

**INTERNAL STABILIZABLE FEEDBACK CONTROLLER  
FOR A FINITE SET OF EQUILIBRIUM SOLUTIONS TO  
THE NAVIER-STOKES EQUATIONS**

BY

IONUȚ MUNTEANU

**Abstract.** We design here a finite-dimensional feedback controller, with support in an arbitrary open set, that locally exponentially stabilizes a finite set of steady-state solutions of the Navier-Stokes equations with no-slip boundary conditions.

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### 1. Preliminaries

The mathematical theory of stabilization of equilibrium solutions to Newtonian fluid flows has been developed as a principal tool to eliminate or attenuate the turbulence. There are lots of works in this field, but in this paper we only refer to the important result obtained in BARBU and TRIGGIANI [1]. In [1] it is proven that the equilibrium solutions to Navier-Stokes equations are exponentially stabilizable by finite-dimensional feedback controllers distributed in spatial domain, which can be taken arbitrarily small and has compact support in the interior of the domain. One must emphasize also that, unlike most stabilizing procedures proposed in literature, the method from [1] works for large Reynolds numbers and no restriction is imposed on the viscosity coefficient  $\nu$ .

**The controlled Navier-Stokes equations.** Consider the controlled Navier-Stokes equations with the non-slip Dirichlet boundary conditions:

$$\begin{aligned}
 & y_t(x, t) - \nu \Delta y(x, t) + (y \cdot \nabla) y(x, t) \\
 & = m(x)u(x, t) + f_e(x) + \nabla p(x, t), \text{ in } Q = \Omega \times (0, \infty), \\
 (1.1) \quad & \nabla \cdot y = 0, \text{ in } Q, \\
 & y = 0, \text{ on } \Sigma = \partial\Omega \times (0, \infty), \\
 & y(x, 0) = y_0(x), \text{ in } \Omega.
 \end{aligned}$$

Here  $\Omega$  is an smooth bounded domain of  $\mathbb{R}^d$ ,  $d = 2, 3$ ;  $m$  is the characteristic function of an open smooth subset  $\omega \subset \Omega$  of positive measure;  $u$  is the control input; and  $y = (y_1, y_2, \dots, y_d)$  is the state (velocity) of the system. The functions  $y_0, f_e \in (L^2(\Omega))^d$  are given, the latter being a body force, while  $p$  is the unknown pressure. The constant  $\nu > 0$  is the viscosity coefficient.

Since the forces are independent of time, we are looking for time independent solutions (steady-state solutions or equilibrium solutions) of the uncontrolled case of the Navier-Stokes equations (1.1), i.e. a function  $y_e = y_e(x)$  (and a function  $p_e = p_e(x)$ ) which satisfies

$$\begin{aligned}
 (1.2) \quad & -\nu \Delta y_e + (y_e \cdot \nabla) y_e = \nabla p_e + f_e, \text{ in } \Omega, \\
 & \nabla \cdot y_e = 0, \text{ in } \Omega, \\
 & y_e = 0, \text{ on } \partial\Omega.
 \end{aligned}$$

The steady-state solution is known to exist for  $d = 2, 3$  (see [2], Theorem 10.1, p.67). Here (see [2], p.8),  $V = \{y \in (H_0^1(\Omega))^d; \nabla \cdot y = 0\}$ , with norm  $\|y\|_V = \|y\| = \{\int_{\Omega} |\nabla y(x)|^2 d\Omega\}^{\frac{1}{2}}$ . And  $H = \{y \in (L^2(\Omega))^d; \nabla \cdot y = 0, y \cdot n = 0 \text{ on } \partial\Omega\}$ , endowed with the same norm as  $(L^2(\Omega))^d$ .

