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GENERALISED CAUCHY-RIEMANN LIGHTLIKE SUBMANIFOLDS OF INDEFINITE COSYMPLECTIC MANIFOLDS

BY

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Abstract. Duggal and Sahin have studied generalised Cauchy-Riemann (GCR) lightlike submanifolds of indefinite Sasakian manifolds. In this paper, we study generalised Cauchy-Riemann (GCR) lightlike submanifold of an indefinite cosymplectic manifold and give a few examples also.

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Introduction

The study of lightlike submanifolds is interesting due to the fact that the intersection of the normal vector bundle and the tangent bundle is non-trivial making it remarkably different from non-degenerate submanifolds. Duggal and Bejancu [4] have studied lightlike submanifolds of indefinite Kaehler manifolds. On the other hand, a general notion of lightlike submanifold and generalized Cauchy-Riemann (GCR) lightlike submanifold of indefinite Sasakian manifold was introduced by Duggal and Sahin ([6], [7]). Recently, we have studied Cauchy-Riemann and screen Cauchy-Riemann lightlike submanifolds of indefinite cosymplectic manifolds [9]. We have shown that there does not exist inclusion relation between these two classes. The objective of this paper is to define a generalised Cauchy-Riemann lightlike submanifold of an indefinite cosymplectic manifold, which includes invariant, screen real, contact CR lightlike subcases and real hypersurfaces.

1. Preliminaries

An odd-dimensional semi-Riemannian manifold $\overline{M}$ is said to be an indefinite almost contact metric manifold if there exist structure tensors $\{\phi, V, \eta, \overline{g}\}$, where ϕ is a (1,1) tensor field, V a vector field, η a 1-form and $\overline{g}$ is the semi-Riemannian metric on $\overline{M}$ satisfying

$$\begin{cases} \phi^2 X = -X + \eta(X)V, \ \eta \circ \phi = 0, \ \phi V = 0, \ \eta(V) = 1 \\ \overline{g}(\phi X, \phi Y) = \overline{g}(X, Y) - \eta(X)\eta(Y), \ \overline{g}(X, V) = \eta(X) \end{cases} \tag{1.1}$$

for any $X, Y \in T\overline{M}$, where $T\overline{M}$ denotes the Lie algebra of vector fields on $\overline{M}$.

An indefinite almost contact metric manifold $\overline{M}$ is called an indefinite cosymplectic manifold if ([3], [8])

$$(\overline{\nabla}_X \phi)Y = 0, \tag{1.2}$$

for any $X, Y \in T\overline{M}$, where $\overline{\nabla}$ denotes the Levi-Civita connection on $\overline{M}$.

A submanifold M^m immersed in a semi-Riemannian manifold $\{\overline{M}^{m+k}, \overline{g}\}$ is called a lightlike submanifold if it admits a degenerate metric g induced from $\overline{g}$ whose radical distribution Rad(TM) is of rank r, where $1 \leq r \leq m$. Now, Rad$(TM) = TM \cap TM^{\perp}$, where

$$TM^{\perp} = \bigcup_{x \in M} \{u \in T_x\overline{M} : \overline{g}(u, v) = 0, \forall v \in T_xM\}. \tag{1.3}$$

Let $S(TM)$ be a *screen distribution* which is a semi-Riemannian complementary distribution of Rad(TM) in TM, that is, $TM = \text{Rad}(TM) \perp S(TM)$. Also, there exists a *screen transversal vector bundle* $S(TM^{\perp})$, which is a semi-Riemannian complementary vector bundle of $Rad(TM)$ in $TM^{\perp}$. Since, for any local basis $\{\xi_i\}$ of Rad(TM), there exists a local frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $[S(TM)]^{\perp}$ such that $\overline{g}(\xi_i, N_j) = \delta_{ij}$ and $\overline{g}(N_i, N_j) = 0$, it follows that there exists a *lightlike transversal vector bundle* $ltr(TM)$ locally spanned by $\{N_i\}$ (cf. [4], page 144). Let tr(TM) be the complementary (but not orthogonal) vector bundle to TM in $T\overline{M}|_M$. Then

$$\begin{cases} tr(TM) = ltr(TM) \perp S(TM^{\perp}) \\ T\overline{M}|_M = S(TM) \perp [Rad(TM) \oplus ltr(TM)] \perp S(TM^{\perp}). \end{cases} \tag{1.4}$$

A submanifold $(M, g, S(TM), S(TM^{\perp}))$ of $\overline{M}$ is said to be

(i) r-lightlike if, $r < min\{m, k\}$;

(ii) coisotropic if, $r = k < m$, $S(TM^{\perp}) = \{0\}$;

(iii) isotropic if, $r = m < k$, $S(TM) = \{0\}$;

(iv) totally lightlike if, $r = m = k$, $S(TM) = \{0\} = S(TM^{\perp})$.

