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CHEN INEQUALITIES FOR SUBMANIFOLDS OF A COSYMPLECTIC SPACE FORM WITH A SEMI-SYMMETRIC METRIC CONNECTION

BY

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Abstract. In this paper, we prove Chen inequalities for submanifolds of a cosymplectic space form of constant φ-sectional curvature $N^{2m+1}(c)$ endowed with a semi-symmetric metric connection, i.e., relations between the mean curvature associated with the semi-symmetric metric connection, scalar and sectional curvatures, k-Ricci curvature and the sectional curvature of the ambient space.

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1. Introduction

The idea of a semi-symmetric linear connection on a differentiable manifold was introduced by FRIEDMANN and SCHOUTENN in [10]. The notion of a semi-symmetric metric connection on a Riemannian manifold was introduced by HAYDEN in [11]. Later in [22], YANO studied some properties of a Riemannian manifold endowed with a semi-symmetric metric connection. In the case of hypersurfaces, in [12] and [13], IMAI found some properties of a Riemannian manifold and a hypersurface of a Riemannian manifold with a semi-symmetric metric connection, respectively. In [20], NAKAO studied submanifolds of a Riemannian manifold with semi-symmetric connections.

In [5], CHEN recalled that one of the basic interests of submanifold theory is to establish simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold. Many famous results in differential geometry can be regarded as results in this respect. The main extrinsic invariant is the squared mean curvature and the main intrinsic invariants include the classical curvature invariants namely the scalar curvature, the sectional curvature or the Ricci curvature. There are also other important modern intrinsic invariants of submanifolds introduced by Chen [8].

Afterwards, many geometers studied similar problems for different submanifolds in various ambient spaces, for example see [3], [4], [7], [9], [16], [17] and [21].

In [14] and [23], submanifolds of cosymplectic space forms satisfying Chen's inequalities were studied.

Recently, in [18] and [19], the first author and Mihai proved Chen inequalities for submanifolds of real space forms with a semi-symmetric metric connection and Chen inequalities for submanifolds of complex space forms and Sasakian space forms endowed with semi-symmetric metric connections, respectively.

Motivated by the studies of the above authors, in this study, we consider Chen inequalities for submanifolds in cosymplectic space forms of constant φ-sectional curvature $N^{2m+1}(c)$ endowed with a semi-symmetric metric connection.

2. Semi-symmetric metric connection

Let N^{n+p} be an $(n+p)$-dimensional Riemannian manifold and $\widetilde{\nabla}$ a linear connection on N^{n+p}. If the torsion tensor $\widetilde{T}$ of $\widetilde{\nabla}$, defined by

$$\widetilde{T}(\widetilde{X},\widetilde{Y}) = \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} \;-\; \widetilde{\nabla}_{\widetilde{Y}}\widetilde{X} \;-\; [\widetilde{X},\widetilde{Y}],$$

for any vector fields $\widetilde{X}$ and $\widetilde{Y}$ on N^{n+p}, satisfies

$$\widetilde{T}(\widetilde{X},\widetilde{Y}) = \omega(\widetilde{Y})\widetilde{X} - \omega(\widetilde{X})\widetilde{Y}$$

for a 1-form ω, then the connection $\widetilde{\nabla}$ is called a *semi-symmetric connection.*

Let g be a Riemannian metric on N^{n+p}. If $\widetilde{\nabla}g = 0$, then $\widetilde{\nabla}$ is called a *semi-symmetric metric connection* on N^{n+p}.

A semi-symmetric metric connection $\widetilde{\nabla}$ on N^{n+p} is given by

$$\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} = \overset{\circ}{\widetilde{\nabla}}_{\widetilde{X}}\widetilde{Y} + \omega(\widetilde{Y})\widetilde{X} - g(\widetilde{X},\widetilde{Y})U,$$

for any vector fields $\widetilde{X}$ and $\widetilde{Y}$ on N^{n+p}, where $\overset{\circ}{\widetilde{\nabla}}$ denotes the Levi-Civita connection with respect to the Riemannian metric g and U is a vector field defined by $g(U,\widetilde{X}) = \omega(\widetilde{X})$, for any vector field $\widetilde{X}$ [22].

