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ON LINEAR OPERATORS PRESERVING ORTHOGONALITY

BY

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Abstract. In this paper we present some relationships among Birkhoff orthogonality and other concepts in convex analysis. In this way we obtain a result of Blanco and Turnšek concerning the linear operators that preserve Birkhoff orthogonality.

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1. Introduction

Let X be a real linear normed space and let us denote by $\perp$ the orthogonality relation (in Birkhoff sense) on X as:

$$x \perp y \Leftrightarrow \|x+ty\| \leq \|x\|, \quad \text{for all } t \in \mathbb{R} \tag{1}$$

(see, for example, [1], [4], [9], [11], [12]).

In general, this orthogonality is neither symmetric nor additive, but it is always homogeneous. Because the function $t \to \|x+ty\|$, $t \in \mathbb{R}$, is convex, the orthogonality can be easily characterized using the directional derivatives of the norm (see [1]).

In 1992, Koldobsky [13] has proved that a linear operator $T : X \to X$ preserves orthogonality if and only if T is an isometry multiplied by a positive constant. In 2006, Blanco and Turnšek [5], extend this result to the case of the linear operators between two linear normed spaces. Moreover, the linear normed spaces X, Y can be considered over C, the field of the complex numbers.

Obviously, if in Koldobsky's result the operator T is the identical operator on a linear space we get that if the orthogonality generated by a norm implies the orthogonality generated by another norm, then that two norms are proportional. A straight demonstration of this result has been given by SCHÖPF [15] using the norm's derivatives.

In the next paragraph we recall a few concepts and basic results in convex analysis (see, for instance [3], [9], [10]), which will be used in the next sections. In the third section we establish some simple properties which characterize the linear operators preserving the orthogonality. Shortly, using a Schöpf's idea, the result of Blanco and Turnšek can be obtained taking into account the monotonicity property of the function $t \to \|Tx + tTy\|_1^2 \cdot \|x + ty\|_2^{-2}$, where T is a linear operator between the real linear normed spaces $(X, \|\cdot\|_1), (Y, \|\cdot\|_2)$ and x, y are two fixed elements in X.

2. Preliminaries

We recall some basic properties of directional derivatives (see, for example, [2], [3], [9], [14]). Let X be a real linear normed space and X^* its dual. If we consider a proper function (that is, there exist $\widetilde{x} \in X$ such that $f(\widetilde{x}) \in \mathbb{R}$) $f : X \to]-\infty, +\infty]$, then the *directional derivatives* (right and left) of f are defined as:

$$f'_{\pm}(x; y) = \lim_{t \to \pm 0} \frac{f(x + ty) - f(x)}{t}, \tag{2}$$

where $x \in Df = \{x \in X; f(x) < \infty\}$, called the *domain* of the function f and $y \in X$.

If f is convex, then $f'_{\pm}(x; y)$ exist for all $x \in Df$, $y \in X$. We also have the next well known properties:

$$f'_{\pm}(x; -y) = -f'_{\mp}(x; y), \quad x \in D(f), \quad y \in X; \tag{3}$$

$$f'_{\pm}(x; \lambda y) = \lambda f'_{\pm}(x; y), \ \lambda \geq 0, x \in D(f), y \in X; \tag{4}$$

$$f'_{-}(x; y) \leq f'_{+}(x; y), \quad x \in D(f), \ y \in X; \tag{5}$$

$$f'_{+}(x; y_1 + y_2) \leq f'_{+}(x; y_1) + f'_{+}(x; y_2), \ x \in D(f), \ y_1, y_2 \in X. \tag{6}$$

The function f is said to be (Gâteaux) *differentiable* at x if $y \to f'_+(x,y)$, $y \in X$, is a linear continuous functional on X. If f is a convex function, then it is differentiable at x whenever f is continuous at x and $f'_-(x,y) = f'_+(x,y)$ for all $y \in X$.

If $f(x) = \frac{1}{2}\|x\|^2$, $x \in X$, we denote its directional derivatives by $n'_\pm(x;y)$. The basic properties of $n'_\pm$ to be frequently used are the following:

$$n'_\pm(x;x) = \|x\|^2, \quad x \in X; \tag{7}$$

$$n'_\pm(x;y) \le \|x\| \cdot \|y\|, \quad x, y \in X; \tag{8}$$

$$n'_+(x;\alpha x + y) = n'_+(x;y) + \alpha\|x\|^2, \quad x \in X, \alpha \in \mathbb{R}. \tag{9}$$

If $n'_-(x,y) = n'_+(x,y)$ for all $x, y \in X$, then the linear normed space X is said to be *smooth*. Obviously, a linear normed space is smooth if and only if its norm $(\frac{1}{2}\|\cdot\|^2)$ is differentiable on X.

