backslash(yv)]/v\cdot\{[x\cdot u\backslash v]/v\}\backslash\{[x\cdot u\backslash z]/v\cdot[u\backslash(xv)]\}.$$

Putting this in autotopic form, we have

$$(\alpha_1(x,u,v), \gamma_1(x,u,v)\mathbb{L}_{[x\cdot u\backslash v]/v}, \gamma_1(x,u,v)) \in AUT(Q).$$

□

Theorem 2.3. *An Osborn loop is universal if and only if some of its principal isotopes are isomorphic to some other principal isotopes.*

Proof. Let $(Q,\cdot,\backslash,/)$ be a universal Osborn loop. We shall use Lemma 2.2. The triple

$$(\alpha_1(x,u,v),\beta_1(x,u,v),\gamma_1(x,u,v))=(R_v\gamma_1\mathbb{R}_{[u\backslash(xv)]},L_{[(u[x\backslash(uv)])/v]}\gamma_1\mathbb{L}_u,\gamma_1)$$

can be written as the following compositions

$$(R_v,L_{[(u[x\backslash(uv)])/v]},I)(\gamma_1,\gamma_1,\gamma_1)(\mathbb{R}_{[u\backslash(xv)]},\mathbb{L}_u,I).$$

Let $(Q,\circ)$ be some principal isotopes of $(Q,\cdot)$ and $(Q,*)$ some other principal isotopes of $(Q,\cdot)$. Let $\phi_1(x,u,v)=[(u[x\backslash(uv)])/v]$, then the composition above can be expressed as:

$$(Q,\cdot)\xrightarrow[\text{principal isotopism}]{(R_v,L_{\phi_1(x,u,v)},I)}(Q,*)\xrightarrow[\text{isomorphism}]{(\gamma_1,\gamma_1,\gamma_1)}(Q,\circ)\xrightarrow[\text{principal isotopism}]{(\mathbb{R}_{[u\backslash(xv)]},\mathbb{L}_u,I)}(Q,\cdot).$$

This means that some principal isotopes $(Q,\circ)$ of $(Q,\cdot)$ are isomorphic to some other principal isotopes $(Q,*)$ of $(Q,\cdot)$. □

Theorem 2.4. *An Osborn loop is universal if and only if the existence of the principal autotopism $(R_v,L_{\phi_1(x,u,v)},I)$, $\phi_1(x,u,v)=[(u[x\backslash(uv)])/v]$ in the loop implies the triple $(\gamma_1\mathbb{R}_{[u\backslash(xv)]},\gamma_1\mathbb{L}_u,\gamma_1)$, where $\gamma_1(x,u,v)=\mathbb{L}_uL_x\mathbb{R}_vR_{[u\backslash(xv)]}$ is an autotopism in the loop, and vice versa.*

Proof. The proof is in line with Theorem 2.3 with a slight adjustment to the composition of the triple $(\alpha_1(x,u,v),\beta_1(x,u,v),\gamma_1(x,u,v))=(R_v\gamma_1\mathbb{R}_{[u\backslash(xv)]},L_{[(u[x\backslash(uv)])/v]}\gamma_1\mathbb{L}_u,\gamma_1)$ which can be re-written as the following compositions $(R_v,L_{[(u[x\backslash(uv)])/v]},I)(\gamma_1\mathbb{R}_{[u\backslash(xv)]},\gamma_1\mathbb{L}_u,\gamma_1)$. Hence, the conclusion follows. □

Theorem 2.5. *If an Osborn loop is universal then, any of its left principal isotopes is isomorphic to some principal isotopes.*

Proof. By Theorem 2.2, if Q is a universal Osborn loop, then

$$\begin{aligned}&(\alpha_1(x,u,v),\gamma_1(x,u,v)\mathbb{L}_{[x\cdot u\backslash v]/v},\gamma_1(x,u,v))\\&=(R_v\mathbb{L}_uL_x\mathbb{R}_v,\gamma_1(x,u,v)\mathbb{L}_{[x\cdot u\backslash v]/v},\gamma_1(x,u,v))\\&=(R_v\gamma_1(x,u,v)\mathbb{R}_{[u\backslash(xv)]},\gamma_1(x,u,v)\mathbb{L}_{[x\cdot u\backslash v]/v},\gamma_1(x,u,v))\in AUT(Q),\end{aligned}$$