We will consider a Riemannian manifold N^{n+p} endowed with a semi-symmetric metric connection $\widetilde{\nabla}$ and the Levi-Civita connection denoted by $\overset{\circ}{\widetilde{\nabla}}$.

Let M^n be an n-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold N^{n+p}. On the submanifold M^n we consider the induced semi-symmetric metric connection denoted by ∇ and the induced Levi-Civita connection denoted by $\overset{\circ}{\nabla}$.

Let $\widetilde{R}$ be the curvature tensor of N^{n+p} with respect to $\widetilde{\nabla}$ and $\overset{\circ}{\widetilde{R}}$ the curvature tensor of N^{n+p} with respect to $\overset{\circ}{\widetilde{\nabla}}$. We also denote by R and $\overset{\circ}{R}$ the curvature tensors of ∇ and $\overset{\circ}{\nabla}$, respectively, on M^n.

The Gauss formulas with respect to ∇, respectively $\overset{\circ}{\nabla}$ can be written as:

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad X,Y \in \chi(M),$$
$$\overset{\circ}{\widetilde{\nabla}}_X Y = \overset{\circ}{\nabla}_X Y + \overset{\circ}{h}(X,Y), \quad X,Y \in \chi(M),$$

where $\overset{\circ}{h}$ is the second fundamental form of M^n in N^{n+p} and h is a $(0,2)$-tensor on M^n. According to the formula (7) from [20] h is also symmetric. The Gauss equation for the submanifold M^n into an $(n+p)$-dimensional Riemannian manifold N^{n+p} is

$$\begin{aligned}\overset{\circ}{\widetilde{R}}(X,Y,Z,W) &= \overset{\circ}{R}(X,Y,Z,W) + g(\overset{\circ}{h}(X,Z),\overset{\circ}{h}(Y,W)) \\ &\quad - g(\overset{\circ}{h}(X,W),\overset{\circ}{h}(Y,Z)).\end{aligned} \tag{2.1}$$

One denotes by $\overset{\circ}{H}$ the mean curvature vector of M^n in N^{n+p}.

Then the curvature tensor $\widetilde{R}$ with respect to the semi-symmetric metric connection $\widetilde{\nabla}$ on N^{n+p} can be written as (see [13])

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) &= \overset{\circ}{\widetilde{R}}(X,Y,Z,W) - \alpha(Y,Z)g(X,W) \\ &\quad + \alpha(X,Z)g(Y,W) - \alpha(X,W)g(Y,Z) + \alpha(Y,W)g(X,Z),\end{aligned} \tag{2.2}$$

for any vector fields $X, Y, Z, W \in \chi(M^n)$, where α is a $(0,2)$-tensor field defined by

$$\alpha(X,Y) = (\overset{\circ}{\tilde{\nabla}}_X \omega)Y - \omega(X)\omega(Y) + \frac{1}{2}\omega(P)g(X,Y), \quad \forall X, Y \in \chi(M).$$

Denote by λ the trace of α.

Let $\pi \subset T_xM^n$, $x \in M^n$, be a 2-plane section. Denote by $K(\pi)$ the sectional curvature of M^n with respect to the induced semi-symmetric metric connection ∇. For any orthonormal basis $\{e_1, ..., e_m\}$ of the tangent space T_xM^n, the scalar curvature τ at x is defined by

$$\tau(x) = \sum_{1 \le i < j \le n} K(e_i \wedge e_j).$$

Recall that the *Chen first invariant* is given by

$$\delta_M(x) = \tau(x) - \inf\{K(\pi) \mid \pi \subset T_xM^n, x \in M^n, \dim \pi = 2\},$$

(see, for example, [8]), where M^n is a Riemannian manifold, $K(\pi)$ is the sectional curvature of M^n associated with a 2-plane section, $\pi \subset T_xM^n, x \in M^n$ and τ is the scalar curvature at x.