Let $\overline{\nabla}$, ∇ and ∇^t denote linear connections on $\overline{M}$, M and the vector bundle tr(TM), respectively. Then according to the decomposition (1.4), the lightlike version of Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \forall X, Y \in \Gamma(TM), \tag{1.5}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^l_X(N) + D^s(X, N), N \in \Gamma(ltr(TM)), \tag{1.6}$$

$$\overline{\nabla}_X W = -A_W X + \nabla^s_X(W) + D^l(X, W), W \in \Gamma(S(TM^{\perp})), \tag{1.7}$$

where h^l, h^s are $\Gamma(ltr(TM))$-valued and $\Gamma(S(TM^{\perp}))$-valued *lightlike second fundamental form* and *screen second fundamental form* of M; $D^l(X, W)$ and $D^s(X, N)$ are the projections of ∇^t on $\Gamma(ltr(TM))$ and $\Gamma(S(TM^{\perp}))$, and ∇^l, ∇^s are linear connections on $\Gamma(ltr(TM))$ and $\Gamma(S(TM^{\perp}))$, respectively. We call ∇^l and ∇^s the lightlike and screen transversal connections on M, and A_N and A_W as shape operators on M with respect to N and W, respectively. Using (1.5) and (1.7), we obtain

$$\overline{g}(h^s(X, Y), W) + \overline{g}(Y, D^l(X, W)) = g(A_W X, Y). \tag{1.8}$$

Let $\overline{P}$ denote the projection of TM on $S(TM)$ and let ∇^*, ∇^{*t} denote the linear connections on $S(TM)$ and $Rad(TM)$, respectively. Then from the decomposition of the tangent bundle of a lightlike submanifold, we have

$$\nabla_X \overline{P}Y = \nabla^*_X \overline{P}Y + h^*(X, \overline{P}Y), \tag{1.9}$$

$$\nabla_X \xi = -A^*_{\xi} X + \nabla^{*t}_X \xi, \tag{1.10}$$

for X, $Y \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$, where h^* and A^* are the second fundamental form and shape operator of the distributions $S(TM)$ and $Rad(TM)$, respectively. From (1.9) and (1.10), we get

$$\overline{g}(h^l(X, \overline{P}Y), \xi) = g(A^*_{\xi} X, \overline{P}Y). \tag{1.11}$$

The curvature tensor $\overline{R}$ of an indefinite cosymplectic space form $\overline{M}(c)$ is given by

$$\begin{aligned}\overline{R}(X,Y)Z = \frac{c}{4}\{&\overline{g}(Y,Z)X - \overline{g}(X,Z)Y + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ \overline{g}(X,Z)\eta(Y)V - \overline{g}(Y,Z)\eta(X)V + \overline{g}(\phi Y,Z)\phi X \\ &+ \overline{g}(\phi Z,X)\phi Y - 2\overline{g}(\phi X,Y)\phi Z\},\end{aligned} \tag{1.12}$$

for any X, Y and Z vector fields on $\overline{M}$ (see [8]).

We have the following definition due to Bejan and Duggal:

Definition 1.1 ([2])**.** A lightlike submanifold $(M, g, S(TM))$ isometrically immersed in a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is minimal if

(i) $h^s = 0$ on $Rad(TM)$;

(ii) $\operatorname{trace} h = 0$, where trace is written with respect to g restricted to $S(TM)$.

Following [5], we have:

Definition 1.2. A lightlike submanifold M of an indefinite cosymplectic manifold $\overline{M}$ is a screen real submanifold if $Rad(TM)$ and $S(TM)$ are, respectively, invariant and anti-invariant with respect to ϕ.

