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A CLASS OF CONTINUOUS HYBRID LINEAR MULTISTEP METHODS FOR STIFF IVPs IN ODEs

BY

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Abstract. In this paper, we present a class of continuous hybrid linear multistep methods (CHLMM) for stiff initial value problems (IVPs) in ordinary differential equations (ODEs). The constuction of these methods are based on the approach of collocation and interpolation. The interval of absolute stability of the method is investigated, using the root locus method. Numerical results of the methods solving stiff IVPs in ODEs are compared with that from the state-of-the-art Ode15s Matlab ODEs code.

Mathematics Subject Classification 2000: 65L05, 65L06.

Key words: continuous hybrid linear multistep methods, collocation and interpolation, initial value problem, stiff stability, root locus.

1. Introduction

Consider the numerical solution of stiff IVPs

$$y' = f(x, y), \quad y(x_0) = y_0, \quad a \le x \le b \tag{1}$$

by a class of continuous hybrid LMM (CHLMM)

$$y(x_n+(t+1)h)=\sum_{j=0}^{k-1}\alpha_j(t)y_{n+j}+\alpha_v(t)y_{n+v}+h\beta_v(t)f(x_{n+v}, y_{n+v}), \tag{2}$$

$$t \in [-1, k-1], 0 \le v \le k$$

$$y(x_n + vh) = \sum_{j=0}^{k}\alpha_j^*(t^*)y_{n+j} + h\beta_k^*(t^*)f_{n+k}, v = t^* + 1, t^* = k - \frac{3}{2}, \tag{3}$$

where y_{n+j} is the numerical approximation to the exact solution $y(x_{n+j})$, $f_{n+j} = f(x_{n+j}, y_{n+j})$, $\{\alpha_j(t), j = 0(1)k-1\}$, $\alpha_v(t)$, $\{\alpha_j^*(t^*), j = 0(1)k\}$, $\beta_v(t)$, and $\beta_k^*(t)$, are continuous coefficients in t presumed to be real and satisfying the normalization condition $\alpha_k(t) = 1$, $\alpha_v^*(t^*) = 1$, $x = x_{n+1} + th$, $t \in [a, b]$ and $h = x_{n+1} - x_n$ is a fixed mesh size.

The problem of stiffness in most ordinary differential equations (ODEs) has posed a lot of computational difficulties in many practical application modeled by ODEs. Stiffness affects the efficiency of numerical methods. Here, we present a class of continuous Hybrid linear multistep methods for stiff IVPs in ODEs. Hybrid LMM was first proposed by [11]. Other authors are, [1, 2, 3, 4, 6, 9, 10, 12, 13, 14, 16, 19, 20, 21, 22]. In fact, the numerical solution of (1) by (2) through collocation and interpolation methods have been well studied in the literature, see for example [23], [25], [5], [7, 8], [26], [16], [24], and [17, 18]. The interval of absolute stability of the CHLMM is investigated using the root locus method discussed in [20, 21] and [3], whose application can be found in [16] and [24] instead of the equivalent boundary locus plot in [7] and [9]. The local truncation error for (2) and (3) are nicely given as

$$(4)\quad L.T.E = [y(x_n + (t+1)h) - \sum_{j=0}^{k-1} \alpha_j(t) y(x_n + jh) - \alpha_v(t) y(_n + vh) - h\beta_v(t) y'(x_n + vh)] \quad \text{and}$$

$$(5)\quad L.T.E = [y(x_n + vh) - \sum_{j=0}^{k} \alpha_j^*(t^*) y(x_n + jh) - h\beta_k^*(t^*) y'(x_n + kh)].$$

The order for (2) and the expression for y_{n+v} in the function f_{n+v} in (2) are $p = k+1$ and $p = k+1$ respectively. Effective implementation of (2) demand the use of the Newton iterative scheme $y_{n+k}^{[s+1]} = y_{n+k}^{[s]} - F'(y_{n+k}^{[s]})^{-1} F(y_{n+k}^{[s]})$, $s = 0, 1, 2, \ldots$, where, $F'(y_{n+k}^{[s]})^{-1}$ is the Jacobian matrix of the vector systems of the method. In particular, for $k - step$ the nonlinear equation $F(y_{n+k}^{[s]}) = 0$. The parameter v is incorporated to provide off step collocation point x_{n+v} in an open interval (x_{n+k-1}, x_{n+k}) and $v = k - \frac{1}{2}$, where k is the step number of the scheme. Formula (2) is zero stable for fixed step size, h case for $k \leq 7$. For $k \geq 8$, no stable process appear to exist. See the root locus plots of the methods in section 4. The motivation to derive the hybrid method (2) is the fact that, it offers the means to by pass the Dahlquist order barrier for A-stable conventional LMM and the fact that continuous solution of the IVPs in ODEs can be obtained. The proposed continuous hybrid LMM in (2) consists of the addition of terms $\alpha_v(t)y_{n+v}$ to the left hand side of the One-leg hybrid LMM in [9] $\sum_{j=0}^{k}\alpha_j y_{n+j} = h\beta_v f(x_{n+v}, y_{n+v})$. Implementation of (2) required us to compute first y_{n+v} in (3) so that the terms y_{n+v} and f_{n+v} in (2) could be evaluated. Considerations as to how this might be done appear in section 5 in this paper.

The Outline of this paper is as follows. We start with the construction of the continuous hybrid LMM in (2) of $k+1$ in section 2. Section 3 deals with the derivations of the continuous hybrid predictor y_{n+v} in the function f_{n+v} of the method in (3). In section 4, we determined the stiff stability of the methods, using the root locus. Finally, in section 5, result of numerical experiments on some stiff test systems are presented and compared with Ode15s code from MATLAB ODE suite in [15].