The following algebraic Lemma is well-known.

Lemma 2.1 ([5]). *Let $a_1, a_2, ..., a_n, b$ be $(n+1)$ $(n \ge 2)$ real numbers such that*

$$\left(\sum_{i=1}^n a_i\right)^2 = (n-1)\left(\sum_{i=1}^n a_i^2 + b\right).$$

Then $2a_1a_2 \ge b$, with equality holding if and only if $a_1 + a_2 = a_3 = ... = a_n$.

Let M^n be an n-dimensional Riemannian manifold, L a k-plane section of T_xM^n, $x \in M^n$, and X a unit vector in L.

We choose an orthonormal basis $\{e_1, ..., e_k\}$ of L such that $e_1 = X$.

One defines [7] the *Ricci curvature* (or *k-Ricci curvature*) of L at X by

$$Ric_L(X) = K_{12} + K_{13} + ... + K_{1k},$$

where K_{ij} denotes, as usual, the sectional curvature of the 2-plane section spanned by e_i, e_j. For each integer k, $2 \le k \le n$, the Riemannian invariant Θ_k on M^n is defined by:

$$\Theta_k(x) = \frac{1}{k-1} \inf_{L,X} Ric_L(X), \quad x \in M^n,$$

where L runs over all k-plane sections in T_xM^n and X runs over all unit vectors in L.

3. Chen first inequality for submanifolds of cosymplectic manifolds

Let N^{2m+1} be a $(2m+1)$-dimensional almost contact manifold endowed with an almost contact structure (φ, ξ, η), that is, φ is a $(1,1)$-tensor field, ξ is a vector field and η is 1-form such that $\varphi^2 X = -X + \eta(X)\xi$, $\eta(\xi) = 1$. Then, $\varphi\xi = 0$ and $\eta \circ \varphi = 0$. The almost contact structure is said to be *normal* if the induced almost complex structure J on the product manifold $N \times \mathbb{R}$ defined by $J(X, \lambda\frac{d}{dt}) = (\varphi X - \lambda\xi, \eta(X)\frac{d}{dt})$ is integrable, where X is tangent to N, t the coordinate of $\mathbb{R}$ and λ a smooth function on $N \times \mathbb{R}$. The condition for being normal is equivalent to vanishing of the torsion tensor $[\varphi, \varphi] + 2d\eta \otimes \xi$, where $[\varphi, \varphi]$ is the Nijenhuis tensor of φ.

Let g be a compatible Riemannian metric with (φ, ξ, η), that is, $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$ or equivalently, $\Phi(X, Y) = g(X, \varphi Y) = -g(\varphi X, Y)$ and $g(X, \xi) = \eta(X)$ for all $X, Y \in TN$. Then N becomes an almost contact metric manifold equipped with an almost contact metric structure (φ, ξ, η, g). If $\Phi = d\eta$, the almost contact structure is a contact structure. A normal contact structure such that the fundamental 2-form Φ and 1-form η are closed is called a *cosymplectic structure*. It can be shown that the cosymplectic structure is characterized by

$$\overset{\circ}{\widetilde{\nabla}}_X \varphi = 0 \quad \text{and} \quad \overset{\circ}{\widetilde{\nabla}}_X \eta = 0, \tag{3.1}$$

(see [2]). From formula (3.1) it follows that $\overset{\circ}{\widetilde{\nabla}}_X \xi = 0$.