2. Generalized Cauchy-Riemann (GCR) lightlike submanifolds

We have the following definition:

Definition 2.1. Let $(M, g, S(TM), S(TM^{\perp}))$ be a real lightlike submanifold, tangent to the structure vector field V and immersed in an indefinite cosymplectic manifold $(\overline{M}, \overline{g})$. Then M is called a generalised Cauchy-Riemann lightlike submanifold of $\overline{M}$ if the following conditions are satisfied:

(A) There exist two subbundles D_1 and D_2 of Rad(TM) on M such that

$$Rad(TM) = D_1 \oplus D_2, \phi(D_1) = D_1, \phi(D_2) \subset S(TM). \tag{2.1}$$

(B) There exist two vector subbundles D_0 and D' of $S(TM)$ such that over M

$$\begin{cases} S(TM) = \{\phi(D_2) \oplus D'\} \perp D_0 \perp \{V\}, \\ \phi D_0 = D_0, \phi(D') = L_1 \perp L_2, \end{cases} \tag{2.2}$$

where D_0 is nondegenerate and L_1 and L_2 are vector subbundles of $S(TM^{\perp})$ and $ltr(TM)$, respectively. Thus, we have the following decomposition:

$$TM = D \oplus D' \perp \{V\}, D = Rad(TM) \perp \phi(D_2) \perp D_0. \quad (2.3)$$

A contact GCR-lightlike submanifold is said to be proper if $D_0 \neq \{0\}$, $D_1 \neq \{0\}$, $D_2 \neq \{0\}$ and $L_1 \neq \{0\}$. Thus, from Definition 2.1, we have:

(a) Condition (A) implies that dim(Rad(TM))≥ 3.

(b) Condition (B) implies that dim(D)≥ 6 and dim(D_2) = dim(L_2). Thus dim(M)≥ 9 and dim($\overline{M}$)≥ 13.

(c) Any proper 9-dimensional contact *GCR*-lightlike submanifold is 3-lightlike.

(d) Contact distribution ($\eta = 0$) and condition (A) imply that index($\overline{M}$)$\geq$4.

Proposition 2.1. *A GCR-lightlike submanifold M of an indefinite cosymplectic manifold $\overline{M}$ is a contact CR* (*respectively, contact SCR lightlike*) *submanifold if and only if $D_1 = \{0\}$* (*respectively, $D_2 = \{0\}$*).

Proof. Let M be a contact CR-lightlike submanifold. Then ϕRad(TM) is a distribution on M such that Rad(TM)$\cap\phi$(Rad(TM)) = $\{0\}$ ([9]). Hence D_2=$Rad(TM)$ and D_1=$\{0\}$, which imply that $ltr(TM) \cap \phi(ltr(TM)) = \{0\}$. Thus it follows that $\phi(ltr(TM)) \subset S(TM)$.

Conversely, suppose that M is a *GCR*-lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$ such that $D_1 = \{0\}$. Then from (2.1), we have D_2 = Rad(TM). Therefore, Rad(TM) $\cap$ ϕ(Rad(TM)) = $\{0\}$, which implies that M is a contact *CR*-lightlike submanifold of $\overline{M}$.

We have:

Proposition 2.2. *There exist no coisotropic, isotropic or totally lightlike proper GCR-lightlike submanifold M of an indefinite cosymplectic manifold $\overline{M}$.*

Proof. If M is isotropic or totally lightlike, then $S(TM) = \{0\}$ and if M is coisotropic then $S(TM^{\perp}) = \{0\}$. Hence, conditions (A) and (B) of Definition 2.1 are not satisfied.

It is easy to see that any contact *CR*-lightlike three-dimensional submanifold is 1-lightlike real hypersurface [9]. Moreover, it has also been proved that contact *SCR*-lightlike submanifolds have invariant and screen real lightlike subcases. Thus, from Proposition 2.1 it follows that *GCR*-lightlike submanifold is an umbrella of real hypersurfaces, invariant, screen real and contact *CR*-lightlike submanifolds.

In what follows, $(R_q^{2m+1}, \phi_0, V, \eta, g)$ denote the manifold R_q^{2m+1} with its usual cosymplectic structure given by

$$(2.4)\quad \begin{cases} \eta = dz, V = \partial z, \\ \overline{g} = \eta \otimes \eta - \sum_{i=1}^{q/2} dx^i \otimes dx^i + dy^i \otimes dy^i + \sum_{i=q+1}^{m} dx^i \otimes dx^i + dy^i \otimes dy^i, \\ \phi_0(\sum_{i=1}^{m}(X_i \partial x^i + Y_i \partial y^i) + Z\partial z) = \sum_{i=1}^{m}(Y_i \partial x^i - X_i \partial y^i), \end{cases}$$

where (x^i, y^i, z) are the Cartesian coordinates.