for all $x,u,v\in Q$. Writing

$$\begin{aligned}&(R_v\gamma_1(x,u,v)\mathbb{R}_{[u\backslash(xv)]},\gamma_1(x,u,v)\mathbb{L}_{[x\cdot u\backslash v]/v},\gamma_1(x,u,v))\\&=(R_v,I,I)(\gamma_1(x,u,v),\gamma_1(x,u,v),\gamma_1(x,u,v))\\&\qquad(\mathbb{R}_{[u\backslash(xv)]},\mathbb{L}_{[x\cdot u\backslash v]/v},I)\in AUT(Q)\end{aligned}$$

such that

$$(Q,\cdot)\xrightarrow[\text{left principal isotopism}]{(R_v,I,I)}(Q,*)\xrightarrow[\text{isomorphism}]{(\gamma_1,\gamma_1,\gamma_1)}(Q,\circ)\xrightarrow[\text{principal isotopism}]{(\mathbb{R}_{[u\backslash(xv)]},\mathbb{L}_{[x\cdot u\backslash v]/v},I)}(Q,\cdot)$$

where $(Q,*)$ is an arbitrary left principal isotope of $(Q,\cdot)$ and $(Q,\circ)$ are some principal isotopes of $(Q,\cdot)$, the conclusion of the theorem follows. □

Theorem 2.6. *If an Osborn loop is universal then, the existence of the principal autotopism* $(R_{[u\backslash(xv)]}, L_{\psi_1(x,u,v)}, I)$*,* $\psi_1(x,u,v) = [x\cdot u\backslash v]/v$ *in the loop implies the triple* $(\gamma_1(x,u,v)^{-1}\mathbb{R}_v, \gamma_1(x,u,v)^{-1}, \gamma_1(x,u,v)^{-1})$*, where* $\gamma_1(x,u,v) = \mathbb{L}_u L_x \mathbb{R}_v R_{[u\backslash(xv)]}$ *is an autotopism in the loop, and vice versa.*

Proof. The proof is in line with Theorem 2.5 with a slight adjustment to the composition by simply considering the inverse composition and reasoning like we did in Theorem 2.4. □

Theorem 2.7. *Let* $\mathcal{Q} = (Q,\cdot,\backslash,/)$ *be an Osborn loop such that the tri-mapping* $\gamma_1(x,u,v) = \mathbb{L}_u L_x \mathbb{R}_v R_{[u\backslash(xv)]}$*. Q is a universal Osborn loop if and only if* $\gamma_1(x,u,v) \in BS(\mathcal{Q})$ *implies* $\mathcal{Q}$ *obeys the identity*

$$yv\cdot\{(u[x\backslash(uv)])/v\}z = yz\ \forall\ x,y,z,u,v\in Q \tag{3}$$

and vice versa.

Proof. The proof is based on Theorem 2.2 and is achieved by using the compositions $(R_v, L_{[(u[x\backslash(uv)])/v]}, I)(\gamma_1\mathbb{R}_{[u\backslash(xv)]}, \gamma_1\mathbb{L}_u, \gamma_1)$ of Theorem 2.4 and hence following by the definition of $BS(\mathcal{Q})$, $\gamma_1(x,u,v)\in BS(\mathcal{Q})$ implies $\mathcal{Q}$ obeys identity (3) and vice versa. □

Theorem 2.8. *Let* $(Q,\cdot,\backslash,/)$ *be an Osborn loop such that the tri-mapping* $\gamma_1(x,u,v) = \mathbb{L}_u L_x \mathbb{R}_v R_{[u\backslash(xv)]}$*. If Q is a universal Osborn loop then, Q is a* $G_\lambda((\gamma_1^{-1})_3)$*-loop implies it obeys the identity*

$$y[u\backslash(xv)]\cdot\{[x\cdot u\backslash v]/v\}z = yz\ \forall\ x,y,z,u,v\in Q \tag{4}$$

and vice versa.

Proof. The proof is based on the composition used in Theorem 2.6. The reasoning used is similar to that in Theorem 2.7. □

Corollary 2.2. *Let* $\gamma_1(x,u,v) = \mathbb{L}_u L_x \mathbb{R}_v R_{[u\backslash(xv)]}$ *be a tri-mapping. There does not exist a non-trivial universal Osborn loop* $\mathcal{Q} = (Q,\cdot,\backslash,/)$ *that is a* $G_\lambda((\gamma_1^{-1})_3)$*-loop or for which* $\gamma_1(x,u,v) \in BS(\mathcal{Q})$.

