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SOME PROPERTIES OF EXTENSION AND DEFORMATION IN NONCOMMUTATIVE HOMOTOPY

BY

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Abstract. The generalized extension problem from the homotopy theory can be translated in the noncommutative language of arbitrary $C*$-algebras and $*$-homomorphisms, where a noncommutative homotopy already exists. In this paper we consider this subject in its general form: first as a problem of homotopy commutative diagram having one of the given arrows a cofibration $*$-homomorphism, and second as a retraction. Both variants of the problem are studied also in the unital case.

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1. Introduction

The extension problems in the (classical) homotopy theory are interesting and occupy the first pages of some books of Homotopy Theory or Algebraic Toplogy since they are not so complicated. Such a book is the book of Hu [5] in which a few paragraphs in the first chapter are dedicated to the problems of extension, deformation or lifting.

A generalized extension problem in homotopy theory can be represented by a diagram

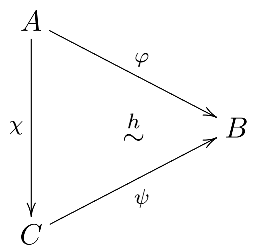
$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle g}\searrow \quad \simeq \quad \nearrow{\scriptstyle h} & \\ & Z & \end{array}$$

where f and g are given maps and h must be found such that $h \circ g \simeq f$.

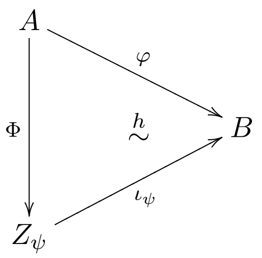
But this generalized problem is equivalent to the (homotopical) extension problem of the map f over the mapping cylinder $M_g \supset X$ of the map g ([5], p. 20). Moreover, it is shown that generalized problem is finally equivalent to a retraction problem, which is apparently the narrowest form of the problem ([5], p. 22).

Since the above generalized extension problem can be easily translated in the noncommutative language of arbitrary C*-algebras and $*$-homomorphisms, where a noncommutative homotopy already exists (see for example [3], [6]), in this Note we aim to consider this subject in the general form: at first as a problem of homotopy commutative diagram having one of given arrows a cofibration $*$-homomorphism, and then as a retraction one.

More concretely, we suppose that some C*-algebras A, B, C and two $*$-homomorphisms $\varphi : A \to B$, $\psi : C \to B$ are given. We shall prove (Theorem 4.1)that there exists a homotopy commutative diagram



if and only if there exists a homotopy commutative diagram



where Z_ψ is the C*-algebra mapping cylinder of the morphism ψ and ι_ψ is a cofibration $*$-homomorphism. And this is equivalent with the property of the C*-algebra A to be a "canonical retract" of the double mapping cylinder $Z_{(\varphi,\psi)}$ of the pair of $*$-homomorphisms (φ, ψ) (Theorem 4.3). Some corollaries are given. Incidentally we prove some new and old properties of mapping cylinders, among which that the double mapping cylinder is a weak homotopy pull-back of B along the pair of $*$-homomorphisms (φ, ψ) (Proposition 3.12). In the context, the notion of C*-triple mapping cylinder is introduced. An application refers to the Čerin's homotopy groups of C*-algebras (Corollary 5.5).

2. Homotopy in noncommutative language

We recall from [3] the definition of the homotopy relation between $*$-homomorphisms.

Given a C*-algebra A and a (locally) compact space Y, we denote by AY the C*-algebra of (vanishing at infinity) continuous functions of Y into A. If $\phi : A \to B$ is a $*$-homomorphism and Y is a (locally) compact space, then ϕ induces a $*$-homomorphism $\phi Y : AY \to BY$, by $(\phi Y)(u) = \phi \circ u, \forall u \in AY$.

If $Y = I = [0, 1]$, then for every $t \in I$, we denote by $\rho_t : AI \to A$ the $*$-homomorphism defined by $\rho_t(u) = u(t), \forall u \in AI$.

