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GRONWALL INEQUALITIES VIA PICARD OPERATORS

BY

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Abstract. In this paper we use some abstract Gronwall lemmas to study Volterra integral inequations in higher dimensions and Fredholm integral inequations.

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1. Introduction

Let $(X, \leq)$ be an ordered set and $A : X \to X$ an operator such that the equation

$$x = A(x) \tag{1.1}$$

has a unique solution, x_A^*. The operatorial inequality problem (see Rus [22]) is the following: Find conditions under which: (i) $x \leq A(x) \Longrightarrow x \leq x_A^*$; (ii) $x \geq A(x) \Longrightarrow x \geq x_A^*$.

To have a "concrete" result for this problem it is necessary to determine the solution x_A^* of the equation(1.1) or to find $y, z \in X$ such that $x_A^* \in [y, z]$. In the last case we ask: (i') $x \leq A(x) \Longrightarrow x \leq z$; (ii') $x \geq A(x) \Longrightarrow x \geq y$.

The aim of this paper is to study this problem in the case of integral operators using some abstract Gronwall lemmas.

2. Notions and notations

Throughout this paper we follow the terminology and notations in [21]. For the convenience of the reader we shall recall some of them.

Let X be a nonempty set and $A : X \longrightarrow X$ an operator. Then $A^0 := 1_X,\ A^1 := A,\ A^{n+1} := A \circ A^n,\ n \in \mathbb{N}$ - the iterate operators of the operator A. $F_A := \{x \in X | A(x) = x\}$ - the fixed point set of the operators A. Let $s(X) := \{(x_n)_{n\in N} | x_n \in X, n \in N\}$. Let $c(x) \subset s(x)$ be a subset of $s(x)$ and $Lim : c(X) \longrightarrow X$ an operator. By definition (Fréchet (1905); see [21]) the triple $(X, c(X), Lim)$ is called an L-space if the following conditions are satisfied:

(i) If $x_n = x, \forall n \in N$, then $(x_n)_{n\in N} \in c(X)$ and $Lim(x_n)_{n\in N} = x$;

(ii) If $(x_n) \in c(X)$ and $Lim(x_n)_{n\in N} = x$, then for all subsequences, $(x_{n_i})_{i\in N}$, of $(x_n)_{n\in N}$ we have that $(x_{n_i})_{i\in N} \in c(X)$ and $Lim(x_{n_i})_{i\in N} = x$. By definition an element of $c(X)$ is a convergent sequence and $x := Lim(x_n)_{n\in N}$ is the limit of this sequence; we write $x_n \to x$ as $n \to \infty$.

In what follows we denote an L-space by $(X, \to)$. For some examples of L-spaces see [21].

3. Abstract Gronwall lemmas

Let $(X, \to)$ be an L-space.

Definition 3.1. An operator $A : X \to X$ is a Picard operator (PO) if

(i) $F_A = \{x_A^*\}$

(ii) $A^n(x) \longrightarrow x_A^*$ as $n \longrightarrow \infty, \forall x \in X$.

For POs on ordered L-spaces we have the following results (see Rus [18], [19], [20] and [21]).

Lemma 3.1 (Abstract Gronwall lemma)**.** *Let $(X, \to, \leq)$ be an ordered L-space and $A : X \longrightarrow X$ an operator. We suppose that*

(i) *A is PO;*

(ii) *A is increasing.*

If we denote by x_A^* the unique fixed point of A, then

(a) $x \leq A(x) \Longrightarrow x \leq x_A^*$;

(b) $x \geq A(x) \Longrightarrow x \geq x_A^*$.

Lemma 3.2 (Abstract Gronwall-comparison lemma; [22]). *Let* $(X, \to, \leq)$ *be an ordered L-space and* $A, B : X \longrightarrow X$ *two operators. We suppose that:*

(i) *A is increasing;*

(ii) *A and B are POs;*

(iii) $A \leq B$.

Then $x \leq A(x) \Longrightarrow x \leq x_B^*$.

Lemma 3.3 ([22]). *Let* $(X, \to)$ *be an L-space and* $A, B : X \longrightarrow X$ *two operators. We suppose that*

(i) *A and B are increasing operators;*

(ii) *A and B are POs;*

(iii) $x = A(x) \Longrightarrow x \leq B(x)$.