We shall denote by  $P : (L^2(\Omega))^d \rightarrow H$  the orthogonal Leray projector (see [2]). Equation (1.1) can be equivalently rewritten in abstract form as

$$(1.3) \quad \frac{dy}{dt} + \nu Ay + By = P(mu + f_e); y(0) = y_0 \in H,$$

since the procedure of applying  $P$  to (1.1) eliminates the pressure from the equations. Here  $Ay = -P\Delta y, \forall y \in \mathcal{D}(A) = (H^2(\Omega))^d \cap V, V = \mathcal{D}(A^{\frac{1}{2}})$ , is a self-adjoint positive definite operator in  $H$  with compact (resolvent)  $A^{-1}$  on  $H$ . Accordingly, the fractional powers  $A^s, 0 < s < 1$ , are well-defined

(see [2], p.9). And  $B : V \rightarrow V'$  is defined as  $By = P[(y \cdot \nabla)y]$ ,  $(By, w) = b(y, y, w)$ ,  $\forall y, w \in V$ , where the trilinear form is defined by  $b(y, z, w) = \int_{\Omega} < y \cdot \nabla z, w >_{\mathbb{R}^d} d\Omega$ ,  $y, w \in H, z \in V$ .

We shall denote by  $(\cdot, \cdot)$  the scalar product in both  $H$  and  $(L^2(\Omega))^d$ . Similarly, we shall denote by the same symbol  $|\cdot|$  the norm of both  $(L^2(\Omega))^d$  and  $H$ .

The problem addressed in [1] is the stabilization of the steady-state solution  $y_e$  by a feedback controller with support in an arbitrary open subset  $\omega \subset \Omega$ . In fact, it is well known that for large Reynolds numbers  $Re = \frac{1}{\nu}$ , such a solution is unstable and so its stabilization is a major problem of fluid dynamics. To this end, we have (see Theorem 2.2, [1]) the next theorem.

**Theorem 1.1.** *There is a finite-dimensional feedback controller  $u = u(x, t)$  of the form*

$$(1.4) \quad u = - \sum_{i=1}^M (R(y - y_e), \psi_i)_{\omega} \psi_i,$$

where  $R : \mathcal{D}(R) \subset H \rightarrow H$  is a certain self-adjoint operator and  $\{\psi_i\}_{i=1}^M$  is a given system of functions, such that once inserted in the N-S system (1.3), exponentially stabilizes the steady-state solution  $y_e$  to (1.1) in a neighborhood

$$(1.5) \quad U_{\rho} = \{y_0 \in \mathcal{D}(A); |A^{\frac{1}{4}}(y_0 - y_e)| < \rho\}$$

of  $y_e$ , for suitable  $\rho > 0$ . More precisely, if  $\rho > 0$  is sufficiently small, then for each  $y_0 \in U_{\rho}$  there exists a weak solution  $y \in L^{\infty}(0, T; H) \cap L^2(0, T; V)$ ,  $\frac{dy}{dt} \in L^{\frac{4}{3}}(0, T; V')$ , for  $d = 3$ , and  $\frac{dy}{dt} \in L^2(0, T; V')$  for  $d = 2$ ,  $\forall T > 0$ , to the closed-loop system

$$(1.6) \quad \frac{dy}{dt} + \nu Ay + By + P \left( m \sum_{i=1}^M (R(y - y_e), \psi_i)_{\omega} \psi_i \right) = Pf_e, t \geq 0, y(0) = y_0,$$

such that the following property holds  $|A^{\frac{1}{4}}(y(t) - y_e)| \leq Ce^{-\gamma t} |A^{\frac{1}{4}}(y_0 - y_e)|$ ,  $t \geq 0$ , for some  $C, \gamma > 0$ .

Concerning the number of the steady-state solutions of (1.3) (equivalently (1.1)), we have due ([2], Theorem 10.4), the next result.