2. Derivation of the continuous hybrid linear multistep methods

The solution of the IVPs in (1) is assumed to be the polynomial

$$y(x) = \sum_{j=0}^{k+1} a_j x^j \tag{6}$$

where $\{a_j\}_{j=0}^{k+1}$ are the real parameter constants to be determined. From (8) we have

$$y'(x) = f(x, y) = \sum_{j=1}^{k+1} j a_j x^{j-1} \tag{7}$$

Collocating (7) at $x = x_{n+v}$ and interpolating (9) at $x = x_{n+j}$, $j = 0(1)k-1$ and $x = x_{n+v}$, we obtain the linear system of equations

$$\begin{pmatrix} 1 & x_n & x_n^2 & \dots & x_n^{k+1} \\ 1 & x_{n+1} & x_{n+1}^2 & \dots & x_{n+1}^{k+1} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & x_{n+k-1} & x_{n+k-1}^2 & \dots & x_{n+k-1}^{k+1} \\ 1 & x_{n+v} & x_{n+v}^2 & \dots & x_{n+v}^{k+1} \\ 0 & 1 & 2x_{n+v} & \dots & (k+1)x_{n+v}^k \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \\ a_k \\ a_{k+1} \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+1} \\ \vdots \\ y_{n+k-1} \\ y_{n+v} \\ f_{n+v} \end{pmatrix}. \tag{8}$$

Solving equation (8) for $a'_j s$ and substituting the resulting values into (6) with $t = (x - x_{n+1})/h$ and setting $x = x_{t+1}$ on the left hand side of (6) yield the values of the continuous coefficients $\{\alpha_j\}_{j=0}^{k-1}(t)$, $\alpha_v(t)$, $\beta_v(t)$ respectively. Fixing $t = k - 1$ into the continuous coefficients $\{\alpha_j\}_{j=0}^{k-1}(t)$, $\alpha_v(t)$, $\beta_v(t)$ for a fixed k, give the values of the discrete coefficients of the method in (2) for $k \leq 7$. For example, see Table 1 in appendix A for the continuous coefficients for method (2) for $k \leq 7$. Table 2. in appendix A shows explicitly the discrete coefficients for method (2) for $k \leq 7$.

3. The derivation of the continuous hybrid predictor

Similarly, the corresponding hybrid predictor

$$(9) \quad y(x_n + vh) = \sum_{j=0}^{k} \alpha_j^*(t) y_{n+j} + h\beta_k^*(t) f_{n+k}, \quad v = t + 1, \quad t = k - 3/2$$

for $y(x_{n+v})$, and $f(x_{n+v})$ in (2) are obtained from the polynomial interpolant

$$(10) \qquad y(x_{n+v}) = \sum_{j=0}^{k+1} b_j x^j.$$

where $\{b\}_{j=0}^{k+1}$ are the real parameter constants to be determined. Following the same procedure in section 2, the unknown continuous coefficients of the hybrid predictors in (2) are obtained. After some simplifications, we obtained a class of continuous hybrid predictors from (2). Table 3 in appendix B below shows the continuous coefficients of the predictor (3) for $k \leq 7$. Table 4. in appendix B gives the discrete coefficients for the predictor (3) for $k \leq 7$.

4. Stability of the methods by plotting the root locus

In this section, we investigate the stability properties of the family of the continuous hybrid linear multistep method (CHLMM) in (2) using the root locus plot discussed in [20, 21]. On substituting the hybrid solution (3) y_{n+v} at point x_{n+v} into the continuous hybrid LMM for a fixed k and t, and applying the resultant method on the scalar test problem $y' = \lambda y$, $Re(\lambda) < 0$, we obtain the continuous hybrid LMM stability polynomials to be

$$\pi(r,z) = r^k - \sum_{j=0}^{k-1} \alpha_j r^j - \alpha_v(\sum_{j=0}^{k} \alpha_j^* r^j + z\beta_k^* r^k) - z\beta_v(\sum_{j=0}^{k} \alpha_j^* r^j + z\beta_k^* r^k),$$

$$z = \lambda h. \tag{11}$$

Plotting $|r_j(z)|$ against z reveals the interval of absolute stability for the methods. The general form of the stability plot is given below in figure 1. Method (2) is said to be stable respectively, if $0 \leq |r_j(z)| \leq 1$ where, $r_j(z)$, $j = 0(1)k$ are roots of the polynomial in (11) with root $|r_j(z)| = 1$ been simple. Plotting the root locus of $\pi(r,z) = 0$, it is observed that the methods in (2) are stiffly stable for $k \leq 7$. The graphs in figures 2-9 below show the loci and thus the interval of absolute/stiff stability of each method for a fixed value of $k \leq 7$. The case of $k \geq 8$ are stiffly unstable, see figure 9 and Table 4.3 respectively.

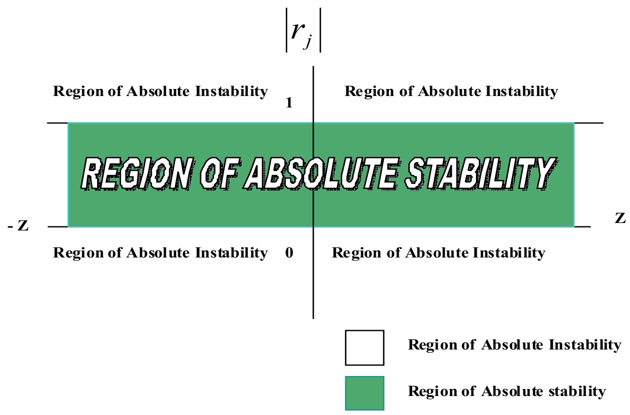


Figure 1: The root locus form of the region of absolute stability/stiff stability. See [20]

Root Locus Plots for the Continuous Hybrid Multistep Methods in (2)