A cosymplectic manifold N^{2m+1} is said to be a *cosymplectic space form* [15] if the φ-sectional curvature is constant c along N^{2m+1} . A cosymplectic space form will be denoted by $N^{2m+1}(c)$. Then the curvature tensor $\widetilde{R}$ of $N^{2m+1}(c)$ can be expressed as

$$\begin{aligned} \overset{\circ}{\widetilde{R}}(X,Y,Z,W) = \frac{c}{4}[&g(X,W)g(Y,Z) - g(X,Z)g(Y,W) \\ &+ g(X,\varphi W)g(Y,\varphi Z) - g(X,\varphi Z)g(Y,\varphi W) - 2g(X,\varphi Y)g(Z,\varphi W) \\ &- \eta(Y)\eta(Z)g(X,W) + \eta(Y)\eta(W)g(X,Z) \\ &-\eta(X)\eta(W)g(Y,Z) + \eta(X)\eta(Z)g(Y,W)]\,. \end{aligned} \tag{3.2}$$

If $N^{2m+1}(c)$ is a cosymplectic space form of constant φ-sectional curvature c with a semi-symmetric metric connection then from (2.2) and (3.2) it follows that

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) &= \frac{c}{4}[g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\\ &+ g(X,\varphi W)g(Y,\varphi Z) - g(X,\varphi Z)g(Y,\varphi W) - 2g(X,\varphi Y)g(Z,\varphi W)\\ &- \eta(Y)\eta(Z)g(X,W) + \eta(Y)\eta(W)g(X,Z)\\ &-\eta(X)\eta(W)g(Y,Z) + \eta(X)\eta(Z)g(Y,W)]\\ &- \alpha(Y,Z)g(X,W) + \alpha(X,Z)g(Y,W)\\ &- \alpha(X,W)g(Y,Z) + \alpha(Y,W)g(X,Z).\end{aligned} \tag{3.3}$$

Let $M^n, n \geq 3$, be an n-dimensional submanifold of an $(2m+1)$-dimensional cosymplectic manifold $N^{n+p}(c)$ of constant φ-sectional curvature c. For any tangent vector field X to M^n, we put

$$\varphi X = PX + FX,$$

where PX and FX are tangential and normal components of φX, respectively and we decompose

$$\xi = \xi^{\top} + \xi^{\perp},$$

where $\xi^{\top}$ and $\xi^{\perp}$denotes the tangential and normal parts of ξ.

Denote by $\Theta^2(\pi) = g^2(Pe_1, e_2)$, where $\{e_1, e_2\}$ is an orthonormal basis of a 2-plane section π, is a real number in $[0,1]$, independent of the choice of e_1, e_2 (see [1]).

For submanifolds of a cosymplectic space form $N^{2m+1}(c)$ of constant φ-sectional curvature c endowed with a semi-symmetric metric connection we establish the following optimal inequality.

Theorem 3.1. *Let $M^n, n \geq 3$, be an n-dimensional submanifold of an $(2m+1)$-dimensional cosymplectic space form $N^{2m+1}(c)$ of constant φ-sectional curvature c endowed with a semi-symmetric metric connection $\widetilde{\nabla}$. We have:*

$$\begin{aligned}\tau(x) - K(\pi) &\leq (n-2)\left[\frac{n^2}{2(n-1)}\|H\|^2 + (n+1)\frac{c}{8} - \lambda\right]\\ &- \frac{c}{4}(3\Theta^2(\pi) - \frac{3}{2}\|P\|^2 + (n-1)\|\xi^{\top}\|^2 - \|\xi_\pi\|^2) - \operatorname{trace}(\alpha_{|_{\pi^\perp}}),\end{aligned} \tag{3.4}$$

where π is a 2-plane section of $T_xM^n, x \in M^n$.