**Theorem 1.2.** *For  $d = 2, 3$ , there exists a dense open set  $\mathcal{O}_{\nu} \subset H$  such that for every  $f_e \in \mathcal{O}_{\nu}$ , the set of solutions of (1.2) is finite and odd in number (the dense set  $\mathcal{O}_{\nu}$  depends on the viscosity coefficient  $\nu$ ).*

Hence, by Theorem 1.2, we have that for each  $f_e \in \mathcal{O}_\nu$ , there exists a finite number of steady-state solutions  $\{y_e^1, y_e^2, \dots, y_e^N\}$  of (1.1). By Theorem 1.1, we have that for each stationary solution  $y_e^i, i = 1, \dots, N$ , there exists a feedback controller, denoted here by  $\Phi_i = \Phi_i(y), i = 1, \dots, N$ , of the form

$$(1.7) \quad \Phi_i = - \sum_{l=1}^{M_i} (R_i(y - y_e^i), \psi_l^i) \omega \psi_l^i,$$

where  $R_i : \mathcal{D}(R_i) \subset H \rightarrow H, i = 1, \dots, N$  are certain self-adjoint operators and  $\{\psi_l^i\}_{l=1}^{M_i}, i = 1, \dots, N$  are given systems of functions (given by Theorem 1.1, corresponding to the steady-state solution  $y_e^i, i = 1, \dots, N$ ), such that the solution to the closed-loop system

$$(1.8) \quad \frac{dy}{dt} + \nu Ay + By = P(m\Phi_i(y)) + Pf_e, \quad t \geq 0, y(0) = y_0,$$

satisfies

$$(1.9) \quad |y(t) - y_e^i|_{\frac{1}{2}} \leq C_i e^{-\gamma_i t} |y_0 - y_e^i|_{\frac{1}{2}}, \quad t \geq 0,$$

whenever  $y_0 \in U_{\rho_i}$  ( $U_{\rho_i}$  given by (1.5)), for some  $\rho_i, C_i, \gamma_i > 0$ , for all  $i = 1, \dots, N$ . Here, we denoted by  $|y|_{\frac{1}{2}} = |A^{\frac{1}{4}}y|, \forall y \in \mathcal{D}(A^{\frac{1}{4}})$ .

Let us consider the sets

$$(1.10) \quad \mathcal{U}_i = \left\{ y_0 \in H; |y_0 - y_e^i|_{\frac{1}{2}} < \frac{\rho_i}{C_i} \right\}, \quad i = 1, \dots, N.$$

Note that we can assume  $C_i$  big enough such that  $\frac{\rho_i}{C_i} < \rho_i$ , hence  $\mathcal{U}_i \subset U_{\rho_i}, \forall i = 1, \dots, N$ . Further, observe that, for  $i \neq j, i, j = 1, \dots, N$  we can assume that  $U_{\rho_i} \cap U_{\rho_j} = \phi$ . Moreover, for  $\epsilon > 0$  small enough, we can assume that

$$(1.11) \quad \left\{ y; |y - y_e^i|_{\frac{1}{2}} < (1 + \epsilon)\rho_i \right\} \cap \left\{ y; |y - y_e^j|_{\frac{1}{2}} < (1 + \epsilon)\rho_j \right\} = \phi,$$

$\forall j \neq i, i, j = 1, \dots, N$  (otherwise we can take smaller  $\rho_i, i = 1, \dots, N$ ).

For an fixed  $\epsilon > 0$  such that relation (1.11) holds true, we introduce the map:  $w : \mathbb{R}_+ \rightarrow [0, 1]$ , defined as

$$w(r) = \begin{cases} 1, & 0 \leq r \leq 1, \\ 0, & r \geq 1 + \epsilon, \\ \text{smooth}, & 1 < r < 1 + \epsilon. \end{cases}$$

Using the map  $w$ , introduced above, we define the functions  $\chi_i : \mathcal{D}(A^{\frac{1}{4}}) \rightarrow [0, 1]$ , as

$$(1.12) \quad \chi_i(y) = w\left(\frac{|y|_{\frac{1}{2}}}{\rho_i}\right), \forall y \in \mathcal{D}(A^{\frac{1}{4}}), i = 1, \dots, N.$$