Example 2.1. Let $\overline{M} = (R_4^{13}, \overline{g})$ be a semi-Euclidean space, where $\overline{g}$ is of signature $(-,-,+,+,+,+,-,-,+,+,+,+,+)$ with respect to the canonical basis

$$(2.5)\quad \{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial z\}.$$

Consider a submanifold M of R_4^{13}, defined by

$$(2.6)\quad \begin{cases} x^4 = x^1 \cos\theta - y^1 \sin\theta, y^4 = x^1 \sin\theta + y^1 \cos\theta, \\ x^2 = y^3, x^5 = \sqrt{1+(y^5)^2}, y^5 \neq \pm 1. \end{cases}$$

Then a local frame of TM is given by

$$(2.7)\quad \begin{cases} Z_1 = \partial x_1 + \cos\theta \partial x_4 + \sin\theta \partial y_4, \\ Z_2 = -\sin\theta \partial x_4 + \partial y_1 + \cos\theta \partial y_4, \\ Z_3 = \partial x_2 + \partial y_3,\ Z_4 = \partial x_3 - \partial y_2,\ Z_5 = \partial x_6, \\ Z_6 = \partial y_6,\ Z_7 = y^5 \partial x_5 + x^5 \partial y_5, \\ Z_8 = \partial x_3 + \partial y_2,\ Z_9 = V = \partial z. \end{cases}$$

Hence, $\mathrm{Rad}(TM) = \mathrm{span}\{Z_1, Z_2, Z_3\}$. Moreover, $\phi_0 Z_1 = -Z_2$, and $\phi_0 Z_3 = Z_4$ which belongs to $\Gamma S(TM)$. Thus $D_1 = \mathrm{span}\ \{Z_1, Z_2\}$, $D_2 = \mathrm{span}\ \{Z_3\}$. Hence, (A) holds. Next, $\phi_0 Z_5 = -Z_6$, which implies that $D_0 = \mathrm{span}\{Z_5, Z_6\}$ is invariant with respect to ϕ_0. By direct calculation, we get

$S(TM^{\perp}) = \text{span}\ \{W = x^5\partial x_5 - y^5\partial y_5\}$ such that $\phi_0(W) = -Z_7$. Hence, $L_1 = S(TM^{\perp})$, and $ltr(TM)$ is spanned by

$$N_1 = -\partial x_1 + \cos\theta\partial x_4 + \sin\theta\partial y_4,$$
$$N_2 = -\sin\theta\partial x_4 - \partial y_1 + \cos\theta\partial y_4, \quad N_3 = -\partial x_2 + \partial y_3$$

such that $\phi_0(N_1) = -N_2$ and $\phi_0(N_3) = Z_8$. Hence, $L_2 = \text{span}\{N_3\}$ and $D' = \text{span}\{\phi_0 N_3, \phi_0 W\}$. Thus, M is a contact GCR-lightlike submanifold of R^{13}_4.

3. Existence and non-existence theorems

We now prove an existence theorem for GCR-lightlike submanifolds in an indefinite cosymplectic space form.

Theorem 3.1. *Let M be a lightlike submanifold of an indefinite cosymplectic space form $\overline{M}(c)$ where $c \neq 0$, with structure vector field tangent to M. Then, M is a GCR-lightlike submanifold of $\overline{M}(c)$ if and only if*

(1) *The maximal invariant subspaces of T_pM, for $p \in M$, define a distribution $D = D_1 \perp D_2 \perp \phi(D_2) \perp D_0$, where $Rad(TM) = D_1 \oplus D_2$, and D_0 is a non-degenerate invariant distribution.*

(2) *There exists a lightlike transversal vector bundle $ltr(TM)$ such that $\overline{g}(\overline{R}(X,Y)\xi, N) = 0$, $\forall X, Y \in \Gamma(D_0), N \in \Gamma(ltr(TM)), \xi \in \Gamma(Rad(TM))$.*

(3) *There exists a vector subbundle $\overline{D}$ on M such that $\overline{g}(\overline{R}(X,Y)W_1, W_2) = 0, \forall W_1, W_2 \in \Gamma(\overline{D})$, where $\overline{D}$ is orthogonal to D and $\overline{R}$ is the curvature tensor of $\overline{M}(c)$.*

Proof. Suppose that M is a GCR-lightlike submanifold of $\overline{M}(c)$ with $c \neq 0$. Then $D = D_1 \perp D_2 \perp \phi(D_2) \perp D_0$ is a maximal invariant subspace. From (1.12), we have

$$\overline{g}(\overline{R}(X,Y)\xi, N) = -\frac{c}{2}\{g(\phi X, Y)\overline{g}(\phi\xi, N)\},$$

for $X, Y \in \Gamma(D_0), N \in \Gamma(ltr(TM)), \xi \in \Gamma(D_2)$. Since $c \neq 0$, $g(\phi X, Y) \neq 0$ and $\overline{g}(\phi\xi, N) = 0$, and therefore, we get

$$\overline{g}(\overline{R}(X,Y)\xi, N) = 0.$$

Similarly, we have $\overline{g}(\overline{R}(X,Y)W_1,W_2) = -\frac{c}{2}\{g(\phi X,Y)\overline{g}(\phi W_1,W_2)\} = 0$ for $X,Y \in \Gamma(D_0)$, and $W_1,W_2 \in \Gamma\phi(L_1)$.