Definition 2.1 ([3], p. 16)**.** If A, B are two C*-algebras, two morphisms of C*-algebras $\eta : A \to B$ and $\phi : A \to B$ are homotopic, written $\eta \overset{h}{\sim} \phi$, if there is a $*$-homomorphism $\Psi : A \to BI$ such that $\rho_0 \circ \Psi = \eta$ and $\rho_1 \circ \Psi = \phi$.

A $*$-homomorphism $\eta : A \to B$ is called a homotopy equivalence when there is a morphism $\xi : B \to A$, such that $\xi \circ \eta$ and $\eta \circ \xi$ are homotopic to the respective identity maps of A and B.

Definition 2.2 ([9])**.** If $\eta : A \to B$ and $\xi : B \to A$ are two $*$-homomorphisms such that $\xi \circ \eta = id_A$ and $\eta \circ \xi \overset{h}{\sim} id_B$, by $*$-homotopy morphism $\Phi : B \to BI$ such that $\rho_t \circ \Phi \circ \eta = \eta, \forall t \in I$, the C*-algebra A will be called a deformation retract of C*-algebra B.

3. Mapping cylinders for $*$-homomorphisms and cofibrations of C*-algebras

For a pair of $*$-homomorphisms $\phi_1 : A_1 \to B$ and $\phi_2 : A_2 \to B$, denote by $A(B; \phi_1, \phi_2)$ the C*-algebra pull-back of B along (ϕ_1, ϕ_2), i.e., $A(B; \phi_1, \phi_2) := \{(a_1, a_2) \in A_1 \oplus A_2 : \phi_1(a_1) = \phi_2(a_2)\}$ ([3], p. 21).

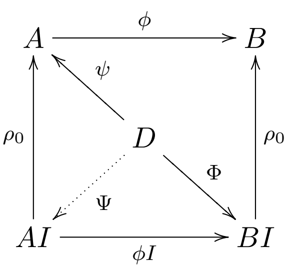
Definition 3.1 ([3], p. 23)**.** The mapping cylinder of the morphism $\phi : A \to B$ is defined as the pull-back $A(B; \phi, \rho_1)$, and it is denoted by Z_ϕ. Hence, $Z_\phi := \{(a, \beta) \in A \oplus BI : \phi(a) = \beta(1)\}$.

Example 3.2. Suppose that $f : X \to Y$ is a continuous map between compact spaces and let M_f be the mapping cylinder of f ([5], p. 18). Then there exists a $*$-isomorphisms between the C*-algebra $C(M_f)$, of all continuous functions defined on the space M_f, and $Z_{C(f)}$, where $C(f) : C(Y) \to C(X)$ is the natural $*$-homomorphism induced by f, $C(f)(u) = u \circ f, \forall u \in C(Y)$. The topological cylinder M_f can be written as $M_f = \{[x,t],[y] : [x,1] = [f(x)], x = inX, t \in I, y \in Y\}$, and C*-algebra $Z_{C(f)}$ is given by $Z_{C(f)} := \{(u,\alpha) \in C(Y) \oplus C(X)I : \alpha(1) = u \circ f\}$. We define $\Phi : Z_{C(f)} \to C(M_f)$ by taking $\Phi((u,\alpha))([x,t]) = \alpha(t)(x)$ and $\Phi((u,\alpha))([y]) = u(y)$. This map is well defined since $\Phi((u,\alpha))([x,1]) = \alpha(1)(x) = u(f(x))$, and it is immediate that it is a $*$-homomorphism. On the other hand, we can define $\Psi : C(M_f) \to Z_{C(f)}$, by taking for $U \in C(M_f)$, $\Psi(U) = (u,\alpha)$, with $u(y) = U([y]), \forall y \in Y$, and $\alpha(t)(x) = U([x,t]), \forall x \in X, \forall t \in I$. This is also well defined and it is a $*$-homomorphism. Now it is immediate that Φ and Ψ are reciprocally inverses. Thus we have obtained a C*-isomorphism $Z_{C(f)} \cong C(M_f)$. If X and Y are only locally compact spaces, this C*-isomorphism becomes $Z_{C_0(f)} \cong C_0(M_f)$.