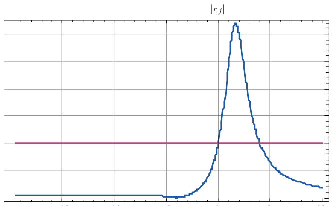


Figure 2: Root Locus Plot for k = 1

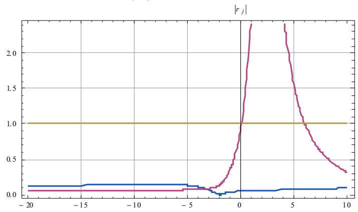


Figure 3: Root Locus Plot for k = 2

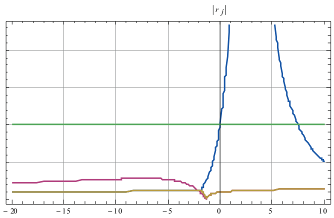


Figure 4: Root Locus Plot for k = 3

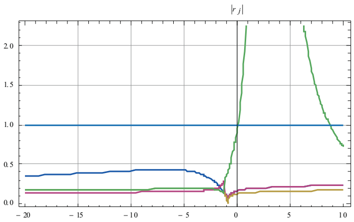


Figure 5: Root Locus Plot for k = 4

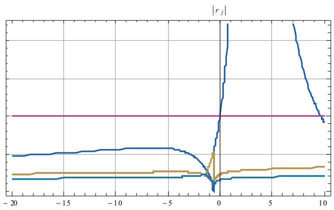


Figure 6: Root Locus Plot for k = 5

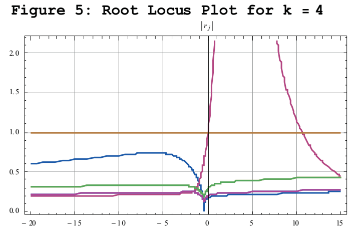


Figure 7: Root Locus Plot for k = 6

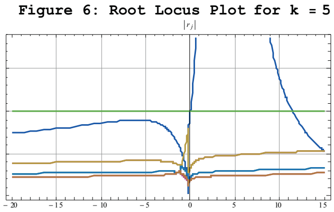


Figure 8: Root Locus Plot for k = 7

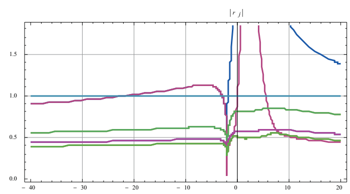


Figure 9*: Root Locus Plot for k = 8

Table 4.1: The continuous and discrete error constants of method(2).

k	t	Continuous Error Constant ($C_{p+1}(t)$)	C_{p+1}
1	0	$\frac{1}{24}(1+t)(1+2t)^2h^3y^{(3)}(x_n)$	$\frac{1}{24}$
2	1	$\frac{1}{96}t(1-3t+4t^3)h^4y^{(4)}(x_n)$	$\frac{1}{48}$
3	2	$\frac{1}{480}(3-2t)^2t(-1+t^2)h^5y^{(5)}(x_n)$	$\frac{1}{80}$
4	3	$\frac{1}{2880}(5-2t)^2t(-2+t)(-1+t)t(1+t)h^6y^{(6)}(x_n)$	$\frac{1}{120}$
5	4	$\frac{1}{20160}(7-2t)^2(-3+t)(-2+t)(-1+t)t(1+t)h^7y^{(7)}(x_n)$	$\frac{1}{168}$
6	5	$\frac{1}{161280}(9-2t)^2(-4+t)(-3+t)(-2+t)(-1+t)t(1+t)h^8y^{(8)}(x_n)$	$\frac{1}{224}$
7	6	$\frac{1}{1451520}(11-2t)^2(-5+t)(-4+t)(-3+t)(-2+t)(-1+t)t(1+t)h^9y^{(9)}(x_n)$	$\frac{1}{288}$
8	7	$\frac{1}{14515200}(13-2t)^2(-6+t)(-5+t)(-4+t)(-3+t)(-2+t)(-1+t)t(1+t)h^{10}y^{(10)}(x_n)$	$\frac{1}{360}$

Table 4.2: The continuous and discrete error constants of predictor(3).

k	t	Continuous Error Constant ($C_{p+1}(t^*)$)	C^*_{p+1}.
1	$\frac{-1}{2}$	$\frac{1}{6}t^2(1+t)h^3y^{(3)}(x_n)$	$\frac{1}{48}$
2	$\frac{1}{2}$	$\frac{1}{24}(-1+t)^2t(1+t)h^4y^{(4)}(x_n)$	$\frac{1}{128}$
3	$\frac{3}{2}$	$\frac{1}{120}(-2+t)^2t(-1+t^2)h^5y^{(5)}(x_n)$	$\frac{1}{256}$
4	$\frac{5}{2}$	$\frac{1}{720}(-3+t)^2(-2+t)(-1+t)t(1+t)h^6y^{(6)}(x_n)$	$\frac{7}{3072}$
5	$\frac{7}{2}$	$\frac{1}{5040}(-4+t)^2(-3+t)(-2+t)(-1+t)t(1+t)h^7y^{(7)}(x_n)$	$\frac{3}{2048}$
6	$\frac{9}{2}$	$\frac{1}{40320}(-5+t)^2(-4+t)(-3+t)(-2+t)(-1+t)t(1+t)h^8y^{(8)}(x_n)$	$\frac{33}{32768}$
7	$\frac{11}{2}$	$\frac{1}{362880}(-6+t)^2(-5+t)(-4+t)(-3+t)(-2+t)(-1+t)t(1+t)h^9y^{(9)}(x_n)$	$\frac{143}{196608}$
8	$\frac{13}{2}$	$\frac{1}{3628800}(-7+t)^2(-6+t)(-5+t)(-4+t)(-3+t)(-2+t)(-1+t)t(1+t)h^{10}y^{(10)}(x_n)$	$\frac{143}{262144}$

where, C_{p+1} and C^*_{p+1} in Table 4.1 and Table 4.2 implies Discrete Error Constants for the CHLMM (2) and Hybrid Predictor (3) respectively.