Conversely, assume that the conditions (1), (2) and (3) are satisfied. Then, (1) implies that D is invariant. From (2) and (1.12), we have

$$\overline{g}(\phi\xi, N) = 0, \tag{3.1}$$

which implies that $\phi\xi \in \Gamma(S(TM))$. Then, from (3.1), we get that $\overline{g}(\xi,\phi N) = 0$. Hence, a part of $\phi ltr(TM)$ also belongs to $S(TM)$. Similarly, from (3) and (1.12), we get

$$\overline{g}(\phi W_1, W_2) = 0, \tag{3.2}$$

which implies that $\phi(\overline{D})$ is orthogonal to $\overline{D}$ and since $\overline{D}$ is non-degenerate, we have $\overline{g}(\phi W_1,\phi W_2) = g(W_1,W_2) \neq 0$. Also, $\overline{g}(\phi\xi,W) = -\overline{g}(\xi,\phi W) = 0$, and therefore, $\phi(\overline{D})$ is orthogonal to Rad(TM), and that $\phi(\overline{D})$ does not belong to $ltr(TM)$. On the other hand, D_0 invariant and non-degenerate imply that $g(\phi W,X) = 0$, for $X \in \Gamma(D_0)$. Thus, $\overline{D}\perp D_0$ and $\phi(\overline{D})\perp D_0$. Moreover, we know that the structure vector field V belongs to $S(TM)$ [1]. Then summing up the above arguments, we conclude that $S(TM) = \{\phi(D_2) \oplus M_1\}\perp\overline{D}\perp D_0\perp\{V\}$, where $\phi(M_1) \subset ltr(TM)$, which completes the proof.

For any $X \in \Gamma(TM)$, we write

$$\phi X = PX + FX, \tag{3.3}$$

where PX and FX are the tangential and transversal parts of ϕX, respectively. Similarly,

$$\phi W = BW + CW, W \in \Gamma(tr(TM)), \tag{3.4}$$

where BW and CW are sections of TM and $tr(TM)$, respectively.

Theorem 3.2. *There exists an induced metric connection on a proper GCR-lightlike submanifold M of an indefinite cosymplectic manifold with structure vctor field tangent to M if and only if for $X \in \Gamma(TM)$, the following hold*

$$P(A^*_{\phi\xi}(X,\phi\xi) - \nabla^{*t}_X\phi\xi) \in \Gamma(Rad(TM)), \xi \in \Gamma(D_1)$$
$$P(h^*(X,\phi\xi) - \nabla^*_X\phi\xi) \in \Gamma(Rad(TM)), \xi \in \Gamma(D_2),$$

and $Bh(X,\phi\xi)=0$, $\xi\in\Gamma(Rad(TM))$.

Proof. Assume that M admits a metric connection ∇. Then, we show that the radical distribution is parallel with respect to ∇ {cf. [4], Theorem 2.4, p.161}. From (1.2), we get that $\overline{\nabla}_X\phi\xi=\phi\overline{\nabla}_X\xi$, and consequently

$$\phi\overline{\nabla}_X\phi\xi=-\overline{\nabla}_X\xi, \tag{3.5}$$

for $X\in\Gamma(TM)$, and $\xi\in\Gamma(Rad(TM))$. Using (1.5) in the above equation, we obtain

$$\phi(\nabla_X\phi\xi+h(X,\phi\xi))=-\nabla_X\xi-h(X,\xi). \tag{3.6}$$

Considering the tangential part of the above equation for $\xi\in\Gamma(D_1)$ and using (1.10), (3.3) and (3.4), we get

$$\nabla_X\xi=PA^*_{\phi\xi}(X,\phi\xi)-P\nabla^{*t}_X\phi\xi-Bh(X,\phi\xi). \tag{3.7}$$