Remark 3.3. In some papers (such as [9]) the term o mapping cylinder of ϕ is used for the C*-algebra $M_\phi := A(B;\phi,\rho_0)$ instead of Z_ϕ.

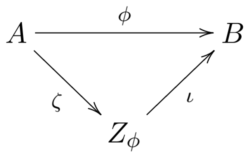
Definition 3.4 ([9])**.** A $*$-homomorphism $\phi : A \to B$ is called a cofibration if for an arbitrary C*-algebra D, and arbitrary $*$-homomorphism $\psi : D \to A$ and a homotopy $*$-homomorphism $\Phi : D \to BI$ for $\phi \circ \psi$, i.e., satisfying $\rho_o \circ \Phi = \phi \circ \psi$, there exists a homotopy $*$-homomorphism $\Psi : D \to AI$ for ψ, i.e., satisfying $\rho_0 \circ \Psi = \psi$, such that $\phi I \circ \Psi = \Phi$.

Schematically, Definition 3.4 can be illustrated by the following diagram.



Proposition 3.5 (see also [9], Corollary 1.9)**.** *For any $*$-homomorphism*

$\phi : A \to B$ there exists a commutative diagram



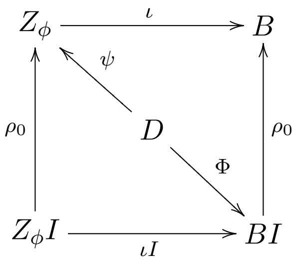
$\phi = \iota \circ \zeta$, with ζ a strong deformation retract $*$-homomorphism and ι a cofibration $*$-homomorphism.

Proof. Define $\zeta : A \to Z_\phi$ by $\zeta(a) = (a, \beta_{\phi(a)})$, with $\beta_{\phi(a)} \in BI$ the constant path $\beta_{\phi(a)}(t) = \phi(a), \forall t \in I, \forall a \in A$, and $\iota : Z_\phi \to B$ by $\iota((a,\beta)) = \beta(0)$. These maps are $*$-homomorphisms and $\phi = \iota \circ \zeta$.

Consider the projection morphism $\rho : Z_\phi \to A$, given by $\rho((a,\beta)) = a$. Then $\rho \circ \zeta = id_A$, and $(\zeta \circ \rho)((a,\beta)) = (a, \beta_{\phi(a)}), \forall (a,\beta) \in Z_\phi$.

Define a homotopy $*$-homomorphism $\Upsilon : Z_\phi \to Z_\phi I$, by $\Upsilon((a,\beta))(t) = (a, \beta_t)$, with $\beta_t \in BI$, the path defined by $\beta_t(\tau) = \beta(t\tau - t + 1), \tau \in I$. this is well defined since $\beta_t(1) = \beta(1) = \phi(a)$. Then $\Upsilon((a,\beta))(0) = (a,\beta_0) = (a, \beta_{\phi(a)}) = ((\zeta \circ \rho)((a,\beta))$, i.e., $\Upsilon \circ \rho_0 = \zeta \circ \rho$, and $\Upsilon((a,\beta))(1) = (a,\beta_1) = (a,\beta)$, i.e., $\Upsilon \circ \rho_1 = id_{Z_\phi}$, so that $\Upsilon : \zeta \circ \rho \overset{h}{\sim} id_{Z_\phi}$. Moreover, $(\rho_t \circ \Upsilon \circ \zeta)(a) = (\rho_t \circ \Upsilon)(a, \beta_{\phi(a)})) = \Upsilon((a, \beta_{\phi(a)}))(t) = (a, \beta_{\phi(a)}) = \zeta(a), \forall a \in A$, so that $\rho_t \circ \Upsilon \circ \zeta = \zeta, \forall t \in I$. Thus, we have obtained that $\zeta : A \to Z_\phi$ is a strong deformation retract $*$-homomorphism.