**Remark 1.1.** Let  $y \in U_{\rho_i}$ , for some  $i = 1, \dots, N$ , we have that

$$\chi_j(y - y_e^j) = \begin{cases} 1, & \text{if } j = i \\ 0, & \text{if } j \neq i. \end{cases}$$

Indeed, let us consider  $y \in U_{\rho_i}$ , for some  $i = 1, \dots, N$ . Hence,  $y \in \{z; |z - y_e^i|_{\frac{1}{2}} < (1 + \epsilon)\rho_i\}$ . Thus, by relation (1.11),  $y \in \{z; |z - y_e^j|_{\frac{1}{2}} \geq (1 + \epsilon)\rho_j\}$ ,  $\forall j \neq i, j = 1, \dots, N$ . It follows immediately from the definition of the functions  $\chi_j, j = 1, \dots, N$  (see (1.12)) that our claim holds true.

The main result in this work (see Theorem 2.1 below) is that the feedback controller  $\Phi(y) = \sum_{i=1}^N \chi_i(y - y_e^i)\Phi_i(y)$  exponentially stabilizes the finite set of the stationary solutions (and not only one steady-state solution) of (1.1). With other words, we show that the same controller  $\Phi$ , defined above, can be used in order to exponentially stabilize any steady-state solution of (1.1). Usually, one considers a steady-state solution and design a feedback controller in order to stabilize it. But when we change the equilibrium solution, the feedback controller constructed does not work anymore, hence another feedback controller must be designed. Here, we prove that the same controller  $\Phi$  works for any steady-state solution.

## 2. Main results

Remember the controlled Navier-Stokes equation (1.3)

$$(2.1) \quad \frac{dy}{dt}(t) + \nu Ay(t) + By(t) = P(mu + f_e), \quad y(0) = y_0.$$

Consider a steady-state solution  $y_e^i$  to (2.1), for some  $i = 1, \dots, N$ . We saw, by Theorem 1.1, that there exists a finite-dimensional feedback controller such that, once inserted into equation (2.1), for every initial data  $y_0$  in a certain set ( $U_{\rho_i}$  more precisely), the solution of the corresponding closed-loop system (2.1) goes exponentially asymptotic to  $y_e^i$ .

Next, we want to construct a finite-dimensional feedback controller,  $\Phi(y)$ , which exponentially stabilizes the system (2.1); more precisely if the

initial data  $y_0$  is sufficiently "close" by any steady-state  $y_e^i$ , then, for the same controller  $\Phi(y)$  introduced into system (2.1), the solution of the corresponding closed-loop system (2.1) goes exponentially asymptotic to  $y_e^i$ .

To this end, using the functions  $\chi_i, i = 1, \dots, N$  given by (1.12) and the finite-dimensional feedback controllers  $\Phi_i, i = 1, \dots, N$  given by (1.7), we construct the next finite-dimensional feedback controller

$$(2.2) \quad \Phi(y) = \sum_{i=1}^N \chi_i(y - y_e^i) \Phi_i(y).$$

Next, we shall prove that, once inserted that feedback controller  $\Phi$  into system (2.1), exponentially stabilizes any steady-state solution  $y_e^i$  to (1.1) in the open set  $\mathcal{U}_i$  given by (1.10) for all  $i = 1, \dots, N$ ; more precisely we have the next theorem.