Similarly, for $\xi\in\Gamma(D_2)$ and using (1.9), (3.3), (3.4) and (3.6), we get

$$\nabla_X\xi=Ph^*(X,\phi\xi)-P\nabla^*_X\phi\xi-Bh(X,\phi\xi). \tag{3.8}$$

Thus, from (3.7), $\nabla_X\xi\in\Gamma(Rad(TM))$ if and only if

$$P(A^*_{\phi\xi}(X,\phi\xi)-\nabla^{*t}_X\phi\xi)\in\Gamma(Rad(TM)), \texttt{and} Bh(X,\phi\xi)=0, \tag{3.9}$$

for $X\in\Gamma(TM)$, and $\xi\in\Gamma(D_1)$. Similarly, from (3.8), $\nabla_X\xi\in\Gamma(Rad(TM))$ if and only if

$$P(h^*(X,\phi\xi)-\nabla^{*t}_X\phi\xi)\in\Gamma(Rad(TM)),\ \text{and}\ Bh(X,\phi\xi)=0, \tag{3.10}$$

for $X\in\Gamma(TM)$, and $\xi\in\Gamma(D_2)$. Hence, the result follows from (3.9) and (3.10).

Following is the non-existence theorem for GCR-lightlike submanifolds:

Theorem 3.3. *There exists no contact totally umbilical proper GCR-lightlike submanifold M with structure vector field tangent to M of an indefinite cosymplectic space form $\overline{M}(c)$ with $c\neq 0$.*

Proof. The proof is similar to the proof of Theorem 4.11 ([9]).

4. Minimal GCR-lightlike submanifolds

In this section, we give an example of a minimal GCR-lightlike submanifold of an indefinite cosymplectic manifold and also prove a characterisation theorem.

Example 4.1. Let $\overline{M} = (R_4^{15}, \overline{g})$ be a semi-Euclidean space, where $\overline{g}$ is of signature $(-,-,+,+,+,+,+,-,-,+,+,+,+,+,+)$ with respect to the canonical basis

$$\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial x_7, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial y_7, \partial z\}. \tag{4.1}$$

Suppose M is a submanifold of R_4^{15}, given by

$$\begin{cases} x^1 = u^1 \cosh\beta, y^1 = -u^2 \cosh\beta \\ x^2 = u^3, y^2 = u^8 \\ x^3 = u^1 \sinh\beta + u^2, y^3 = -u^2 \sinh\beta + u^1 \\ x^4 = u^3, y^4 = u^9 \\ x^5 = \cos u^4 \cosh u^5, y^5 = \sin u^4 \sinh u^5 \\ x^6 = \cos u^6 \cosh u^7, y^6 = \cos u^6 \sinh u^7 \\ x^7 = \sin u^6 \cosh u^7, y^7 = \sin u^6 \sinh u^7 \\ z = u^{10}. \end{cases} \tag{4.2}$$

Then a local frame for TM is given by

$$\begin{cases} Z_1 & = \cosh\beta\partial x_1 + \sinh\beta\partial x_3 + \partial y_3, \\ Z_2 & = \partial x_3 - \cosh\beta\partial y_1 - \sinh\beta\partial y_3 \\ Z_3 & = \partial x_2 + \partial x_4, Z_4 = -\sin u^4 \cosh u^5 \partial x_5 + \cos u^4 \sinh u^5 \partial y_5 \\ Z_5 & = \cos u^4 \sinh u^5 \partial x_5 + \sin u^4 \cosh u^5 \partial y_5, \\ Z_6 & = -\sin u^6 \cosh u^7 \partial x_6 + \cos u^6 \cosh u^7 \partial x_7 - \sin u^6 \sinh u^7 \partial y_6 \\ & \quad + \cos u^6 \sinh u^7 \partial y_7 \\ Z_7 & = \cos u^6 \sinh u^7 \partial x_6 + \sin u^6 \sinh u^7 \partial x_7 + \cos u^6 \cosh u^7 \partial y_6 \\ & \quad + \sin u^6 \cosh u^7 \partial y_7 \\ Z_8 & = \partial y_2, Z_9 = \partial y_4, Z_{10} = \partial z = V. \end{cases} \tag{4.3}$$