Proof. Let $\mathcal{Q} = (Q,\cdot,\backslash,/)$ be an arbitrary non-trivial universal Osborn loop. According to Theorem 2.7 or Theorem 2.8, if $\gamma_1(x,u,v) \in BS(\mathcal{Q})$ or $\mathcal{Q}$ is a $\mathrm{G}_\lambda((\gamma_1^{-1})_3)$-loop then it obeys identity (3) or (4). Put $y = z = v = u = e$ in identity (3), then $x = e$. Which is a contradiction. Put $y = z = e$ and $u = x$ in identity (4), then $v = e$. Which is also a contradiction. □

Remark 2.2. There is no non-trivial group or Moufang loop or universal WIPL or VD-loop or CC-loop Q that is a $\mathrm{G}_\lambda((\gamma_1^{-1})_3)$-loop or for which $\gamma_1(x,u,v) \in BS(\mathcal{Q})$ when $\gamma_1(x,u,v) = \mathbb{R}_v R_{[u\backslash(xv)]}\mathbb{L}_u L_x$.

2.2. Left universality of Osborn loops

Theorem 2.9. *A loop* $(Q,\cdot,\backslash,/)$ *is a left universal Osborn loop if and only if it obeys the identity*

$$\underbrace{[x\cdot yz]/v\cdot(xv) = [x\cdot yv]/v\cdot\{\{x\cdot[(x\backslash v)/v\cdot z]\}/v\cdot(xv)\}}_{OS_5^\lambda} \qquad or$$

$$\underbrace{[x\cdot(yz)]/v\cdot(xv) = [x\cdot(yv)]/v\cdot\{\{[v/(xv)]\cdot(xz)\}/v\cdot(xv)\}}_{OS_3^\lambda}.$$

Proof. The method of the proof of this theorem is similar to the method used to prove Theorem 2.1 by just using the arbitrary left principal isotope $\mathfrak{Q} = (Q,\blacktriangle,\nwarrow,\nearrow)$ such that $x\blacktriangle y = xR_v^{-1}\cdot y = (x/v)\cdot y\ \forall\ v \in Q$. □

Lemma 2.3. *A quasigroup is left isotopic to a left universal Osborn loop if and only if it obeys the identity* OS_5^λ *or* OS_3^λ.

Proof. The method of the proof of this lemma is similar to the method used to prove Lemma 2.1. □

Corollary 2.3. *A quasigroup is left isotopic to a Moufang loop or CC-loop or VD-loop or universal WIPL if and only if it obeys the identity* OS_5^λ *or* OS_3^λ.

Remark 2.3. Not all CC-quasigroups are left isotopic to groups or Moufang loops or VD-loops.

Lemma 2.4. *Let Q be a loop with multiplication group $\mathcal{M}ult(Q)$. Q is a left universal Osborn loop if and only if the triple $(\alpha_1(x,v),\beta_1(x,v),\gamma_1(x,v)) \in AUT(Q)$ or $(\alpha_1(x,v), L_xL_{[v/(xv)]}\mathbb{L}_x\gamma_1(x,v),\gamma_1(x,v)) \in AUT(Q)$, for all $x,v \in Q$ where $\alpha_1(x,v) = R_vL_x\mathbb{R}_v$, $\beta_1(x,v) = L_{[(x\backslash v)/v]}L_x\mathbb{R}_vR_{[xv]}$ and $\gamma_1(x,v) = L_x\mathbb{R}_vR_{[xv]}$ are elements of $\mathcal{M}ult(Q)$.*

Proof. This is gotten from Theorem 2.9 by just writing identity OS_5^λ or OS_3^λ in autotopic form. □

Theorem 2.10. *Let Q be a loop with multiplication group $\mathcal{M}ult(Q)$. If Q is a left universal Osborn loop, then the triple $(\alpha_1(x,v),\gamma_1(x,v)\mathbb{L}_{[x\cdot v]/v}, \gamma_1(x,v)) \in AUT(Q)$, for all $x,v \in Q$ where $\alpha_1(x,v) = R_vL_x\mathbb{R}_v$ and $\gamma_1(x,v) = L_x\mathbb{R}_vR_{[xv]}$ are elements of $\mathcal{M}ult(Q)$.*