3. Characterizations of linear operators that preserve orthogonality

If we consider the definition (1) of orthogonality relation, we observe that $t = 0$ is a minimum point of the convex function $t \to \|x+ty\|$, $t \in \mathbb{R}$. Taking into account the characterization of minimum points for convex functions on $\mathbb{R}$ and using the directional derivatives [1], [14], we get that:

$$x \perp y \text{ if and only if } n'_-(x;y) \le 0 \le n'_+(x;y). \tag{10}$$

Particularly, it follows:

$$\text{if } n'_+(x;y) = 0 \text{ or } n'_-(x;y) = 0, \quad \text{then } x \perp y. \tag{11}$$

Taking into account the relations (3) and (9), we can easily obtain from (11) that:

$$x \perp \frac{n'_+(x;y)}{\|x\|^2} x - y \text{ for any } x \in X \setminus \{0\},\ y \in Y. \tag{12}$$

Let us denote by $T : X_1 \to X_2$ a linear operator, where X_1 and X_2 are two real linear normed spaces and by $\perp_1, \perp_2$ the orthogonality generated on X_1, respectively X_2.

In the next paragraph we will specify some properties using the directional derivatives of the norms on the spaces X_1, X_2. We denote these derivatives by $n'_{1\pm}$, respectively $n'_{2\pm}$. These properties will prove the property that preserves the orthogonality, namely:

$$\text{if } x\perp_1 y, \text{ then } Tx\perp_2 Ty, \quad x, y \in X_1. \tag{13}$$

If we have two linearly independent vectors $x, y \in X_1$, let us denote by:

$$F_{x,y}(t) = \|Tx + tTy\|_2^2 \cdot \|x + ty\|_1^{-2}, \; t \in \mathbb{R}. \tag{14}$$

Theorem 1. *If $T : X_1 \to X_2$ is a linear operator, then the following properties are equivalent:*

(*i*) *T preserves the orthogonality;*

(*ii*) *if $n'_{1+}(x; y) = 0$, $x, y \in X_1$, then $n'_{2+}(Tx; Ty) \geq 0$;*

(*iii*) *the function $F_{x,y}$ is non-decreasing on $\mathbb{R}$;*

(*iv*) *the function $F_{x,y}$ is constant on $\mathbb{R}$;*

(*v*) *there exist $k > 0$ such that $\|Tx\|_2 = k\|x\|_1$, for all $x \in X_1$ (that is T is an isometry multiplied by a positive constant).*

Proof. (i)→(ii). Let us consider the linear operator T with property (13). If $n'_{1+}(x; y) = 0$, and taking into account the property (11), we obtain that $x\perp_1 y$. Therefore, $Tx\perp_2 Ty$ and so, using (10) we get $n'_{2+}(Tx; Ty) \geq 0$,, that is (ii) is fulfilled.

(ii)→(iii). The function $F_{x,y}$ is right derivable on $\mathbb{R}$ because the norms of the linear normed spaces X_1, X_2 have directional derivatives. More than this, after an usual computation, we obtain:

$$\begin{aligned}(F_{x,y})'_r(t) = [&\|x + ty\|_1^2 \cdot n'_{2+}(Tx + tTy; Ty) \\ &- \|Tx + tTy\|_2^2 \cdot n'_{1+}(x + ty; y)] \cdot \|x + ty\|_1^{-2}.\end{aligned} \tag{15}$$

On the other hand, using (9), we have:

$$n'_{1+}\left(u; \frac{n'_{1+}(u; v)}{\|u\|^2} u - v\right) = 0, \text{ for all } u \in X_1\backslash\{0\}, y \in X_2. \tag{16}$$

Thus, by hypothesis (ii) it follows:

$$n'_{2+}\left(Tu; \frac{n'_{1+}(u;v)}{\|u\|^2}Tu - Tv\right) \geq 0, \tag{17}$$

that is,

$$\frac{n'_{1+}(u;v)}{\|u\|^2}\|Tu\|_2^2 - n'_{2+}(Tu;Tv) \geq 0. \tag{18}$$

Now, taking $u = x + ty$, $v = y$ we obtain that $(F_{x,y})'_r(t) \leq 0$, for all $t \in \mathbb{R}$. This means that $F_{x,y}$ is a decreasing function on $\mathbb{R}$ (see, for example, [6], p.21).

(iii)$\to$(iv). We observe that $F_{x,y}(-t) = F_{x,-y}(t)$ and so $t \to F_{x,y}(-t)$, $t \in \mathbb{R}$ is also a decreasing function. On the other hand, this function is non-decreasing because $t \to F_{x,y}(t)$, $t \in \mathbb{R}$ is decreasing. Finally $F_{x,y}$ is constant on $\mathbb{R}$, that is (iv) is satisfied.