Then $x \leq A(x) \Longrightarrow x \leq x_B^*$.

In what follows we shall apply the above lemmas to integral inequalities. For other applications see Buica [3], Craciun [6], Lungu [11], Lungu and Rus [12], Rus [18]-[21]. For other integral inequalities see Agarwal and Thandapani [1], Bainov and Simeonov [2], Corduneanu [4], Corduneanu [5], Dragomir and Ionescu [7], Flett [8], Lakshmikantham and Leela [10], Lakshmikantham, Leela and Martynyuk [11], Mitrinović, Pečarić and Fink [14], Pachpatte [15], [16], Polynian and Manzhirov [17], Ver Eecke [23].

4. Volterra integral inequations in higher dimensions

4.1. Volterra integral equations

In what follows we consider the following integral equation

$$\begin{aligned} u(x_1, x_2, \ldots, x_n) &= \alpha + \int_0^{x_1} K_1(s_1, x_2, \ldots, x_m) u(s_1, x_2, \ldots, x_m) ds_1 \\ &+ \int_0^{x_1}\int_0^{x_2} K_2(s_1, s_2, x_3, \ldots, x_m) u(s_1, s_2, x_3, \ldots, x_m) ds_1 ds_2 + \ldots \\ &+ \int_0^{x_1} \ldots \int_0^{x_m} K_m(s_1, \ldots, s_m) u(s_1, \ldots, s_m) ds_1 \ldots ds_m, \end{aligned} \tag{4.1}$$

where $\alpha > 0, a_i > 0$, $i = \overline{1,m}$, $D = \prod_{i=1}^{m}[0,a_i]$, $K_i \in C(D)$, $i = \overline{1,m}$ and $M_{k_i} > 0$ is such that $|K_i(x)| \leq M_{k_i}$, $\forall x \in D, i = \overline{1,m}$.

Let $A : C(D) \longrightarrow C(D)$ be the operator defined by $A(u)(x_1,\ldots,x_m) :=$ second part of (4.1). If we consider the Banach space $(C(D), \|\cdot\|_B)$ where $\|\cdot\|$ is a Bielecki norm $\|u\|_B := \max\limits_{x\in D}(|u(x)|e^{-\tau x_1}), \tau > 0$; then the operator A is Lipschitz with the constant $L_A = \frac{1}{\tau}(M_{k_1} + M_{k_2}a_2 + \cdots + M_{k_m}a_2 \ldots a_m)$. Thus A is a contraction with respect to $\|\cdot\|_B$, for $\tau > 0$ suitable chosen. So, A is PO in $(C(D), \xrightarrow{\|\cdot\|_B})$. From the above considerations we have

Theorem 4.1. *If $K_i \in C(D), i = \overline{1,m}$, then the equation* (4.1) *has in $C(D)$ a unique solution u^*, and $A^n(u)$ converge uniformly to u^* as $n \longrightarrow \infty$, for all $u \in C(D)$.*

Theorem 4.2. *We suppose that $\alpha > 0, K_i \in C(D,\mathbb{R}_+), i = \overline{1,m}$. Then*

(*a*) *$u^*(x) > 0, \forall x \in D$;*

(*b*) *If $K_i(x_1,\ldots,x_m)$ is increasing with respect to $x_{i+1},\ldots,x_m, i = \overline{1,m-1}$, then u^* is increasing.*

Proof. (a) Let $u_0 \in C(D,\mathbb{R}_+)$ and $u_n := A^n(u_0), n \in \mathbb{N}^*$. Then from (4.1) we have that $u_n(x) \geq \alpha > 0, \forall x \in D$ and u_n is increasing. By Theorem 4.1, $(u_n)_{n\in\mathbb{N}}$ converges uniformly to $u^*(u_n \xrightarrow{unif} u^*)$. So, we have (a).