Proof. This follows by using identity OS_5^λ or OS_3^λ of Theorem 2.9 the way identity OS_5' or OS_3' of Theorem 2.1 was used to prove Theorem 2.2.□

Theorem 2.11. *An Osborn loop is left universal if and only if some of its principal isotopes are isomorphic to some left principal isotopes.*

Proof. Let $(Q,\cdot,\backslash,/)$ be a left universal Osborn loop. We shall use Lemma 2.4. The triple

$$(\alpha_1(x,v),\beta_1(x,v),\gamma_1(x,v)) = (R_v\gamma_1\mathbb{R}_{[xv]}, L_{[(x\backslash v)/v]}\gamma_1,\gamma_1)$$

can be written as the following compositions

$$(R_v, L_{[(x\backslash v)/v]}, I)(\gamma_1,\gamma_1,\gamma_1)(\mathbb{R}_{[xv]}, I, I).$$

Let $(Q,*)$ be some principal isotopes of $(Q,\cdot)$ and $(Q,\circ)$ be some left principal isotopes of $(Q,\cdot)$. Let $\phi_1(x,v) = [(x\backslash v)/v]$, then the composition above can be expressed as:

$$(Q,\cdot) \xrightarrow[\text{principal isotopism}]{(R_v,L_{\phi_1(x,v)},I)} (Q,*) \xrightarrow[\text{isomorphism}]{(\gamma_1,\gamma_1,\gamma_1)} (Q,\circ) \xrightarrow[\text{left principal isotopism}]{(\mathbb{R}_{(xv)},I,I)} (Q,\cdot).$$

This means that some principal isotopes $(Q,*)$ of $(Q,\cdot)$ are isomorphic to some left principal isotopes $(Q,\circ)$ of $(Q,\cdot)$. □

Theorem 2.12. *An Osborn loop is left universal if and only if the existence of the principal autotopism $(R_v, L_{\phi_1(x,v)}, I)$, $\phi_1(x,v) = [(x\backslash v)/v]$ in the loop implies the triple $(\gamma_1\mathbb{R}_{(xv)},\gamma_1,\gamma_1)$, where $\gamma_1(x,v) = L_x\mathbb{R}_vR_{(xv)}$ is an autotopism in the loop and vice versa.*

Proof. The proof is in line with Theorem 2.11 with a slight adjustment to the composition of the triple

$$(\alpha_1(x,v),\beta_1(x,v),\gamma_1(x,v)) = (R_v\gamma_1\mathbb{R}_{(xv)}, L_{[(x\backslash v)/v]}\gamma_1,\gamma_1)$$

which can be re-written as the following compositions

$$(R_v, L_{[(x\backslash v)/v]}, I)(\gamma_1\mathbb{R}_{(xv)},\gamma_1,\gamma_1).$$

Hence, the conclusion follows. □

Theorem 2.13. *If an Osborn loop is left universal then, any arbitrary left principal isotope of it is isomorphic to some principal isotopes of it.*

Proof. By Theorem 2.10, if Q is a left universal Osborn loop, then $(\alpha_1(x,v),\gamma_1(x,v)\mathbb{L}_x,\gamma_1(x,v)) = (R_v\gamma_1(x,v)\mathbb{R}_{(xv)},\gamma_1(x,v)\mathbb{L}_x,\gamma_1(x,v)) \in AUT(Q)$, for all $x,u,v\in Q$. Splitting this into composition of isotopism like it was done in the proof of Theorem 2.5, we get

$$(Q,\cdot)\xrightarrow[\text{left principal isotopism}]{(R_v,I,I)}(Q,*)\xrightarrow[\text{isomorphism}]{(\gamma_1,\gamma_1,\gamma_1)}(Q,\circ)\xrightarrow[\text{principal isotopism}]{(\mathbb{R}_{(xv)},\mathbb{L}_x,I)}(Q,\cdot)$$

where $(Q,*)$ is an arbitrary left principal isotope of $(Q,\cdot)$ and $(Q,\circ)$ are some principal isotopes of $(Q,\cdot)$. The conclusion of the theorem follows. □

Theorem 2.14. *Let $\mathcal{Q} = (Q,\cdot,\backslash,/)$ be an Osborn loop such that the bi-mapping $\gamma_1(x,v) = L_x\mathbb{R}_vR_{(xv)}$. $\mathcal{Q}$ is a left universal Osborn loop if and only if $\gamma_1(x,v)\in BS_\lambda(\mathcal{Q})$ implies $\mathcal{Q}$ obeys the identity*

$$yv\cdot[(x\backslash v)/v]z = yz\ \forall\ x,y,z,v\in Q \tag{5}$$

and vice versa.