**Theorem 2.1.** *Let  $f_e \in \mathcal{O}_\nu$ . Once plugged the finite-dimensional feedback controller  $\Phi$  given by (2.2) in the  $N$ - $S$  system (2.1), it exponentially stabilizes any steady-state solution  $y_e^i, i = 1, \dots, N$  to (1.1) in the neighborhood  $\mathcal{U}_i, i = 1, \dots, N$ , given by (1.10), of  $y_e^i$ . More precisely, for each  $y_0 \in \mathcal{U}_i, i = 1, \dots, N$  there exists a weak solution  $y \in L^\infty(0, T; H) \cap L^2(0, T; V), \frac{dy}{dt} \in L^{\frac{4}{3}}(0, T; V')$ , for  $d = 3$ , and  $\frac{dy}{dt} \in L^2(0, T; V')$  for  $d = 2, \forall T > 0$ , to the closed-loop system*

$$(2.3) \quad \frac{dy}{dt} + \nu Ay + By = P \left( m \sum_{j=1}^N \chi_j(y - y_e^j) \Phi_j(y) \right) + Pf_e, \quad t \geq 0, y(0) = y_0,$$

such that the following property holds  $|y(t) - y_e^i|_{\frac{1}{2}} \leq C_i e^{-\gamma_i t} |y_0 - y_e^i|_{\frac{1}{2}}, t \geq 0$ , for  $C_i, \gamma_i > 0$  given by (1.9). Here  $\mathcal{O}_\nu$  is given by Theorem 1.2.

**Proof.** Let  $y_0 \in \mathcal{U}_i$ , for some  $i \in \{1, \dots, N\}$ .

In order to prove the theorem we consider the next delay equation for  $\lambda > 0$

$$(2.4) \quad \begin{aligned} & \frac{dy_\lambda}{dt}(t) + \nu Ay_\lambda(t) + By_\lambda(t) \\ & = P \left( m \sum_{j=1}^N \chi_j(y_\lambda(t - \lambda) - y_e^j) \Phi_j(y_\lambda(t)) \right) + Pf_e, \\ & t > 0, y_\lambda(t) = y_0, t \in [-\lambda, 0]. \end{aligned}$$

For  $t \in [0, \lambda]$ ,  $y_\lambda(t - \lambda) = y_0$ . Since  $y_0 \in \mathcal{U}_i$ , we have that  $|y_\lambda(t - \lambda) - y_e^i|_{\frac{1}{2}} < \frac{\rho_i}{C_i} < \rho_i, \forall t \in [0, \lambda]$  (see (1.10)). Thus  $y_\lambda(t - \lambda) \in U_{\rho_i}, \forall t \in [0, \lambda]$ . It follows then, by Remark 1.1, that

$$\chi_j(y_\lambda(t - \lambda) - y_e^j) = \begin{cases} 1, & \text{if } j = i \\ 0, & \text{if } j \neq i, \end{cases}$$

for all  $t \in [0, \lambda]$ . So, on  $[0, \lambda]$ , equation (2.4) becomes

$$(2.5) \quad \begin{aligned} \frac{dy_\lambda}{dt}(t) + \nu Ay_\lambda(t) + By_\lambda(t) &= P(m\Phi_i(y_\lambda(t))) + Pf_e, t \in (0, \lambda), \\ y_\lambda(0) &= y_0. \end{aligned}$$

By Theorem 1.1 and relations (1.8) and (1.9), we get that, there exists a weak solution  $y_\lambda(t)$  on  $[0, \lambda]$  to the system (2.5), which satisfies

$$(2.6) \quad |y_\lambda(t) - y_e^i|_{\frac{1}{2}} \leq C_i e^{-\gamma_i t} |y_0 - y_e^i|_{\frac{1}{2}}, t \in [0, \lambda].$$

Remember that  $y_0 \in \mathcal{U}_i$ , this means that we have

$$(2.7) \quad |y_0 - y_e^i|_{\frac{1}{2}} < \frac{\rho_i}{C_i}.$$

It follows by (2.6) and (2.7), that  $|y_\lambda(t) - y_e^i|_{\frac{1}{2}} < \rho_i, t \in [0, \lambda]$ , this means that  $y_\lambda(t) \in U_{\rho_i}, t \in [0, \lambda]$ . So

$$(2.8) \quad y_\lambda(t - \lambda) \in U_{\rho_i}, t \in [\lambda, 2\lambda].$$