Table 4.3: The step-number, scaled variable t and interval of **absolute stability of z for the method in (2) and (3).**

k	t	Interval of Absolute Stability of z for CHLMM (2)+ Predictor(3)	Order(2)	Order(3)
1	0	$(-\infty, 0) \cup (4, \infty)$	2	2
2	1	$(-\infty, 0) \cup (6, \infty)$	3	3
3	2	$(-\infty, 0) \cup (7.46, \infty)$	4	4
4	3	$(-\infty, 0) \cup (8.667, \infty)$	5	5
5	4	$(-\infty, 0) \cup (9.7, \infty)$	6	6
6	5	$(-\infty, 0) \cup (10.2, \infty)$	7	7
7	6	$(-\infty, 0) \cup (11.46, \infty)$	8	8
8	7	Unstable	9	9

Table 4.4: The step-number, the range of t for which the **CHLMM (2) and predictor (3) is zero-stable.**

k	The range of t for which the CHLMM (2)+ Predictor(3) is zero stable
1	$\{t : t\varepsilon[-\infty, \infty]\}$
2	$\{t : t\varepsilon(-\infty, -1.85) \cup (-1, 1.6) \cup (1.728, \infty)\}$
3	$\{t : t\varepsilon(-\infty, -1.53) \cup (0, 2.515) \cup (2.6, \infty)\}$
4	$\{t : t\varepsilon(-\infty, -1.92) \cup (1, 3.432) \cup (3.5452, \infty)\}$
5	$\{t : t\varepsilon(-\infty, 1.50725) \cup (0, 1) \cup (1.29, 1.46) \cup (2, 4.3754) \cup (4.5041, \infty)\}$
6	$\{t : t\varepsilon(-\infty, 0) \cup (10.2, \infty) \cup (2.6, \infty)\}$
7	$\{t : t\varepsilon(0, 1.01) \cup (2, 3) \cup (3.35, 3.83) \cup (4, 6.296) \cup (6.45269, \infty)\}$
8	Unstable

5. Numerical experiments

In this section the implementation of the CHLMM in (2) discussed in sections 2 and 3 of this paper on the stiff initial value problems will be considered.

Problem 1: Linear problem in [7]

$$y' = \begin{pmatrix} -0.1 & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 \\ 0 & 0 & -100 & 0 \\ 0 & 0 & 0 & -1000 \end{pmatrix} y, y(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \; y(x) = \begin{pmatrix} e^{-0.1x} \\ e^{-10x} \\ e^{-100x} \\ e^{-1000x} \end{pmatrix}$$

Problem 2: Nonlinear chemical problem in [7] and [15]

$$y_1' = -0.04y_1 + 10^4 y_2 y_3,$$
$$y_2' = -400y_1 + 10^4 y_2 y_3 - 3 \times 10^7 y_2^2,$$
$$y_3' = 3 \times 10^7 y_2^2,$$
$$y(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

with x being the range [0,10] for problem [1] and $h = 0.0001$ for problem [2], $x \in 0(0.0001)3$. In solving the initial value problems above, set up the continuous form of the methods from the continuous coefficients table of interest, CHLMM in (2) for $k = 1$ is

$$(12)\quad y(x_n+(t+1)h) = (1+4t+4t^2)y_n+(-4t-4t^2)y_{n+\frac{1}{2}}+h(1+3t+2t^2)f_{n+\frac{1}{2}}$$

from table (1) in Appendix A. The local truncation error and the order for (14) is

$$(13)\qquad C_3(t) = \frac{(1+t)(1+2t)^2 h^3 y^{(3)}(x)}{24}, \quad p = 2.$$

Setting $t = 0$ in (14) gives the equivalent discrete form of the CHLMM in (2) to be

$$(14)\qquad y_{n+1} = y_n + hf_{n+\frac{1}{2}}, \quad p = 2$$

Similarly, from table (3) in Appendix B, we obtained the equivalent discrete form of the continuous hybrid predictor in (3) for $k = 1$ and $t = -\frac{1}{2}$ respectively to be

$$y_{n+\frac{1}{2}} = \frac{1}{4}y_n + \frac{3}{4}y_{n+1} - \frac{h}{4}f_{n+1}, \quad p = 2 \tag{15}$$

It has been noted by [7], [9], [12, 13], and [20], that linear multistep methods suitable for stiff ODEs must be implicit and must therefore require a scheme to resolve the implicitness of the methods. Applying discrete methods (2) and (3) respectively to the initial value problem above leads to solving implicit set of equations which demands the use of Newton Raphson iterative scheme,

$$y_{n+k}^{[s+1]} = y_{n+k}^{[s]} - F'(y_{n+k}^{[s]})^{-1}F(y_{n+k}^{[s]}), \quad s = 0, 1, 2, ... \tag{16}$$

where $F'(y_{n+k}^{[s]})^{-1}$ is the Jacobian matrix of the vector systems of the method. In particular, for $k = 1$ the nonlinear equation $F(y_{n+k}^{[s]}) = 0$, where

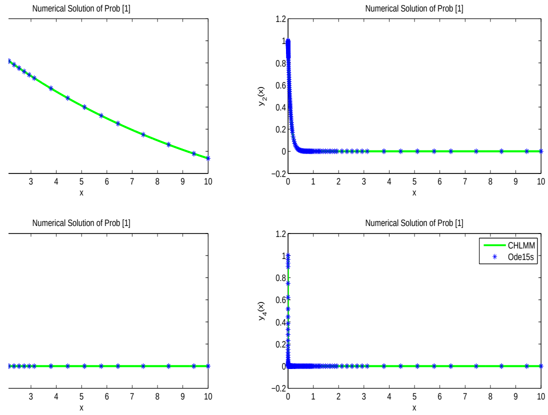
$$F(y_{n+1}^{[s]}) = y_{n+1}^{[s]} - y_n - hf_{(}x_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}^{[s]}) = 0 \tag{17}$$

$$y_{n+\frac{1}{2}}^{[s]} = \frac{1}{4}y_n + \frac{3}{4}y(x_{n+1}^{[s]}) - \frac{h}{4}f(x_{n+1}, y_{n+1}^{[s]}). \tag{18}$$

In this regard $y_{n+1}^{[0]}$ is given from the trapezoidal rule

$$y_{n+1}^{[0]} = y_n + \frac{h}{2}(f_{n+1} + f_n), \quad s = 0, 1, 2, ... \tag{19}$$

as an initial guess for y_{n+1} in (19). Let L be the Lispschitz constant of $f(x, y)$ with respect to y. For non-stiff problems, where L is small, the step size is usually determined by accuracy conditions. However, for stiff problems where L is large, the step size is severely restricted by stability constraint. Terminations of the iteration (16) occur whenever we observe that $|y_{n+1}^{[s+1]} - y_{n+1}^{[s]}| \leqslant \text{TOL}$, where TOL is the order of the unit round off error of the computer, which may be assumed by the user. However figure



[h]
Figure 10: The plot of numerical solutions of Problem 1 and Ode15s in [15].