(b) Follows from (a) and from (4.1). □

4.2. Lower solutions of (4.1)

Now we consider the inequation

$$u \leq A(u). \tag{4.2}$$

By definition a solution of (4.2) is a lower solution of (4.1). We have

Theorem 4.3. *We suppose that the conditions of Theorem* 4.2 *are satisfied. If $u \in C(D,\mathbb{R}_+)$ is a lower solution of* (4.1), *then*

$$u(x_1,\ldots,x_m) \leq \alpha \exp\Big(\int_0^{x_1} K_1(s_1,x_2,\ldots,x_m)ds_1 + \cdots + \int_0^{x_1}\ldots\int_0^{x_m} K_m(s_1,\ldots,s_m)ds_1\ldots ds_m\Big).$$

Proof. First, we remark that under the conditions of Theorem 4.3 the operator A is increasing. Let $u \in C(D, \mathbb{R}_+)$ be a lower solution of (4.1). Then by Lemma 3.1, $u \leq u^*$. From the Theorem 4.2 we have that

$$\begin{aligned} u^*(s_1, s_2, x_3, \ldots, x_m) &\leq u^*(s_1, x_2, \ldots, x_m), \ldots, u^*(s_1, \ldots, s_m) \\ &\leq u^*(s_1, x_2, \ldots, x_m), \end{aligned}$$

for all $s, x \in D, s \leq x$. Hence we have

$$\begin{aligned} u^*(x) = A(u^*)(x) \leq \alpha + \int_0^{x_1} K_1(s_1, x_2, \ldots, x_m) u(s_1, x_2, \ldots, x_m) ds_1 \\ + \int_0^{x_1}\int_0^{x_2} K_2(s_1, s_2, x_3, \ldots, x_m) u(s_1, x_2, \ldots, x_m) ds_1 ds_2 + \ldots \\ + \int_0^{x_1} \ldots \int_0^{x_m} K_m(s_1, \ldots, s_m) u(s_1, x_2, \ldots, x_m) ds_1 \ldots ds_m. \end{aligned} \tag{4.3}$$

Consider the operator $B : C(D) \longrightarrow C(D)$ defined by $B(u)(x) :=$ last part of (4.3). It is clear that the operator B is PO on $(C(D), \overset{unif.}{\longrightarrow})$ and is increasing. Let u_B^* be the unique fixed point of B. Thus we have (see 4.3), $u^* = A(u^*) \leq B(u^*)$. From Lemma 3.3, we have that $u^* \leq u_B^*$. From the definition of B we have that

$$\begin{aligned} \frac{\partial u_B^*(x)}{\partial x_1} = \Big[K_1(x) + \int_0^{x_2} K_2(x_1, s_2, x_3, \ldots, x_m) ds_2 + \cdots + \\ \int_0^{x_2} \ldots \int_0^{x_m} K_m(x_1, s_2, \ldots, s_m) ds_2 \ldots ds_m\Big] u_B^*(x) \end{aligned} \tag{4.4}$$

and $u^*(0, x_2, \ldots, x_m) = \alpha$. From (4.4) we have the Theorem 4.3. □

Remark 4.1. For $m = 2$ we have a result of Kim [9].

Remark 4.2. Theorem 4.3 remains true if (4.1) is replaced by

$$\begin{aligned} u(x_1, x_2, \ldots, x_m) = \alpha + \int_0^{x_1} K_1(s_1, x_2, \ldots, x_m) f_1(u(s_1, x_2, \ldots, x_m)) ds_1 + \ldots \\ + \int_0^{x_1} \ldots \int_0^{x_m} K_m(s_1, \ldots, s_m) f_m(u(s_1, \ldots, s_m)) ds_1 \ldots ds_m, \end{aligned}$$

where $f_i(u) \leq u$, f_i increasing and Lipschitz.

Remark 4.3. As in [8] and [13] we can use the above results to study the solutions and lower solutions of Darboux problem $\frac{\partial^m u}{\partial x_1 \ldots \partial x_m} = f(x, u(x))$, $u(x_1, \ldots, x_{m-1}, 0) = \varphi_1(x_1, \ldots, x_{m-1}), \ldots, u(0, x_2, \ldots, x_m) = \varphi_m(x_2, \ldots, x_m)$, where $f \in C(D \times \mathbb{R})$, $f(x, \cdot) : \mathbb{R} \longrightarrow \mathbb{R}$ is L_f-Lipschitz and increasing and φ_i, $i = \overline{1, m}$ are continuous.