Thus, proceeding as before, i.e using the definition of the functions  $\chi_j, j = 1, \dots, N$ , we deduce by (2.8) that

$$\chi_j(y_\lambda(t - \lambda) - y_e^j) = \begin{cases} 1, & \text{if } j = i \\ 0, & \text{if } j \neq i, \end{cases}$$

for all  $t \in [\lambda, 2\lambda]$ . This, together with the previous step (see (2.5)), yields that on  $[0, 2\lambda]$ , equation (2.4) becomes

$$(2.9) \quad \begin{aligned} \frac{dy_\lambda}{dt}(t) + \nu Ay_\lambda(t) + By_\lambda(t) &= P(m\Phi_i(y_\lambda(t))) + Pf_e, t \in (0, 2\lambda), \\ y_\lambda(0) &= y_0. \end{aligned}$$

Applying Theorem 1.1 once again we obtain that there exists a solution to (2.9) on  $[0, 2\lambda]$  which remains in  $U_{\rho_i}$ .

Continuing with this argument, we can finally conclude that, the system (2.4), for  $y_0 \in \mathcal{U}_i$ , has a solution  $y_\lambda$ . Moreover,  $y_\lambda$  is in fact the weak solution to the system

$$(2.10) \quad \frac{dy_\lambda}{dt}(t) + \nu Ay_\lambda(t) + By_\lambda(t) = P(m\Phi_i(y_\lambda(t))) + Pf_e, t > 0,$$

$$y_\lambda(0) = y_0,$$

for which we know (via Theorem 1.1, see relations (1.8) and (1.9) also) that satisfies

$$(2.11) \quad |y_\lambda(t) - y_e^i|_{\frac{1}{2}} \leq C_i e^{-\gamma_i t} |y_0 - y_e^i|_{\frac{1}{2}}, t \in [0, \infty).$$

In virtue of (2.11), the fact that  $y_0 \in \mathcal{U}_i$  and the definition of the functions  $\chi_j, j = 1, \dots, N$ , we have that

$$\chi_j(y_\lambda(t - \lambda) - y_e^j) = \begin{cases} 1, & \text{if } j = i \\ 0, & \text{if } j \neq i, \end{cases}$$

for all  $t \geq 0$ . This means that, in fact, the functions  $\chi_j(y_\lambda(t - \lambda) - y_e^j)$  are constant (1 if  $j = i$  and 0 if  $j \neq i$ ) for all  $t \geq 0$ .

We conclude that: for every  $\lambda > 0$  there exists a weak solution  $y_\lambda$  to the delay system (2.4) and these solutions are all equal to the weak solution of the system

$$(2.12) \quad \frac{dy}{dt}(t) + \nu Ay(t) + By(t) = P(m\Phi_i(y(t))) + Pf_e, t > 0,$$

$$y(0) = y_0,$$

i.e.  $y_\lambda \equiv y, \forall \lambda > 0$ ,  $y$  solution to the above closed-loop system. The functions  $\chi_j(y_\lambda(t - \lambda) - y_e^j), j = 1, \dots, N$  are all constant for all  $\lambda > 0$ . Hence we can pass to the limit  $\lambda \rightarrow 0$  in (2.4), and obtain that there exists a weak solution to the closed-loop system

$$(2.13) \quad \frac{dy}{dt} + \nu Ay + By = P\left(m \sum_{j=1}^N \chi_j(y - y_e^j) \Phi_j(y)\right) + Pf_e,$$

$$t \geq 0, y(0) = y_0 \in \mathcal{U}_i,$$

which satisfies the exponential decay claimed. This completes the proof.  $\square$



**Remark 2.1.** One may connect the result from Theorem 2.1 with the long time behaviour characteristics of the solutions to the N-S equations. To see this, let us consider again the controlled N-S equations

$$(2.14) \quad \frac{dy}{dt}(t) + \nu Ay(t) + By(t) = P(\Phi(y)) + f, t > 0$$

$$y(0) = y_0,$$

where  $f = Pf_e$  and  $\Phi$  is given by (2.2).