Proof. The proof is based on Theorem 2.4 and is achieved by using the compositions $(R_v, L_{[(x\backslash v)/v]}, I)(\gamma_1\mathbb{R}_{(xv)},\gamma_1,\gamma_1)$ of Theorem 2.12. □

Theorem 2.15. *Let $(Q,\cdot,\backslash,/)$ be an Osborn loop such that the bi-mapping $\gamma_1(x,v) = L_x\mathbb{R}_vR_{(xv)}$. If Q is a universal Osborn loop then, Q is a $G_\lambda((\gamma_1^{-1})_3)$-loop implies it obeys the identity*

$$y(xv)\cdot xz = yz\ \forall\ x,y,z,v\in Q \tag{6}$$

and vice versa.

Proof. The proof is based on the composition used in Theorem 2.6. The reasoning used is similar to that in Theorem 2.7. □

Corollary 2.4. *Let $\gamma_1(x,v) = L_x\mathbb{R}_v R_{(xv)}$ be a bi-mapping. There does not exist a non-trivial left universal Osborn loop $\mathcal{Q}$ that is a $G_\lambda((\gamma_1^{-1})_3)$-loop or for which $\gamma_1(x,v) \in BS_\lambda(\mathcal{Q})$.*

Proof. Let $(Q,\cdot,\backslash,/)$ be an arbitrary non-trivial left universal Osborn loop. According to Theorem 2.14, if in $\mathcal{Q}$, $\gamma_1(x,v) \in BS_\lambda(\mathcal{Q})$ then it obeys identity (5). Put $y = z = v = e$ in identity (5), then $x = e$. Which is a contradiction.

Also, according to Theorem 2.15, if $\mathcal{Q}$ is a $G_\lambda((\gamma_1^{-1})_3)$-loop, then it obeys identity (6). Put $x = y = z = e$ in identity (6), then $v = e$. Which is a contradiction. □

Remark 2.4. There is no non-trivial group or Moufang loop or universal WIPL or VD-loop or CC-loop that is a $G_\lambda((\gamma_1^{-1})_3)$-loop or for which $\gamma_1(x,v) \in BS_\lambda(\mathcal{Q})$ when $\gamma_1(x,v) = L_x\mathbb{R}_v R_{(xv)}$.

2.3. Right universality of Osborn loops

Theorem 2.16. *A loop $(Q,\cdot,\backslash,/)$ is a right universal Osborn loop if and only if it obeys the identity*

$$\underbrace{[x\cdot u\backslash(yz)][u\backslash x] = [x\cdot u\backslash y]\cdot u\backslash\{\{x\cdot u\backslash[(u[x\backslash u])z]\}\cdot u\backslash x\}}_{OS_5\rho} \qquad or$$

$$\underbrace{[x\cdot u\backslash(yz)][u\backslash x] = [x\cdot u\backslash y]\cdot u\backslash\{\{[u/(u\backslash x)]\cdot u\backslash(xz)\}\cdot u\backslash x\}}_{OS_3^\rho}.$$

Proof. The method of the proof of this theorem is similar to the method used to prove Theorem 2.1 by just using the arbitrary right principal isotope $\mathfrak{Q} = (Q, \blacktriangle, \nwarrow, \nearrow)$ such that $x\blacktriangle y = x\cdot yL_u^{-1} = x\cdot(u\backslash y)\ \forall\ u\in Q$. □

Lemma 2.5. *A quasigroup is right isotopic to a right universal Osborn loop if and only if it obeys the identity OS_5^ρ or OS_3^ρ.*

Proof. The method of the proof of this lemma is similar to the method used to prove Lemma 2.1. □

Corollary 2.5. *A quasigroup is right isotopic to a Moufang loop or CC-loop or VD-loop or universal WIPL if and only if it obeys the identity* OS_5^ρ *or* OS_3^ρ.