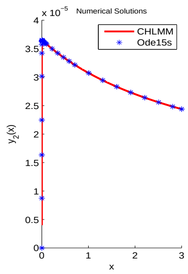


Figure 11: The plot of numerical solutions of $y_2(x)$ of Problem 2.

10 and figure 11 below show the plot of the numerical results of the methods when applied to the linear problem 1 and nonlinear problem 2.

Finally, in this paper, we have derived a class of continuous hybrid linear multistep methods (2) which is of order $p = k+1$, and stiffly stable for $k \leqslant 7$ using collocation and interpolation process. The root locus plot in figure 9 revealed that the instability of the methods in (2) set in when $k \geqslant 8$. The numerical solution graphs in figure 10 and figure 11, of the methods in (14) and (15) coincide and show that the methods in (2) is compared with the state-of the-art of ***MATLAB ode15s*** code in [15], on Problem 1 and Problem 2 respectively.

Appendix A

Table 1 : The Continuous Coefficients for the CHLMM in (2) for $k \le 7$.

k	t	j	$\alpha_j(t)$	$\beta_j(t)$
1	0	0	$1+4t+4t^2$	0
		$\frac{1}{2}$	$-4t-4t^2$	$1+3t+2t^2$
		1	1	0
2	1	0	$-\frac{t}{9}+\frac{4t^2}{9}-\frac{4t^3}{9}$	0
		1	$1-3t+4t^3$	0
		$\frac{3}{2}$	$\frac{28t}{9}-\frac{4t^2}{9}-\frac{32t^3}{9}$	$-\frac{2t}{3}+\frac{2t^2}{3}+\frac{4t^3}{3}$
		2	0	0
3	2	0	$-\frac{9t}{50}+\frac{21t^2}{50}-\frac{8t^3}{25}+\frac{2t^4}{25}$	0
		1	$1-\frac{4t}{3}-\frac{5t^2}{9}+\frac{4t^3}{3}-\frac{4t^4}{9}$	0
		2	$\frac{9t}{2}-\frac{3t^2}{2}-4t^3+2t^4$	0
		$\frac{5}{2}$	$-\frac{224t}{75}+\frac{368t^2}{225}+\frac{224t^3}{75}-\frac{368t^4}{225}$	$\frac{4t}{5}-\frac{8t^2}{15}-\frac{4t^3}{5}+\frac{8t^4}{15}$
		3	1	0
4	3	0	$-\frac{25t}{147}+\frac{115t^2}{294}-\frac{31t^3}{98}+\frac{16t^4}{147}-\frac{2t^5}{147}$	0
		1	$1-\frac{13t}{10}-\frac{11t^2}{25}+\frac{61t^3}{50}-\frac{14t^4}{25}+\frac{2t^5}{25}$	0
		2	$\frac{25t}{9}-\frac{5t^2}{6}-\frac{37t^3}{18}+\frac{4t^4}{3}-\frac{2t^5}{9}$	0
		3	$-\frac{25t}{6}+\frac{10t^2}{3}+\frac{7t^3}{2}-\frac{10t^4}{3}+\frac{2t^5}{3}$	0
		$\frac{7}{2}$	$\frac{6304t}{2205}-\frac{9008t^2}{3675}-\frac{25888t^3}{11025}+\frac{9008t^4}{3675}-\frac{5632t^5}{11025}$	$-\frac{16t}{21}+\frac{24t^2}{35}+\frac{64t^3}{105}-\frac{24t^4}{35}+\frac{16t^5}{105}$
		4	1	0