(iv)$\to$(v). As $\lim_{t\to\infty} F_{x,y}(t) = \frac{\|Ty\|_2^2}{\|y\|_1^2}$ and $F_{x,y}(0) = \frac{\|Tx\|_2^2}{\|x\|_1^2}$ we get that

$$\frac{\|Ty\|_2}{\|y\|_1} = \frac{\|Tx\|_2}{\|x\|_1}, \text{ for all } x, y \in X_1 \setminus \{0\} \tag{19}$$

and this means (v) is satisfied. The implication (v)$\to$(i) is obvious. Thus, the proof of theorem is complete.

Corollary 1. *Two orthogonalities on the same real linear normed space X are equivalent if and only if the corresponding norms are proportional.*

Proof. We take $X_1 = X_2 = X$ and T the identical operator.

Remark 1. The statement (ii) is obviously equivalent to the following statement:

(ii′) $n'_-(x,y) = 0$, $x, y \in X_1$, then $n'_-(Tx,Ty) \leq 0$.

Indeed, if $x \perp y$, then $x \perp -y$.

Remark 2. Let us consider the next special orthogonalities:

$$x \perp^s y \text{ if } n'_+(x;y) = 0 \text{ (called Birkhoff \textit{strong orthogonality})}, \tag{20}$$

respectively

$$x \perp^w y \text{ if } n'_+(x;y) \geq 0 \text{ (called Birkhoff \textit{weak orthogonality})}. \tag{21}$$

Obviously, $\perp^s$ is stronger than $\perp$, while $\perp^w$ is weaker than $\perp$. So, taking into account the hypothesis (ii) a linear operator preserves the Birkhoff's orthogonality if and only if satisfies the next weaker condition:

$$Tx \perp_2^w Ty \text{ if } x \perp_1^s y, \quad x, y \in X. \tag{22}$$

Particularly, *if* $\perp_1$ *and* $\perp_2$ *are two orthogonality relations on the same linear real space* X*, then* $\perp_1 \to \perp_2$ *if and only if* $\perp_1^s \to \perp_2^w$.

Other special orthogonalities of Birkhoff type were defined by Chmielisnski in [7], [8]. In these papers, he establishes some results concerning mappings preserving orthogonality.

Remark 3. The proof of Theorem 1 can be considered as a new proof of Blanco and Turnišek's result in the real case. Their proof use some special properties of support functionals.

Now, we consider the following three statements:

(i) *a linear operator* $T : X_1 \to X_2$*, where* X_1, X_2 *are two real linear normed spaces, preserves the orthogonality relation if and only if is an isometry multiplied by a positive constant*;

(ii) *a linear operator* $T : X \to X$*, where* X *is a real linear normed space, preserves the orthogonality relation if and only if is an isometry multiplied by a positive constant*;

(iii) *two orthogonality relations on the same real linear normed space* X*, generated by two norms on* X*, are equivalent if and only if these two norms are proportional.*

Theorem 2. *The statements* (*i*), (*ii*), (*iii*) *are equivalent.*

Proof. Obviously (i)→(ii)→(iii). Now, we suppose the statement (iii) is true and let us consider an arbitrary linear operator $T : X_1 \to X_2$ that preserves the orthogonality relation. Exactly as Blanco and Turnšek proved (see [5], Theorem 3.1), the operator must be injective. Indeed, if we suppose by contrary that exists an element $\widetilde{x} \in X_1$, $\|\widetilde{x}\| = 1$, such that $T\widetilde{x} = 0$, then taking an arbitrary element $y \in X_1$, $\|y\| = 1$, by (1), it follows that the elements $\alpha\widetilde{x} + y$ and $\widetilde{x}$ are not orthogonal, whenever $\alpha > 2$. Now, using (9) and (11) we obtain that the elements $\alpha\widetilde{x} + y$ and $\beta(\alpha\widetilde{x} + \widetilde{y}) + \widetilde{x}$ are orthogonal if $\beta = -n'_+(\alpha\widetilde{x} + y, \widetilde{x})\|\alpha\widetilde{x} + y\|^{-2}$, $\alpha > 2$. Moreover, we have $\beta \neq 0$. Since, by hypothesis, the linear operator T preserves the orthogonality relation, we also have $T(\widetilde{\alpha}\widetilde{x}+y)\perp_2(T\widetilde{x}+\beta T(\widetilde{\alpha}\widetilde{x}+y))$, that is $Ty\perp_2 Ty$. Hence, $Ty = 0$, for all $y \in X$, that is T is the null operator, and so, we arrive at a contradiction with the property (13) of T. Therefore, T is necessarily injective, Thus, if we denote

$$\|x\|_1' = \|Tx\|_2, \qquad x \in X_1,$$

we obtain that $\|\cdot\|_1'$ is also a norm on X_1 and $\perp_1$ implies $\perp_1'$. Using (iii), we get that the norms $\|\cdot\|_1$, $\|\cdot\|_1'$ are proportional and this proves that T is a scalar multiple of a linear isometry, meaning (i) is satisfied, so the proof is finished.

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