A first energy-type equality is obtained by taking the scalar product of (2.14) with  $y$ . Using the orthogonality property (see [3])  $b(y, z, z) = 0$ ,  $\forall y \in V, \forall z \in (H^1(\Omega))^d$ , we see that  $(By, y) = 0$  and there remains

$$(2.15) \quad \frac{1}{2} \frac{d}{dt} |y|^2 + \nu \|y\|^2 = (f, y) + (P(\Phi y), y).$$

We know that  $|y| \leq \frac{1}{\sqrt{\lambda_1}} \|y\|$ ,  $\forall y \in V$ , where  $\lambda_1$  is the first eigenvalue of  $A$ . Hence, we can majorize the right-hand side of (2.15) by

$$\frac{1}{\sqrt{\lambda_1}} |f| \|y\| + (P(\Phi y), y) \leq \frac{\nu}{2} \|y\|^2 + \frac{1}{2\nu\lambda_1} |f|^2 + C|y|^2,$$

and we obtain

$$(2.16) \quad \frac{d}{dt} |y|^2 + (\nu\lambda_1 - 2C)|y|^2 \leq \frac{1}{\nu\lambda_1} |f|^2.$$

By the definition of  $\Phi$  (see (2.2)), we can take  $\omega$  with the measure sufficiently small such that  $\nu\lambda_1 - 2C > 0$ .

Using the classical Gronwall lemma, we obtain via (2.16), that

$$(2.17) \quad |y(t)|^2 \leq |y_0|^2 e^{-\alpha t} + \frac{1}{\nu^2 \lambda_1^2} |f|^2 (1 - e^{-\alpha t}),$$

where  $\alpha = \nu\lambda_1 - 2C > 0$ .

Thus

$$(2.18) \quad \limsup_{t \rightarrow \infty} |y(t)| \leq \rho_0, \rho_0 = \frac{1}{\nu\lambda_1} |f|.$$

Further, we define the semigroup  $S(t) : y_0 \rightarrow y(t)$ . We infer from (2.17) that the balls  $B_H(0, \rho)$  of  $H$  with  $\rho \geq \rho_0$  are positively invariants for the

semigroup  $S(t)$ , and these balls are absorbing for any  $\rho > \rho_0$ . We choose  $\rho'_0 > \rho_0$  and denote by  $\mathcal{B}_0$  the ball  $B_H(0, \rho'_0)$ . Any set  $\mathcal{B}$  bounded in  $H$  is included in a ball  $B_H(0, R)$  of  $H$ . It is easy to deduce from (2.17) that  $S(t)\mathcal{B} \subset \mathcal{B}_0$  for  $t \geq t_0(\mathcal{B}, \rho'_0)$ . Hence,  $\mathcal{B}_0$  is an absorbing set for the semigroup  $S(t)$ . Further we define  $X = \bigcap_{t>0} S(t)\mathcal{B}_0$ . Then,  $X$  is compact, invariant under  $S(t)$ , connected and  $\lim_{t \rightarrow \infty} \text{dist}(S(t)y_0, X) = 0, \forall y_0 \in H$ .  $X$  is called the universal attractor of the equation (2.14) (for more details see [2]).

To conclude, we have that  $\{y_e^1, \dots, y_e^N\} \subset X$ , and, in virtue of Theorem 2.1, given  $y_0 \in H$ , if there exists  $t' \geq 0$  such that  $S(t')y_0 \in \mathcal{U}_i$  for some  $i = 1, \dots, N$ , then  $S(t)y_0 \rightarrow y_e^i$ , in  $H$ , when  $t \rightarrow \infty$ . Using this, one may develop these ideas in order to have a better picture of the attractor  $X$ .

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Received: 3.V.2010    Institute of Mathematics "O. Mayer" of the Romanian Academy,  
Iași, 700506,  
ROMANIA  
ionutmunteanu23@yahoo.com