Continuation of Table 1

k	t	j	$\alpha_j(t)$	$\beta_j(t)$
5	4	0	$-\frac{49t}{324}+\frac{707t^2}{1944}-\frac{313t^3}{972}+\frac{29t^4}{216}-\frac{13t^5}{486}+\frac{t^6}{486}$	0
		1	$1-\frac{59t}{42}-\frac{27t^2}{98}+\frac{365t^3}{294}-\frac{209t^4}{294}+\frac{8t^5}{49}-\frac{2t^6}{147}$	0
		2	$\frac{147t}{50}-\frac{119t^2}{100}-2t^3+\frac{33t^4}{20}-\frac{11t^5}{25}+\frac{t^6}{25}$	0
		3	$-\frac{49t}{18}+\frac{133t^2}{54}+\frac{107t^3}{54}-\frac{43t^4}{18}+\frac{20t^5}{27}-\frac{2t^6}{27}$	0
		4	$\frac{49t}{12}-\frac{35t^2}{8}-\frac{31t^3}{12}+\frac{101t^4}{24}-\frac{3t^5}{2}+\frac{t^6}{6}$	0
		$\frac{9}{2}$	$-\frac{38\,912t}{14\,175}+\frac{897\,152t^2}{297\,675}+\frac{20\,032t^3}{11\,907}-\frac{19\,136t^4}{6615}+\frac{316\,352t^5}{297\,675}-\frac{36\,032t^6}{297\,675}$	$\frac{32t}{45}-\frac{752t^2}{945}-\frac{80t^3}{189}+\frac{16t^4}{21}-\frac{272t^5}{945}+\frac{32t^6}{945}$
		5	0	0
6	5	0	$-\frac{81t}{605}+\frac{819t^2}{2420}-\frac{1577t^3}{4840}+\frac{227t^4}{1452}-\frac{581t^5}{14\,520}+\frac{19t^6}{3630}-\frac{t^7}{3630}$	0
		1	$1-\frac{55t}{36}-\frac{61t^2}{648}+\frac{2461t^3}{1944}-\frac{563t^4}{648}+\frac{505t^5}{1944}-\frac{t^6}{27}+\frac{t^7}{486}$	0
		2	$\frac{162t}{49}-\frac{171t^2}{98}-\frac{403t^3}{196}+\frac{313t^4}{147}-\frac{437t^5}{588}+\frac{17t^6}{147}-\frac{t^7}{147}$	0
		3	$-\frac{81t}{25}+\frac{333t^2}{100}+\frac{197t^3}{100}-\frac{187t^4}{60}+\frac{377t^5}{300}-\frac{16t^6}{75}+\frac{t^7}{75}$	0
		4	$3t-\frac{43t^2}{12}-\frac{319t^3}{216}+\frac{119t^4}{36}-\frac{325t^5}{216}+\frac{5t^6}{18}-\frac{t^7}{54}$	0
		5	$-\frac{81t}{20}+\frac{207t^2}{40}+\frac{67t^3}{40}-\frac{113t^4}{24}+\frac{281t^5}{120}-\frac{7t^6}{15}+\frac{t^7}{30}$	0
		$\frac{11}{2}$	$\frac{3\,529\,216t}{1\,334\,025}-\frac{41\,074\,048t^2}{12\,006\,225}-\frac{37\,896\,896t^3}{36\,018\,675}+\frac{7\,450\,304t^4}{2\,401\,245}-\frac{56\,558\,912t^5}{36\,018\,675}+\frac{1\,274\,176t^6}{4\,002\,075}-\frac{833\,024t^7}{36\,018\,675}$	$\frac{3\,529\,216t}{1\,334\,025}-\frac{41\,074\,048t^2}{12\,006\,225}-\frac{37\,896\,896t^3}{36\,018\,675}+\frac{7\,450\,304t^4}{2\,401\,245}-\frac{56\,558\,912t^5}{36\,018\,675}+\frac{1\,274\,176t^6}{4\,002\,075}-\frac{833\,024t^7}{36\,018\,675}$
		6	1	0

Continuation of Table 1

k	t	j	$\alpha_j(t)$	$\beta_j(t)$
7	6	0	$-\frac{121\,t}{1014}+\frac{19\,217\,t^2}{60\,840}-\frac{39\,761\,t^3}{121\,680}+\frac{1637\,t^4}{9360}-\frac{1291\,t^5}{24\,336}+\frac{1121\,t^6}{121\,680}-\frac{t^7}{1170}+\frac{t^8}{30\,420}$	0
		1	$1-\frac{1087\,t}{660}+\frac{1327\,t^2}{14\,520}+\frac{14\,t^3}{11}-\frac{2471\,t^4}{2420}+\frac{889\,t^5}{2420}-\frac{339\,t^6}{4840}+\frac{5\,t^7}{726}-\frac{t^8}{3630}$	0
		2	$\frac{605\,t}{162}-\frac{4697\,t^2}{1944}-\frac{2689\,t^3}{1296}+\frac{1157\,t^4}{432}-\frac{1471\,t^5}{1296}+\frac{307\,t^6}{1296}-\frac{2\,t^7}{81}+\frac{t^8}{972}$	0
		3	$-\frac{605\,t}{147}+\frac{8327\,t^2}{1764}+\frac{125\,t^3}{63}-\frac{535\,t^4}{126}+\frac{131\,t^5}{63}-\frac{17\,t^6}{36}+\frac{23\,t^7}{441}-\frac{t^8}{441}$	0
		4	$\frac{121\,t}{30}-\frac{3179\,t^2}{600}-\frac{559\,t^3}{400}+\frac{1867\,t^4}{400}-\frac{41\,t^5}{16}+\frac{251\,t^6}{400}-\frac{11\,t^7}{150}+\frac{t^8}{300}$	0
		5	$-\frac{121\,t}{36}+\frac{5071\,t^2}{1080}+\frac{13\,t^3}{15}-\frac{731\,t^4}{180}+\frac{29\,t^5}{12}-\frac{227\,t^6}{360}+\frac{7\,t^7}{90}-\frac{t^8}{270}$	0
		6	$\frac{121\,t}{30}-\frac{2101\,t^2}{360}-\frac{115\,t^3}{144}+\frac{3581\,t^4}{720}-\frac{2249\,t^5}{720}+\frac{617\,t^6}{720}-\frac{t^7}{9}+\frac{t^8}{180}$	0
		$\frac{11}{2}$	$-\frac{94\,375\,936\,t}{36\,891\,855}+\frac{22\,689\,584\,128\,t^2}{6\,087\,156\,075}+\frac{12\,495\,872\,t^3}{26\,351\,325}-\frac{23\,531\,264\,t^4}{7\,432\,425}+\frac{23\,314\,432\,t^5}{11\,594\,583}-\frac{23\,094\,784\,t^6}{41\,409\,225}+\frac{11\,420\,672\,t^7}{156\,080\,925}-\frac{22\,545\,664\,t^8}{6\,087\,156\,075}$	$\frac{512\,t}{819}-\frac{123\,776\,t^2}{135\,135}-\frac{64\,t^3}{585}+\frac{128\,t^4}{165}-\frac{640\,t^5}{1287}+\frac{896\,t^6}{6435}-\frac{64\,t^7}{3465}+\frac{128\,t^8}{135\,135}$
		7	1	0

Table 2 : The Discrete Coefficients for the CHLMM in (2).