Proof. From [20], the Gauss equation with respect to the semi-symmetric metric connection is

$$\widetilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(h(X,Z),h(Y,W)) - g(h(Y,Z),h(X,W)). \tag{3.5}$$

Let $x \in M^n$ and $\{e_1, e_2, ..., e_n\}$ and $\{e_{n+1}, ..., e_{2m+1}\}$ be orthonormal basis of T_xM^n and $T_x^\perp M^n$, respectively. For $X = W = e_i, Y = Z = e_j$, $i \neq j$, from the equation (3.3) it follows that:

$$\tilde{R}(e_i,e_j,e_j,e_i) = \frac{c}{4} + \frac{3c}{4}g^2(Pe_j,e_i) - \frac{c}{4}\left\{\eta(e_i)^2 + \eta(e_j)^2\right\} - \alpha(e_i,e_i) - \alpha(e_j,e_j). \tag{3.6}$$

From (3.5) and (3.6) we get

$$\frac{c}{4} + \frac{3c}{4}g^2(Pe_j,e_i) - \frac{c}{4}\left\{\eta(e_i)^2 + \eta(e_j)^2\right\} - \alpha(e_i,e_i) - \alpha(e_j,e_j) = R(e_i,e_j,e_j,e_i) + g(h(e_i,e_j),h(e_i,e_j)) - g(h(e_i,e_i),h(e_j,e_j)).$$

By summation after $1 \leq i,j \leq n$, it follows from the previous relation that

$$2\tau + \|h\|^2 - n^2\|H\|^2 = -2(n-1)\lambda + (n^2-n)\frac{c}{4} + \frac{3c}{4}\|P\|^2 - \frac{c}{2}(n-1)\|\xi^\top\|^2. \tag{3.7}$$

We take

$$\varepsilon = 2\tau - \frac{n^2(n-2)}{n-1}\|H\|^2 + 2(n-1)\lambda - (n^2-n)\frac{c}{4} - \frac{3c}{4}\|P\|^2 + \frac{c}{2}(n-1)\|\xi^\top\|^2. \tag{3.8}$$

Then, from (3.7) and (3.8) we get

$$n^2\|H\|^2 = (n-1)\left(\|h\|^2 + \varepsilon\right). \tag{3.9}$$

Let $x \in M^n$, $\pi \subset T_xM^n$, $\dim\pi = 2$, $\pi = span\{e_1, e_2\}$. We define $e_{n+1} = \frac{H}{\|H\|}$ and from the relation (3.9) we obtain:

$$(\sum_{i=1}^{n} h_{ii}^{n+1})^2 = (n-1)(\sum_{i,j=1}^{n}\sum_{r=n+1}^{2m+1}(h_{ij}^r)^2 + \varepsilon),$$

or equivalently,

$$(\sum_{i=1}^{n} h_{ii}^{n+1})^2 = (n-1)\left[\sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{i,j=1}^{n}\sum_{r=n+2}^{2m+1}(h_{ij}^{r})^2 + \varepsilon\right].$$

By using the algebraic Lemma we have from the previous relation

$$2h_{11}^{n+1}h_{22}^{n+1} \geq \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{i,j=1}^{n}\sum_{r=n+2}^{2m+1}(h_{ij}^{r})^2 + \varepsilon.$$

If we denote by $\xi_\pi = pr_\pi \xi$ we can write (see [19])