Remark 2.5. Not all CC-quasigroups are right isotopic to groups or Moufang loops or VD-loops.

Lemma 2.6. *Let Q be a loop with multiplication group* $\mathcal{M}ult(Q)$. *Q is a right universal Osborn loop if and only if the triple* $(\alpha_1(x,u),\beta_1(x,u),\gamma_1(x,u)) \in AUT(Q)$ *or the triple* $(\alpha_1(x,u), L_x\mathbb{L}_uL_{[u/(u\backslash x)]}\mathbb{L}_xL_u\gamma_1(x,u)\mathbb{L}_u, \gamma_1(x,u)) \in AUT(Q)$, *for all* $x,u\in Q$ *where*

$$\alpha_1(x,u)=\mathbb{L}_uL_x,\ \ \beta_1(x,u)=L_{[u(x\backslash u)]}\mathbb{L}_uL_xR_{[u\backslash x]}\mathbb{L}_u$$

and $\gamma_1(x,u)=\mathbb{L}_uL_xR_{[u\backslash x]}$ *are elements of* $\mathcal{M}ult(Q)$.

Proof. This is gotten from Theorem 2.16 by just writing identity OS_5^ρ or OS_3^ρ in autotopic form. □

Theorem 2.17. *Let Q be a loop with multiplication group* $\mathcal{M}ult(Q)$. *If Q is a right universal Osborn loop, then the triple* $(\alpha_1(x,u),\gamma_1(x,u)\mathbb{L}_{[x\cdot u^\rho]}, \gamma_1(x,u))\in AUT(Q)$, *for all* $x,u\in Q$ *where* $\alpha_1(x,u)=\mathbb{L}_uL_x$ *and* $\gamma_1(x,u)=\mathbb{L}_uL_xR_{[u\backslash x]}$ *are elements of* $\mathcal{M}ult(Q)$.

Proof. This follows by using identity OS_5^ρ or OS_3^ρ in Theorem 2.16 the way identity OS_5' or OS_3' was used in Theorem 2.1. □

Theorem 2.18. *An Osborn loop is right universal if and only if some of its right principal isotopes are isomorphic to some principal isotopes.*

Proof. Let $(Q,\cdot,\backslash,/)$ be a right universal Osborn loop. We shall use Lemma 2.6. The triple

$$(\alpha_1(x,u),\beta_1(x,u),\gamma_1(x,u)) = (\gamma_1\mathbb{R}_{[u\backslash x]}, L_{(u[x\backslash u])}\gamma_1\mathbb{L}_u,\gamma_1)$$

can be written as the following compositions $(I, L_{[u(x\backslash u)]}, I)(\gamma_1,\gamma_1,\gamma_1)(\mathbb{R}_{[u\backslash x]}, \mathbb{L}_u, I)$. Let $(Q,\circ)$ be some right principal isotopes of $(Q,\cdot)$ and $(Q,*)$ some principal isotopes of $(Q,\cdot)$. Then, the composition above can be expressed as:

$$(Q,\cdot)\xrightarrow[\text{right principal isotopism}]{(I,L_{[u(x\backslash u)]},I)}(Q,*)\xrightarrow[\text{isomorphism}]{(\gamma_1,\gamma_1,\gamma_1)}(Q,\circ)\xrightarrow[\text{principal isotopism}]{(\mathbb{R}_{[u\backslash x]},\mathbb{L}_u,I)}(Q,\cdot).$$

This means that some right principal isotopes $(Q,\circ)$ of $(Q,\cdot)$ are isomorphic to some principal isotopes $(Q,*)$ of $(Q,\cdot)$. □

Theorem 2.19. *An Osborn loop is right universal if and only if the existence of the principal autotopism* $(R_{[u\backslash x]}, L_u, I)$*, in the loop implies the triple* $(\gamma_1^{-1}, \gamma_1^{-1}\mathbb{L}_{[u(x\backslash u)]}, \gamma_1^{-1})$ *where* $\gamma_1(x,u) = \mathbb{L}_u L_x R_{[u\backslash x]}$ *is an autotopism in the loop and vice versa.*