We see that M is a 3-lightlike submanifold with $\mathrm{Rad}(TM)=span\{Z_1, Z_2, Z_3\}$, $\phi_0 Z_1 = Z_2$ and $\phi_0(Z_3) = -Z_8 - Z_9 \in \Gamma(S(TM))$. Thus $D_1 = span\{Z_1, Z_2\}$

and $D_2 = span\{Z_3\}$. On the other hand, $\phi_0(Z_4)=Z_5$ and $D_0=span\{Z_4, Z_5\}$ is invariant. Moreover, since $\phi_0 Z_6$ and $\phi_0 Z_7$ are non-null and perpendicular to TM, we can choose $S(TM^{\perp}) = span\{\phi_0 Z_6, \phi_0 Z_7\}$. Furthermore, the lightlike transversal bundle $ltr(TM)$ is spanned by

$$\begin{cases} N_1 = -\cosh\beta\partial x_1 - \sinh\beta\partial x_3 + \partial y_3 \\ N_2 = \partial x_3 + \cosh\beta\partial y_1 + \sinh\beta\partial y_3 \\ N_3 = -\partial x_2 + \partial x_4 \end{cases} \tag{4.4}$$

and $\phi_0 N_1 = N_2$, $\phi_0 N_3 = Z_8 - Z_9 \in \Gamma(S(TM))$. Thus, we have $\phi_0 D' = span\{\phi_0 Z_6, \phi_0 Z_7, \phi_0 N_3\}$. Hence, we conclude that M is a contact GCR-lightlike submanifold of R^{15}_4. Then, a quasi orthonormal basis for $\overline{M}$ along M is given by

$$\begin{cases} \xi_1 = Z_1, \;\; \xi_2 = Z_2, \;\; \xi_3 = Z_3, \\ \phi_0\xi_3 = -(\partial y_2 + \partial y_4), \;\; \phi_0 N_3 = (\partial y_2 - \partial y_4) \\ e_1 = \frac{1}{\sqrt{\cosh^2 u^5 - \cos^2 u^4}} Z_4, \;\; e_2 = \frac{1}{\sqrt{\cosh^2 u^5 - \cos^2 u^4}} Z_5 \\ e_3 = \dfrac{1}{\sqrt{\cosh^2 u^7 + \sinh^2 u^7}} Z_6, \\ e_4 = \dfrac{1}{\sqrt{\cosh^2 u^7 + \sinh^2 u^7}} Z_7, \;\; V = Z_{10} \\ W_1 = \dfrac{1}{\sqrt{\cosh^2 u^7 + \sinh^2 u^7}} \phi_0 Z_6, \\ W_2 = \dfrac{1}{\sqrt{\cosh^2 u^7 + \sinh^2 u^7}} \phi_0 Z_7 N_1, N_2, N_3, \end{cases} \tag{4.5}$$

where $\varepsilon_1 = g(e_1, e_1) = 1$, $\varepsilon_2 = g(e_2, e_2) = 1$, $\varepsilon_3 = g(e_3, e_3) = 1$, and $\varepsilon_4 = g(e_4, e_4) = 1$. By direct calculation and using the Gauss formula (1.5), we get

$$\begin{cases} h(\xi_1, \xi_1) = h(\xi_2, \xi_2) = h(\xi_3, \xi_3) = h(e_1, e_1) = h(e_2, e_2) = 0, \\ h(\phi_0\xi_3, \phi_0\xi_3) = h(\phi_0 N_3, \phi_0 N_3) = h^l(e_3, e_3) = h^l(e_4, e_4) = 0, \\ h^s(e_3, e_3) = -\dfrac{1}{(\cosh^2 u^7 + \sinh^2 u^7)^{\frac{3}{2}}} W_2, \\ h^s(e_4, e_4) = \dfrac{1}{(\cosh^2 u^7 + \sinh^2 u^7)^{\frac{3}{2}}} W_2. \end{cases} \tag{4.6}$$

Therefore,

$$\begin{aligned}\operatorname{trace} h_{g|S(TM)} &= \varepsilon_1 h^s(e_3,e_3) + \varepsilon_2 h^s(e_4,e_4)\\ &= h^s(e_3,e_3) + h^s(e_4,e_4) = 0.\end{aligned} \tag{4.7}$$

Thus M is a minimal proper contact GCR-lightlike submanifold of R_4^{15}. Now, we prove a characterisation theorem for minimal proper contact GCR-lightlike submanifold. We use a quasi orthonormal frame for M given by $\{\xi_1,\dots,\xi_q,e_1,\dots,e_m,V,W_1,\dots,W_n,N_1,\dots,N_q\}$ where $\{\xi_1,\dots,\xi_q,\ e_1,\dots,e_m,\ V\}\in\Gamma(TM)$ such that $\{\xi_1,\dots,\xi_{2p}\}$, $\{\xi_{2p+1},\dots,\xi_q\}$ and $\{e_1,\dots,e_{2l}\}$ form a basis for D_1, D_2 and D_0, respectively. Moreover, take $\{W_1,\dots,W_k\}$ as a basis for L_1 and $\{N_{2p+1},\dots,N_q\}$ as a basis for L_2. Thus, we have a quasi orthonormal basis for M, $\{\xi_1,\dots,\xi_{2p},\xi_{2p+1},\dots,\xi_q,\phi\xi_{2p+1},\dots,\phi\xi_q,$ $e_1,\dots,e_l,\phi e_1,\dots,\phi e_l,\phi W_1,\dots,\phi W_k,\phi N_{2p+1},\dots,\phi N_q\}$.