$$-\eta(e_1)^2 - \eta(e_2)^2 = -\left\|\xi_\pi\right\|^2.$$

The Gauss equation for $X = W = e_1, Y = Z = e_2$ gives

$$\begin{aligned}
K(\pi) = R(e_1,e_2,e_2,e_1) &= \frac{c}{4} + \frac{3c}{4}g^2(Pe_1,e_2) - \frac{c}{4}\left\|\xi_\pi\right\|^2 \\
&- \alpha(e_1,e_1) - \alpha(e_2,e_2) + \sum_{r=n+1}^{2m+1}[h_{11}^r h_{22}^r - (h_{12}^r)^2] \\
&\geq \frac{c}{4} + \frac{3c}{4}g^2(Pe_1,e_2) - \frac{c}{4}\left\|\xi_\pi\right\|^2 - \alpha(e_1,e_1) - \alpha(e_2,e_2) \\
&+ \frac{1}{2}[\sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{i,j=1}^{n}\sum_{r=n+2}^{2m+1}(h_{ij}^r)^2 + \varepsilon] \\
&+ \sum_{r=n+2}^{2m+1} h_{11}^r h_{22}^r - \sum_{r=n+1}^{2m+1}(h_{12}^r)^2 \\
&= \frac{c}{4} + \frac{3c}{4}g^2(Pe_1,e_2) - \frac{c}{4}\left\|\xi_\pi\right\|^2 - \alpha(e_1,e_1) - \alpha(e_2,e_2) \\
&+ \frac{1}{2}\sum_{i\neq j}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{r=n+2}^{2m+1}(h_{ij}^r)^2 + \frac{1}{2}\varepsilon \\
&+ \sum_{r=n+2}^{2m+1} h_{11}^r h_{22}^r - \sum_{r=n+1}^{2m+1}(h_{12}^r)^2 \\
&= \frac{c}{4} + \frac{3c}{4}g^2(Pe_1,e_2) - \frac{c}{4}\left\|\xi_\pi\right\|^2 - \alpha(e_1,e_1) - \alpha(e_2,e_2)
\end{aligned}$$

$$+\frac{1}{2}\sum_{i\neq j}(h_{ij}^{n+1})^2+\frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{i,j>2}(h_{ij}^r)^2+\frac{1}{2}\sum_{r=n+2}^{2m+1}(h_{11}^r+h_{22}^r)^2$$
$$+\sum_{j>2}[(h_{1j}^{n+1})^2+(h_{2j}^{n+1})^2]+\frac{1}{2}\varepsilon$$
$$\geq\frac{c}{4}+\frac{3c}{4}g^2(Pe_1,e_2)-\frac{c}{4}\|\xi_\pi\|^2-\alpha(e_1,e_1)-\alpha(e_2,e_2)+\frac{\varepsilon}{2},$$

which implies

$$K(\pi)\geq\frac{c}{4}+\frac{3c}{4}g^2(Pe_1,e_2)-\frac{c}{4}\|\xi_\pi\|^2-\alpha(e_1,e_1)-\alpha(e_2,e_2)+\frac{\varepsilon}{2}.$$

Denote by

$$\alpha(e_1,e_1)+\alpha(e_2,e_2)=\lambda-\operatorname{trace}(\alpha_{|_{\pi^\perp}}),$$

(see [19]). From (3.8) it follows

$$K(\pi)\geq\tau-(n-2)\left[\frac{n^2}{2(n-1)}\|H\|^2+(n+1)\frac{c}{8}-\lambda\right]$$
$$+\frac{c}{4}\left(3\Theta^2(\pi)-\frac{3}{2}\|P\|^2+(n-1)\|\xi^\top\|^2-\|\xi_\pi\|^2\right)+\operatorname{trace}(\alpha_{|_{\pi^\perp}}),$$

which represents the inequality to prove. □

Corollary 3.2. *Under the same assumptions as in Theorem* 3.1 *if* ξ *is tangent to* M^n*, we have*

$$\tau(x)-K(\pi)\leq(n-2)\left[\frac{n^2}{2(n-1)}\|H\|^2+(n+1)\frac{c}{8}-\lambda\right]$$
$$-\frac{c}{4}\left(3\Theta^2(\pi)-\frac{3}{2}\|P\|^2+n-1-\|\xi_\pi\|^2\right)-\operatorname{trace}(\alpha_{|_{\pi^\perp}}).$$

If ξ *is normal to* M^n*, we have*

$$\tau(x)-K(\pi)\leq(n-2)\left[\frac{n^2}{2(n-1)}\|H\|^2+(n+1)\frac{c}{8}-\lambda\right]$$
$$-\frac{c}{4}\left(3\Theta^2(\pi)-\frac{3}{2}\|P\|^2\right)-\operatorname{trace}(\alpha_{|_{\pi^\perp}}).$$

Recall the following important result (Proposition 1.2) from [12].