k	t	v	β_v	α_7	α_6	α_5	α_4	α_3	α_2	α_v	α_1	α_0
1	0	$\frac{1}{2}$	1	0	0	0	0	0	0	0	1	1
2	1	$\frac{3}{2}$	$\frac{4}{3}$	0	0	0	0	0	1	$-\frac{8}{9}$	2	$-\frac{1}{9}$
3	2	$\frac{5}{2}$	$\frac{8}{5}$	0	0	0	0	1	3	$-\frac{128}{75}$	$-\frac{1}{3}$	$\frac{1}{25}$
4	3	$\frac{7}{2}$	$\frac{64}{35}$	0	0	0	1	4	$-\frac{2}{3}$	$-\frac{9088}{3675}$	$\frac{4}{25}$	$-\frac{1}{49}$
5	4	$\frac{9}{2}$	$\frac{128}{63}$	0	0	1	5	$-\frac{10}{9}$	$\frac{2}{5}$	$-\frac{63488}{19845}$	$-\frac{5}{49}$	$\frac{1}{81}$
6	5	$\frac{11}{2}$	$\frac{512}{231}$	0	1	6	$-\frac{5}{3}$	$\frac{4}{5}$	$-\frac{15}{49}$	$-\frac{3116032}{800415}$	$\frac{2}{27}$	$-\frac{1}{121}$
7	6	$\frac{13}{2}$	$\frac{1024}{429}$	1	7	$-\frac{7}{3}$	$\frac{7}{5}$	$-\frac{5}{7}$	$\frac{7}{27}$	$-\frac{88113152}{19324305}$	$-\frac{7}{121}$	$\frac{1}{169}$

Appendix B

Table 3 :
The Continuous Coefficients of the Hybrid Predictor in (3) for k ≤ 7.

k	t	j	$\alpha_j(t)$	$\beta_j(t)$
1	$-\frac{1}{2}$	0	t^2	0
		$\frac{1}{2}$	1	0
		1	$1-t^2$	$t+t^2$
2	$\frac{1}{2}$	0	$-\frac{t}{4}+\frac{t^2}{2}-\frac{t^3}{4}$	0
		1	$1-t-t^2+t^3$	0
		$\frac{3}{2}$	1	0
		2	$\frac{5t}{4}+\frac{t^2}{2}-\frac{3t^3}{4}$	$-\frac{t}{2}+\frac{t^3}{2}$
3	$\frac{3}{2}$	0	$-\frac{2t}{9}+\frac{4t^2}{9}-\frac{5t^3}{18}+\frac{t^4}{18}$	0
		1	$1-t-\frac{3t^2}{4}+t^3-\frac{t^4}{4}$	0
		2	$2t-\frac{3t^3}{2}+\frac{t^4}{2}$	0
		$\frac{5}{2}$	1	0
		3	$-\frac{7t}{9}+\frac{11t^2}{36}+\frac{7t^3}{9}-\frac{11t^4}{36}$	$\frac{t}{3}-\frac{t^2}{6}-\frac{t^3}{3}+\frac{t^4}{6}$
4	$\frac{5}{2}$	0	$-\frac{3t}{16}+\frac{13t^2}{32}-\frac{29t^3}{96}+\frac{3t^4}{32}-\frac{t^5}{96}$	0
		1	$1-\frac{7t}{6}-\frac{5t^2}{9}+\frac{10t^3}{9}-\frac{4t^4}{9}+\frac{t^5}{18}$	0
		2	$\frac{9t}{4}-\frac{3t^2}{8}-\frac{13t^3}{8}+\frac{7t^4}{8}-\frac{t^5}{8}$	0
		3	$-\frac{3t}{2}+t^2+\frac{4t^3}{3}-t^4+\frac{t^5}{6}$	0
		$\frac{7}{2}$	1	0
		4	$\frac{29t}{48}-\frac{137t^2}{288}-\frac{149t^3}{288}+\frac{137t^4}{288}-\frac{25t^5}{288}$	$-\frac{t}{4}+\frac{5t^2}{24}+\frac{5t^3}{24}-\frac{5t^4}{24}+\frac{t^5}{24}$
5	$\frac{7}{2}$	0	$-\frac{4t}{25}+\frac{28t^2}{75}-\frac{19t^3}{60}+\frac{t^4}{8}-\frac{7t^5}{300}+\frac{t^6}{600}$	0
		1	$1-\frac{4t}{3}-\frac{17t^2}{48}+\frac{115t^3}{96}-\frac{61t^4}{96}+\frac{13t^5}{96}-\frac{t^6}{96}$	0
		2	$\frac{8t}{3}-\frac{8t^2}{9}-\frac{11t^3}{6}+\frac{49t^4}{36}-\frac{t^5}{3}+\frac{t^6}{36}$	0
		3	$-2t+\frac{5t^2}{3}+\frac{37t^3}{24}-\frac{13t^4}{8}+\frac{11t^5}{24}-\frac{t^6}{24}$	0
		4	$\frac{4t}{3}-\frac{4t^2}{3}-\frac{11t^3}{12}+\frac{31t^4}{24}-\frac{5t^5}{12}+\frac{t^6}{24}$	0
		$\frac{9}{2}$	1	0
		5	$-\frac{38t}{75}+\frac{1931t^2}{3600}+\frac{157t^3}{480}-\frac{149t^4}{288}+\frac{431t^5}{2400}-\frac{137t^6}{7200}$	$\frac{t}{5}-\frac{13t^2}{60}-\frac{t^3}{8}+\frac{5t^4}{24}-\frac{3t^5}{40}+\frac{t^6}{120}$