Theorem 4.1. *Let M be a proper contact GCR-lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. Then M is minimal if and only if*

$$\operatorname{trace} A_{W_j|S(TM)} = 0, \operatorname{trace} A^*_{\xi_k|S(TM)} = 0, \overline{g}(Y, D^l(X,W)) = 0 \tag{4.8}$$

for $X,Y\in\Gamma(Rad(TM))$, and $W\in\Gamma(S(TM^\perp)$.

Proof. We know that $h^l=0$ on Rad(TM) [2]. Hence, from Definition 1.1, a GCR-lightlike submanifold is minimal if and only if $\sum_{i=1}^{2l}\varepsilon_i h(e_i,e_i)+\sum_{j=2p+1}^{q} h(\phi\xi_j,\phi\xi_j)+\sum_{j=2p+1}^{q} h(\phi N_j,\phi N_j)+\sum_{l=1}^{k}\varepsilon_l h(\phi W_l,\phi W_l)=0$, and $h^s=0$ on Rad(TM). Now from (1.8), we have that $h^s=0$ on Rad(TM) if and only if $\overline{g}(Y,D^l(X,W))=0$, for $X,Y\in\Gamma(\mathrm{Rad}(TM))$, and $W\in\Gamma(S(TM^\perp))$.

On the other hand,

$$\begin{aligned}\operatorname{trace} h|_{S(TM)} &= \frac{1}{q}\sum_{a=1}^{q}\sum_{j=2p+1}^{q}\overline{g}(h^l(\phi\xi_j,\phi\xi_j),\xi_a)N_a + \overline{g}(h^l(\phi N_j,\phi N_j),\xi_a)N_a\\ &+\frac{1}{n}\sum_{j=2p+1}^{q}\sum_{b=1}^{n}\varepsilon_b\{\overline{g}(h^s(\phi\xi_j,\phi\xi_j),W_b)W_b+\overline{g}(h^s(\phi N_j,\phi N_j),W_b)W_b\}\\ &+\sum_{b=1}^{n}\varepsilon_b\frac{1}{n}\{\sum_{i=1}^{2l}\overline{g}(h^s(e_i,e_i),W_b)W_b+\sum_{l=1}^{k}\overline{g}(h^s(\phi W_l,\phi W_l),W_b)W_b\}\\ &+\sum_{c=1}^{q}\frac{1}{q}\{\sum_{i=1}^{2l}\overline{g}(h^l(e_i,e_i),\xi_c)N_c+\sum_{l=1}^{k}\overline{g}(h^l(\phi W_l,\phi W_l),\xi_c)N_c\}.\end{aligned}$$

Using (1.8) and (1.11) in the above equation, we get

$$\begin{aligned}
\operatorname{trace} h|_{S(TM)} &= \frac{1}{q}\sum_{a=1}^{q}\sum_{j=2p+1}^{q} g(A^{*}_{\xi_a}\phi\xi_j,\phi\xi_j)N_a + g(A^{*}_{\xi_a}\phi N_j,\phi N_j)N_a \\
&+ \frac{1}{n}\sum_{j=2p+1}^{q}\sum_{b=1}^{n}\varepsilon_b\{g(A_{W_b}\phi\xi_j,\phi\xi_j)W_b + g(A_{W_b}\phi N_j,\phi N_j)W_b\} \\
&+ \sum_{b=1}^{n}\varepsilon_b\frac{1}{n}\{\sum_{i=1}^{2l} g(A_{W_b}e_i,e_i)W_b + \sum_{l=1}^{k} g(A_{W_b}\phi W_l,\phi W_l)W_b\} \\
&+ \sum_{c=1}^{q}\frac{1}{q}\{\sum_{i=1}^{2l} g(A^{*}_{\xi_c}e_i,e_i)N_c + \sum_{l=1}^{k} g(A^{*}_{\xi_c}\phi W_l,\phi W_l)N_c\}
\end{aligned} \tag{4.9}$$

and, our assertion follows from the above equation.

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