Proposition 3.3. *The mean curvature H of M^n with respect to the semi-symmetric metric connection coincides with the mean curvature $\overset{\circ}{H}$ of M^n with respect to the Levi-Civita connection if and only if the vector field U is tangent to M^n.*

Remark 3.4. According to the formula (7) from [20] (see also Proposition 3.3), it follows that $h = \overset{\circ}{h}$ if U is tangent to M^n. In this case inequality (3.4) becomes

$$\tau(x) - K(\pi) \leq (n-2)\left[\frac{n^2}{2(n-1)}\left\|\overset{\circ}{H}\right\|^2 + (n+1)\frac{c}{8} - \lambda\right] - \frac{c}{4}\left(3\Theta^2(\pi) - \frac{3}{2}\|P\|^2 + (n-1)\|\xi^\top\|^2 - \|\xi_\pi\|^2\right) - \operatorname{trace}(\alpha_{|_{\pi^\perp}}).$$

Proposition 3.5. *If the vector field U is tangent to M^n, then the equality case of inequality (3.4) holds at a point $x \in M^n$ if and only if there exists an orthonormal basis $\{e_1, e_2, ..., e_n\}$ of T_xM^n and an orthonormal basis $\{e_{n+1}, ..., e_{n+p}\}$ of $T_x^\perp M^n$ such that the shape operators of M^n in $N^{2m+1}(c)$ at x have the following forms:*

$$A_{e_{n+1}} = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & \cdots & 0 \\ 0 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu \end{pmatrix}, \quad a + b = \mu,$$

$$A_{e_r} = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 & \cdots & 0 \\ h_{12}^r & -h_{11}^r & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad n+2 \leq i \leq 2m+1,$$

where we denote by $h_{ij}^r = g(h(e_i, e_j), e_r)$, $1 \leq i, j \leq n$ and $n+2 \leq r \leq 2m+1$.

Proof. The equality case holds at a point $x \in M^n$ if and only if it achieves the equality in all the previous inequalities and we have the equality in the Lemma.

$$
\begin{aligned}
&h_{ij}^{n+1} = 0, \forall i \neq j, i, j > 2,\\
&h_{ij}^{r} = 0, \forall i \neq j, i, j > 2, r = n+1, ..., 2m+1,\\
&h_{11}^{r} + h_{22}^{r} = 0, \forall r = n+2, ..., 2m+1,\\
&h_{1j}^{n+1} = h_{2j}^{n+1} = 0, \forall j > 2,\\
&h_{11}^{n+1} + h_{22}^{n+1} = h_{33}^{n+1} = ... = h_{nn}^{n+1}.
\end{aligned}
$$

We may chose $\{e_1, e_2\}$ such that $h_{12}^{n+1} = 0$ and we denote by $a = h_{11}^{r}$, $b = h_{22}^{r}$, $\mu = h_{33}^{n+1} = ... = h_{nn}^{n+1}$. □

It follows that the shape operators take the desired forms.

4. *k*-Ricci curvature for submanifolds of cosymplectic space forms

We first state a relationship between the sectional curvature of a submanifold M^n of a cosymplectic space form $N^{2m+1}(c)$ of constant φ-sectional curvature c endowed with a semi-symmetric metric connection $\widetilde{\nabla}$ and the squared mean curvature $\|H\|^2$. Using this inequality, we prove a relationship between the k-Ricci curvature of M^n (intrinsic invariant) and the squared mean curvature $\|H\|^2$ (extrinsic invariant), as another answer of the basic problem in submanifold theory which we have mentioned in the introduction.

In this section we suppose that the vector field U is tangent to M^n.