Proof. The proof is in line with Theorem 2.18 with a slight adjustment to the composition of the triple

$$(\alpha_1(x,u), \beta_1(x,u), \gamma_1(x,u)) = (\gamma_1\mathbb{R}_{[u\backslash x]}, L_{(u[x\backslash u])}\gamma_1\mathbb{L}_u, \gamma_1)$$

which can be re-written as the following compositions $(R_{[u\backslash x]}, L_u, I)(\gamma_1^{-1}, \gamma_1^{-1}\mathbb{L}_{[u(x\backslash u)]}, \gamma_1^{-1})$. Hence, the conclusion follows. □

Theorem 2.20. *If an Osborn loop* $\mathcal{Q}$ *is right universal then,* $\gamma_1(x,u) \in BS(\mathcal{Q})$ *where* $\gamma_1(x,u) = \mathbb{L}_u L_x R_{[u\backslash x]}$.

Proof. By Theorem 2.17, if Q is a right universal Osborn loop, then

$$\begin{aligned}&(\alpha_1(x,u), \gamma_1(x,u)\mathbb{L}_{[xu^\rho]}, \gamma_1(x,u))\\ &= (\gamma_1(x,u)\mathbb{R}_{[u\backslash x]}, \gamma_1(x,u)\mathbb{L}_{[xu^\rho]}, \gamma_1(x,u)) \in AUT(Q),\end{aligned}$$

for all $x, u \in Q$. Which means that $\gamma_1(x,v) \in BS(\mathcal{Q})$. □

Lemma 2.7. *If an Osborn loop* Q *is right universal then,* $\gamma_1(x,u) = \mathbb{L}_u L_x R_{[u\backslash x]} \in AUM(Q)$ *if and only if* Q *obeys the identity* $y[u\backslash x] \cdot [xu^\rho]z = yz$*, for all* $x, u, y, z \in Q$*. Hence,* Q *is an abelian group.*

Proof. By Theorem 2.17, if Q is a right universal Osborn loop, then $(\alpha_1(x,u), \gamma_1(x,u)\mathbb{L}_{[x\cdot u^\rho]}, \gamma_1(x,u)) = (\gamma_1(x,u)\mathbb{R}_{[u\backslash x]}, \gamma_1(x,u)\mathbb{L}_{[x\cdot u^\rho]}, \gamma_1(x,u)) \in AUT(Q)$, for all $x, u \in Q$. By breaking this triple appropriately into two, the claim follows. In the equation $y[u\backslash x] \cdot [xu^\rho]z = yz$, if $u = y = z = e$, then $x^2 = e$ which means Q is an Osborn loop of exponent 2, thence, an abelian group. □

Theorem 2.21. *Let* $\mathcal{Q} = (Q, \cdot, \backslash, /)$ *be an Osborn loop such that the bi-mapping* $\gamma_1(x,u) = \mathbb{L}_u L_x R_{[u\backslash x]}$*.* $\mathcal{Q}$ *is a right universal Osborn loop if and only if* $\gamma_1(x,u) \in BS_\rho(\mathcal{Q})$ *implies it obeys the identity*

$$y(u\backslash x) \cdot uz = yz \ \forall\ x, y, z, u, \in Q \tag{7}$$

and vice versa.

Proof. The proof is based on Theorem 2.6 and is achieved by using the compositions $(R_{[u\backslash x]}, L_u, I)(\gamma_1^{-1}, \gamma_1^{-1}\mathbb{L}_{[u(x\backslash u)]}, \gamma_1^{-1})$ of Theorem 2.19. And hence, $\gamma_1(x,u) \in BS_\rho(\mathcal{Q})$ implies $\gamma_1(x,u)^{-1} \in BS_(\mathcal{Q})$ which implies it obeys identity (7) and vice versa. □

Corollary 2.6. *Let $\gamma_1(x,u) = \mathbb{L}_u L_x R_{[u\backslash x]}$ be a bi-mapping. There does not exist a non-trivial right universal Osborn loop $\mathcal{Q}$ for which $\gamma_1(x,u) \in BS_\rho(\mathcal{Q})$.*

Proof. Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be an arbitrary non-trivial right universal Osborn loop. According to Theorem 2.21, if in $\mathcal{Q}$, $\gamma_1(x,u) \in BS_\rho(\mathcal{Q})$ then it obeys identity (7). Put $y = z = u = e$ in identity (7), then $x = e$. Which is a contradiction. □

Remark 2.6. There is no non-trivial group or Moufang loop or universal WIPL or VD-loop or CC-loop $\mathcal{Q}$ for which $\gamma_1(x,u) \in BS_\rho(\mathcal{Q})$ where $\gamma_1(x,u) = \mathbb{L}_u L_x R_{[u\backslash x]}$.