Continuation of Table 3

k	t	j	$\alpha_j(t)$	$\beta_j(t)$
6	$\frac{9}{2}$	0	$-\frac{5t}{36}+\frac{149t^2}{432}-\frac{1399t^3}{4320}+\frac{65t^4}{432}-\frac{t^5}{27}+\frac{t^6}{216}-\frac{t^7}{4320}$	0
		1	$1-\frac{89t}{60}-\frac{91t^2}{600}+\frac{749t^3}{600}-\frac{49t^4}{60}+\frac{7t^5}{30}-\frac{19t^6}{600}+\frac{t^7}{600}$	0
		2	$\frac{25t}{8}-\frac{145t^2}{96}-\frac{127t^3}{64}+\frac{23t^4}{12}-\frac{61t^5}{96}+\frac{3t^6}{32}-\frac{t^7}{192}$	0
		3	$-\frac{25t}{9}+\frac{295t^2}{108}+\frac{193t^3}{108}-\frac{139t^4}{54}+\frac{53t^5}{54}-\frac{17t^6}{108}+\frac{t^7}{108}$	0
		4	$\frac{25t}{12}-\frac{115t^2}{48}-\frac{107t^3}{96}+\frac{107t^4}{48}-\frac{23t^5}{24}+\frac{t^6}{6}-\frac{t^7}{96}$	0
		5	$-\frac{5t}{4}+\frac{37t^2}{24}+\frac{23t^3}{40}-\frac{17t^4}{12}+\frac{2t^5}{3}-\frac{t^6}{8}+\frac{t^7}{120}$	0
		$\frac{11}{2}$	1	0
		6	$\frac{53t}{120}-\frac{4033t^2}{7200}-\frac{2701t^3}{14\,400}+\frac{23t^4}{45}-\frac{361t^5}{1440}+\frac{353t^6}{7200}-\frac{49t^7}{14\,400}$	$-\frac{t}{6}+\frac{77t^2}{360}+\frac{49t^3}{720}-\frac{7t^4}{36}+\frac{7t^5}{72}-\frac{7t^6}{360}+\frac{t^7}{720}$

k	t	j	$\alpha_j(t)$	$\beta_j(t)$
7	$\frac{11}{2}$	0	$-\frac{6t}{49}+\frac{157t^2}{490}-\frac{137t^3}{420}+\frac{431t^4}{2520}-\frac{17t^5}{336}+\frac{43t^6}{5040}-\frac{3t^7}{3920}+\frac{t^8}{35\,280}$	0
		1	$1-\frac{97t}{60}+\frac{17t^2}{360}+\frac{2737t^3}{2160}-\frac{4249t^4}{4320}+\frac{371t^5}{1080}-\frac{137t^6}{2160}+\frac{13t^7}{2160}-\frac{t^8}{4320}$	0
		2	$\frac{18t}{5}-\frac{111t^2}{50}-\frac{41t^3}{20}+\frac{1507t^4}{600}-\frac{247t^5}{240}+\frac{83t^6}{400}-\frac{t^7}{48}+\frac{t^8}{1200}$	0
		3	$-\frac{15t}{4}+\frac{67t^2}{16}+\frac{23t^3}{12}-\frac{2185t^4}{576}+\frac{43t^5}{24}-\frac{113t^6}{288}+\frac{t^7}{24}-\frac{t^8}{576}$	0
		4	$\frac{10t}{3}-\frac{77t^2}{18}-\frac{137t^3}{108}+\frac{821t^4}{216}-\frac{869t^5}{432}+\frac{205t^6}{432}-\frac{23t^7}{432}+\frac{t^8}{432}$	0
		5	$-\frac{9t}{4}+\frac{123t^2}{40}+\frac{53t^3}{80}-\frac{1289t^4}{480}+\frac{37t^5}{24}-\frac{31t^6}{80}+\frac{11t^7}{240}-\frac{t^8}{480}$	0
		6	$\frac{6t}{5}-\frac{17t^2}{10}-\frac{17t^3}{60}+\frac{527t^4}{360}-\frac{71t^5}{80}+\frac{169t^6}{720}-\frac{7t^7}{240}+\frac{t^8}{720}$	0
		$\frac{13}{2}$	1	
		7	$-\frac{1159t}{2940}+\frac{100\,133t^2}{176\,400}+\frac{103t^3}{1260}-\frac{48\,901t^4}{100\,800}+\frac{761t^5}{2520}-\frac{4133t^6}{50\,400}+\frac{37t^7}{3528}-\frac{121t^8}{235\,200}$	$\frac{t}{7}-\frac{29t^2}{140}-\frac{t^3}{36}+\frac{127t^4}{720}-\frac{t^5}{9}+\frac{11t^6}{360}-\frac{t^7}{252}+\frac{t^8}{5040}$

Table 4 : The Discrete Coefficients for the Hybrid Predictor in (3).

k	t	v	β_k	α_v	α_7	α_6	α_5	α_4	α_3	α_2	α_1	α_0
1	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{4}$	1	0	0	0	0	0	0	$\frac{3}{4}$	$\frac{1}{4}$
2	$\frac{1}{2}$	$\frac{3}{2}$	$-\frac{3}{16}$	1	0	0	0	0	0	$\frac{21}{32}$	$\frac{3}{8}$	$-\frac{1}{32}$
3	$\frac{3}{2}$	$\frac{5}{2}$	$-\frac{5}{32}$	1	0	0	0	0	$\frac{115}{192}$	$\frac{15}{32}$	$-\frac{5}{64}$	$\frac{1}{96}$
4	$\frac{5}{2}$	$\frac{7}{2}$	$-\frac{35}{256}$	1	0	0	0	$\frac{1715}{3072}$	$\frac{35}{64}$	$-\frac{35}{256}$	$\frac{7}{192}$	$-\frac{5}{1024}$
5	$\frac{7}{2}$	$\frac{9}{2}$	$-\frac{63}{512}$	1	0	0	$\frac{5397}{10240}$	$\frac{315}{512}$	$-\frac{105}{512}$	$\frac{21}{256}$	$-\frac{45}{2048}$	$\frac{7}{2560}$
6	$\frac{9}{2}$	$\frac{11}{2}$	$-\frac{231}{2048}$	1	0	$\frac{20559}{40960}$	$\frac{693}{1024}$	$-\frac{1155}{4096}$	$\frac{77}{512}$	$-\frac{495}{8192}$	$\frac{77}{5120}$	$-\frac{7}{4096}$
7	$\frac{11}{2}$	$\frac{13}{2}$	$-\frac{429}{4096}$	1	$\frac{275847}{573440}$	$\frac{3003}{4096}$	$-\frac{3003}{8192}$	$\frac{1001}{4096}$	$-\frac{2145}{16384}$	$\frac{1001}{20480}$	$-\frac{91}{8192}$	$\frac{33}{28672}$

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