5. Volterra-Fredholm integral inequations

5.1. Volterra-Fredholm integral equations

In what follows we consider the following integral equation

$$u(x_1, x_2) = \alpha + \int_0^{x_1} \int_0^b F(x_1, x_2, s_1, s_2, u(s_1, s_2)) ds_1 ds_2 \tag{5.1}$$

where $F \in C(D \times D \times \mathbb{R})$, $D = [0, a] \times [0, b], a > 0, b > 0$.

Let us consider the operator $A : C(D) \to C(D)$ defined by $A(u)(x_1, x_2) :=$ second part of (5.1). We consider the Banach space $(C(D), \|\cdot\|_B)$ where $\|\cdot\|_B$ is a Bielecki norm $\|\cdot\|_B := \max_{x \in D}\{|u(x_1, x_2)| e^{-\tau x_1}\}$, $\tau > 0$. If we suppose that $F(x, s, \cdot) : \mathbb{R} \longrightarrow \mathbb{R}$ is Lipschitz with a constant L, for all $(x, s) \in D \times D$, then the operator A is Lipschitz with the constant $L_A = \frac{bL}{\tau}$. Thus A is a contraction with respect to $\|\cdot\|_B$, for $\tau > L \cdot b$. So, A is PO in $(C(D), \stackrel{unif.}{\longrightarrow})$.

From the above considerations we have

Theorem 5.1. *We suppose that*

(i) $F \in C(D \times D \times \mathbb{R})$,

(ii) there exists $L > 0$ *such that* $|F(x, s, v) - F(x, s, w)| \leq L|v - w|$, *for all* $x, s \in D, v, w \in \mathbb{R}$.

Then the equation (5.1) *has in* $C(D)$ *a unique solution* u^* *and* $A^n(u) \stackrel{unif.}{\longrightarrow} u^*$ *as* $n \longrightarrow \infty$, *for all* $u \in C(D)$.

In addition we suppose that

(iii) $F(x, s, \cdot) : \mathbb{R} \longrightarrow \mathbb{R}$ is increasing, for all $x, s \in D$,

(iv) there exists $K \in C(D \times D, \mathbb{R}_+)$ such that $F(x, s, v) \leq K(x, s)v$, for all $x, s \in D, v \in \mathbb{R}$.

We consider the equation

$$u(x_1,x_2)=\alpha+\int_0^{x_1}\int_0^{b}K(x_1,x_2,s_1,s_2)u(s_1,s_2)ds_1ds_2. \tag{5.2}$$

Let $B:C(D)\longrightarrow C(D)$ be defined by $B(u)(x_1,x_2):=$ second part of (5.2).

It is clear that the operator B is PO on $(C(D),\overset{unif.}{\longrightarrow})$.

Let u_B^* the unique fixed point of B. Let $u\in C(D)$ be a lower solution of (5.1). Then by Lemma 3.1, $u\le u^*$. Moreover we have $u^*=A(u^*)\le B(u^*)$.

From Lemma 3.3, we have that $u^*\le u_B^*$. So, we have

Theorem 5.2. *We suppose that the conditions* (i)-(iv) *are satisfied. If* $u\in C(D)$ *is a lower solution of* (5.1)*, then*

$$u(x_1,x_2)\le\alpha+\alpha\int_0^{x_1}\int_0^{b}\Gamma(x_1,x_2,s_1,s_2)ds_1ds_2,$$

where Γ *is the resolvent kernel of the equation* (5.2)*.*

Example 5.1. We consider the following integral equation

$$u(x_1,x_2)=\alpha+\int_0^{x_1}\int_0^{b}e^{x_1-x_2-s_1+s_2}u^p(s_1,s_2))ds_1ds_2,\ \alpha>1 \tag{5.3}$$