3. Concluding remarks and future studies

Using the bi-mapping $\gamma_1(x,u) = \mathbb{L}_u L_x R_{[u\backslash x]}$ of Theorem 2.20 in some existing results of Adeniran [1] and Chiboka [4] on the Bryant Schneider groups of right universal Osborn loops like CC-loops and extra loops respectively, more equations and information can be deduced. For example, Theorem 2.2 of Chiboka [4] claims that in an extra loop $(L,\cdot)$, corresponding to every mapping $\theta \in BS(L,\cdot)$ is a unique pair of right pseudo-automorphisms. So for the bi-mapping $\gamma_1(x,u) = \mathbb{L}_u L_x R_{[u\backslash x]}$, the mappings $\vartheta_1 = \mathbb{L}_u L_x R_{u^{-1}x} L_{x^{-1}u^2x^{-1}}$ and $\varphi_1 = \mathbb{L}_u L_x R_{u^{-1}x} R_{x^{-1}u^2x^{-1}}$ are right pseudo-automorphisms with companions $c_1 = x^{-1}uxu^{-2}x$ and $c_2 = xu^{-1}x^{-1}u^2x^{-1}$ respectively. Also, in Chiboka [6], the author showed that in an extra loop $(L,\cdot)$, the middle inner mapping $T(x) = R_x L_x^{-1} \in BS(L,\cdot)$, for all $x \in L$. $T(x)$ is a mono-mapping but $\gamma_1(x,u)$ is a bi-mapping. Multiplying them, more elements of Bryant Schneider group of an extra loop can be gotten.

We need to identify the subgroup(s) of the multiplications group to which the bi-mappings and tri-mappings(which are not special mappings) of Corollary 2.2 and Corollary 2.4 belong to.

REFERENCES

1. Adeniran, J.O. – *The study of properties of certain class of loops via their Bryant-Schneider group*, Ph.D Thesis, University of Agriculture, Abeokuta, 2002.
2. Bruck, R.H.; Paige, L.J. – *Loops whose inner mappings are automorphisms*, Ann. of Math., 63 (1956), 308–323.
3. Bryant, B.F.; Schneider, H. – *Principal loop-isotopes of quasigroups*, Canad. J. Math., 18 (1966), 120–125.
4. Chiboka, V.O. – *The Bryant-Schneider group of an extra loop*, Zb. Rad. (Kragujevac), 18 (1996), 9–20.
5. Chiboka, V.O.; Solarin, A.R.T. – *Autotopism characterization of G-loops*, An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi Secţ. I a Mat., 39 (1993), 19–26.
6. Chiboka, V.O. – *The inner mappings of extra loops*, Kragujevac J. Math., 24 (2002), 55–60.
7. Jaiyéọlá, T.G.; Adéníran, J.O. – *New identities in universal Osborn loops*, Quasigroups Related Systems, 17 (2009), 55–76.
8. Kinyon, M.K. – *A survey of Osborn loops*, Milehigh conference on loops, quasigroups and non-associative systems, University of Denver, Denver, Colorado, 2005.
9. Kinyon, M.K.; Kunen, K.; Phillips, J.D. – *Diassociativity in conjugacy closed loops*, Comm. Algebra, 32 (2004), 767–786.
10. Kinyon, M.K.; Kunen, K.; Phillips, J.D. – *Every diassociative A-loop is Moufang*, Proc. Amer. Math. Soc., 130 (2002), 619–624.
11. Kunen, K. – *The structure of conjugacy closed loops*, Trans. Amer. Math. Soc., 352 (2000), 2889–2911.
12. Kunen, K. – *G-loops and permutation groups*, J. Algebra, 220 (1999), 694–708.
13. Robinson, D.A. – *The Bryant-Schneider group of a loop*, Ann. Soc. Sci. Bruxelles Sér. I, 94 (1980), 69–81.

Received: 17.II.2010
Revised: 11.V.2010

Department of Mathematics,
Obafemi Awolowo University,
Ile Ife,
NIGERIA
tjayeola@oauife.edu.ng