Theorem 4.1. *Let $M^n, n \geq 3$, be an n-dimensional submanifold of an $(2m+1)$-dimensional a cosymplectic space form $N^{2m+1}(c)$ of constant φ-sectional curvature c endowed with a semi-symmetric metric connection $\widetilde{\nabla}$ such that the vector field U is tangent to M^n. Then we have*

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} + \frac{2}{n}\lambda - \frac{c}{4} - \frac{3c}{4n(n-1)}\|P\|^2 + \frac{c}{2n}\|\xi^{\top}\|^2. \tag{4.1}$$

Proof. Let $x \in M^n$ and $\{e_1, e_2, ..., e_n\}$ and orthonormal basis of T_xM^n. The relation (3.7) is equivalent with

$$n^2\|H\|^2 = 2\tau + \|h\|^2 + 2(n-1)\lambda - (n^2-n)\frac{c}{4} - \frac{3c}{4}\|P\|^2 + \frac{c}{2}(n-1)\|\xi^{\top}\|^2. \tag{4.2}$$

We choose an orthonormal basis $\{e_1, ..., e_n, e_{n+1}, ..., e_{n+p}\}$ at x such that e_{n+1} is parallel to the mean curvature vector $H(x)$ and $e_1, ..., e_n$ diagonalize the shape operator $A_{e_{n+1}}$. Then the shape operators take the forms

$$A_{e_{n+1}} \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{pmatrix},$$

$$A_{e_r} = (h_{ij}^r),\ i, j = 1, ..., n; r = n+2, ..., 2m+1, \text{trace } A_{e_r} = 0.$$

From (4.2), we get

$$n^2 \|H\|^2 = 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{2m+1} \sum_{i,j=1}^{n} (h_{ij}^r)^2 + 2(n-1)\lambda$$
$$- (n^2 - n)\frac{c}{4} - \frac{3c}{4} \|P\|^2 + \frac{c}{2}(n-1)\|\xi^\top\|^2. \tag{4.3}$$

Since $\sum_{i=1}^{n} a_i^2 \geq n \|H\|^2$, hence we obtain

$$n^2 \|H\|^2 \geq 2\tau + n \|H\|^2 + 2(n-1)\lambda - (n^2 - n)\frac{c}{4} - \frac{3c}{4} \|P\|^2 + \frac{c}{2}(n-1)\|\xi^\top\|^2.$$

Last inequality represents (4.1). □

Using Theorem 4.1, we obtain the following result:

Theorem 4.2. *Let $M^n, n \geq 3$, be an n-dimensional submanifold of an $(2m+1)$-dimensional cosymplectic space form $N^{2m+1}(c)$ of constant φ-sectional curvature c endowed with a semi-symmetric metric connection $\widetilde{\nabla}$, such that the vector field U is tangent to M^n. Then, for any integer k, $2 \leq k \leq n$, and any point $x \in M^n$, we have*

$$\|H\|^2 (x) \geq \Theta_k(x) + \frac{2}{n}\lambda - \frac{c}{4} - \frac{3c}{4n(n-1)} \|P\|^2 + \frac{c}{2n}\|\xi^\top\|^2. \tag{4.4}$$

Proof. Let $\{e_1, ...e_n\}$ be an orthonormal basis of T_xM. Denote by $L_{i_1 \dots i_k}$ the k-plane section spanned by $e_{i_1}, ..., e_{i_k}$. By the definitions, one has

$$\tau(L_{i_1 \dots i_k}) = \frac{1}{2} \sum_{i \in \{i_1, \dots, i_k\}} Ric_{L_{i_1 \dots i_k}}(e_i), \tag{4.5}$$

$$\tau(x) = \frac{1}{C_{n-2}^{k-2}} \sum_{1 \leq i_1 < \dots < i_k \leq n} \tau(L_{i_1 \dots i_k}). \tag{4.6}$$

From (4.1), (4.5) and (4.6), one derives $\tau(x) \geq \frac{n(n-1)}{2}\Theta_k(x)$, which implies (4.4). $\square$

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