Evidently,

$$u(x_1,x_2)\le\alpha+\int_0^{x_1}\int_0^{b}e^{x_1-x_2-s_1+s_2}u(s_1,s_2)ds_1ds_2 \tag{5.4}$$

and the operators A and B are:

$$A:=\alpha+\int_0^{x_1}\int_0^{b}(e^{x_1-x_2-s_1+s_2},u^p(s_1,s_2))ds_1ds_2, \tag{5.5}$$

$$B:=\alpha+\int_0^{x_1}\int_0^{b}e^{x_1-x_2-s_1+s_2}u(s_1,s_2)ds_1ds_2. \tag{5.6}$$

Let $u\in C(D)$ be a lower solution of (5.3), then $u\le u^*$, where u^* is the unique solution of (5.3). If u_B^* is the unique fixed point of B, then $u^*\le u_B^*$.

But, u_B^* can be obtained by successive approximation. So,

$$K_1(x_1,x_2,s_1,s_2) = K(x_1,x_2,s_1,s_2) = e^{x_1-x_2-s_1+s_2}$$
$$K_2(x_1,x_2,s_1,s_2) = \int_{s_1}^{x_1}\int_0^b e^{x_1-x_2-z_1+z_2}\cdot e^{z_1-z_2-s_1+s_2}dz_1dz_2$$
$$= b\frac{(x_1-s_1)}{1!}e^{x_1-x_2-s_1+s_2}$$
$$K_3(x_1,x_2,s_1,s_2) = b^2\frac{(x_1-s_1)^2}{2!}e^{x_1-x_2-s_1+s_2}$$
....
$$K_n(x_1,x_2,s_1,s_2) = \int_{s_1}^{x_1}\int_0^b K(x_1,x_2,z_1,z_2)K_{n-1}(z_1,z_2,s_1,s_2)dz_1dz_2$$
$$K_n(x_1,x_2,s_1,s_2) = b^{n-1}\frac{(x_1-s_1)^{n-1}}{(n-1)!}e^{x_1-x_2-s_1+s_2}.$$

In this case the resolvent kernel is

$$\Gamma(x_1,x_2,s_1,s_2) = \sum_{n=0}^{\infty} b^n\frac{(x_1-s_1)^n}{n!}e^{x_1-x_2-s_1+s_2} = e^{(b+1)(x_1-s_1)-s_1+s_2}.$$

Then $u_B^*(x_1,x_2) = \alpha + \frac{\alpha}{b+1}(e^{b-x_2}-e^{-x_2})(e^{(b+1)x_1}-1)$ and $u^*(x_1,x_2) \leq \alpha + \frac{\alpha}{b+1}(e^{b-x_2}-e^{-x_2})(e^{(b+1)x_1}-1)$.

Remark 5.1. Let X be an ordered Banach space, $\Omega \subset \mathbb{R}^m$ a bounded domain, $a>0$, $g \in C(D,X), F \in C(D\times D\times X, X)$, where $D := [0,a)\times\bar{\Omega}$. We consider the following integral equation

$$u(t,x) = g(t,x) + \int_0^t\int_\Omega F(t,x,\xi,s)d\xi ds. \tag{5.7}$$

In a similar way we have

Theorem 5.3. *We suppose that*

(*i*) *There exist* $L>0$ *such that* $\|F(t,x,\xi,s,v) - F(t,x,\xi,s,w)\|_X \leq L\|v-w\|_X$, *for all* $t,\xi\in[0,\alpha], x,s\in\bar{\Omega},\ v,w\in X$*;*

(*ii*) $F(t,x,\xi,s,\cdot): X \longrightarrow X$ *is increasing, for all* $t,\xi\in[0,a], x,s\in\bar{\Omega}$*;*

(*iii*) *there exists a continuous and linear positive operator* $K(t,x,\xi,s) : X \longrightarrow X$ *for each* $t,\xi \in [0,a), x,s \in \bar{\Omega}$*, such that* $F(t,x,\xi,s,v) \leq K(t,x,\xi,s)v$*, for all* $t,\xi \in [0,a],\ x,s \in \bar{\Omega}$.

Then the equation (5.7) has in $C(D,X)$ a unique solution and if $u \in C(D,X)$ is a lower solution of (5.7), then

$$u(t,x) \leq g(t,x) + \int_0^t \int_\Omega \Gamma(t,x,\xi,s)g(\xi,s)d\xi ds,$$

where Γ is the resolvent kernel of the equation

$$u(t,x) = g(t,x) + \int_0^t \int_\Omega K(t,x,\xi,s)g(\xi,s)d\xi ds.$$

REFERENCES

1. Agarwal, R.P.; Thandapani, E. – *Some inequalities of Gronwall type*, An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi, Mat., 28 (1982), 71–75.
2. Baĭnov, D.; Simeonov, P. – *Integral Inequalities and Applications*, Mathematics and its Applications, 57, Kluwer Academic Publishers Group, Dordrecht, 1992.
3. Buică, A. – *Gronwall-type nonlinear integral inequalities*, Mathematica, 44 (2002), 19–23.
4. Corduneanu, A. – *O inegalitate de tip Gronwall în două variabile independente,* Seminarul Itinerant de Ecuaţii Funcţionale, aproximare şi convexitate, Cluj-Napoca, 1982, 63–66.
5. Corduneanu, C. – *Integral Equations and Applications*, Cambridge University Press, Cambridge, 1991.
6. Crăciun, C. – *On some Gronwall inequalities*, Semin. Fixed Point Theory Cluj-Napoca, 1 (2000), 31–33.
7. Dragomir, S.S.; Ionescu, N.M. – *On nonlinear integral inequalities of two independent variables*, Studia Univ. Babeş-Bolyai Math., 34 (1989), 11–17.
8. Flett, T.M. – *Differential Analysis*, Differentiation, differential equations and differential inequalities, Cambridge University Press, Cambridge-New York, 1980.
9. Kim, B.-I. – *On some Gronwall type integral inequalities and their applications*, Bull. Korean Math. Soc., 41 (2004), 559–571.

10. LAKSHMIKANTHAM, V.; LEELA, S. – *Differential and Integral Inequalities: Theory and Applications*, Academic Press, New York-London, 1969.

11. LAKSHMIKANTHAM, V.; LEELA, S.; MARTYNYUK, A.A. – *Stability Analysis of Nonlinear Systems*, Monographs and Textbooks in Pure and Applied Mathematics, 125, Marcel Dekker, Inc., New York, 1989.

12. LUNGU, N. – *On some Gronwall-Bihari-Wendorff-type inequalities*, International Conference on Nonlinear Operators, Differential Equations and Applications (Cluj-Napoca, 2001), Semin. Fixed Point Theory Cluj-Napoca, 3 (2002), 249–254.

13. LUNGU, N.; RUS, I.A. – *Hyperbolic differential inequalities,* Libertas Math., 21(2001), 35–40.

14. MITRINOVIĆ, D.S.; PEČARIĆ, J.E.; FINK, A.M. – *Inequalities Involving Functions and Their Integrals and Derivatives*, Mathematics and its Applications (East European Series), 53, Kluwer Academic Publishers Group, Dordrecht, 1991

15. PACHPATTE, B.G. – *On some new inequalities related to certain inequalities in the theory of differential equations*, J. Math. Anal. Appl., 189 (1995), 128–144.

16. PACHPATTE, B.G. – *Inequalities for Differential and Integral Equations*, Mathematics in Science and Engineering, 197, Academic Press, Inc., San Diego, CA, 1998.

17. POLYANIN, A.D.; MANZHIROV, A.V. – *Handbook of Integral Equations*, CRC Press, Boca Raton, FL, 1998.

18. RUS, I.A. – *Weakly Picard mappings*, Comment. Math. Univ. Carolin., 34 (1993), 769–773.

19. RUS, I.A. – *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, 2001.

20. RUS, I.A. – *Weakly Picard operators and applications*, Semin. Fixed Point Theory Cluj-Napoca, 2 (2001), 41–57.

21. RUS, I.A. – *Picard operators and applications*, Sci. Math. Jpn., 58 (2003), 191–219.

22. RUS, I.A. – *Abstract Gronwall lemmas and applications,* Carpathian J. of Math., 20(2004), 125–134.

23. VER EECKE, P. – *Applications du Calcul Différentiel*, Mathématiques, Presses Universitaires de France, Paris, 1985.

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