Now we prove that $\iota : Z_\phi \to B$ verifies the definition of a cofibration $*$-homomorphism. Suppose that the following commutative diagram is given



and we need to define a homotopy $*$-homomorphism $\Psi : D \to Z_\phi I$ for ψ.

If for $d \in D$ we have $\psi(d) = (a, u), a \in A, u \in BI$, with $u(1) = \phi(a)$, then $(\iota \circ \psi)(d) = u(0)$. On the other hand, $(\rho_0 \circ \Psi)(d) = \Psi(d)(0)$, hence we have $u(0) = \Psi(d)(0)$.

We shall define Ψ as $\Psi(d)(t) = (a, u_t)$, with $u_t \in BI, t \in I$, satisfying $u_t(1) = \phi(a)$, in order that $(a, u_t) \in Z_\phi$. Moreover, the condition $\rho_0 \circ \Psi = \psi$ implies $\Psi(d)(0) = (a, u_0)$, so that the equality $u_0 = u$ is necessary. An finally, since $\iota I \circ \Psi = \Phi$, we have $\iota I(\Psi(d))(t) = \Phi(d)(t) \Rightarrow \iota(\Psi(d))(t) = \Psi(d)(t)$, so that is also necessary that the condition $u_t(0) = \Phi(d)(t)$ be fulfilled.

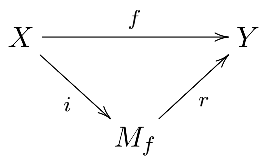
Then the underlined conditions are all satisfied by the path

$$u_t(\tau) = \begin{cases} \Phi(d)((t - 2\tau)), & \text{if } 0 \le \tau \le \frac{t}{2}, \\ u(\frac{2\tau - t}{2 - t}), & \text{if } \frac{t}{2} \le \tau \le 1. \end{cases}$$

Thus, $\iota : Z_\phi \to B$ is a cofibration $*$-homomorphism, and this finishes the proof. □

Remark 3.6. Proposition 3.5 corresponds to the following statement in the commutative case ([10], Ch. I, §4, Th. 12).

Given a continuous map $f : X \to Y$ there exists a commutative diagram



with r a strong deformation retract and i, defined by $i(x) = [x, 0]$, a cofibration.

Corollary 3.7. *If $\phi : A \to B$ is a homotopy equivalence of C^*-algebras, then $\iota : Z_\phi \to B$ is also a homotopy equivalence of C^*-algebras.*

Remark 3.8. In order to avoid some confusions in the case of using more morphisms, we make the convention to denote the morphisms from Proposition 3.5 and its proof by $\zeta_\phi : A \to Z_\phi$, $\iota_\phi : Z_\phi \to B$ and $\rho_\phi : Z_\phi \to A$, respectively.

Definition 3.9. Let $\varphi : A \to B$, $\psi : C \to B$ be $*$-homomorphisms.

a) The relative mapping cylinder of φ with respect to ψ is the C*-algebra $Z_\varphi(\psi) := A(B; \varphi, \psi \circ \rho_1) = \{(a, \gamma) \in A \oplus CI : \psi(\gamma(1)) = \varphi(a)\}$.

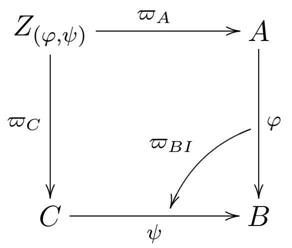
b) The double mapping cylinder of the pair (φ, ψ) is the C*-algebra $Z_{(\varphi,\psi)} := \{(a, c, \beta) \in A \oplus C \oplus BI : \beta(0) = \varphi(a), \beta(1) = \psi(c)\}$.

Remark 3.10. Let $\varphi \overset{h}{\sim} \varphi' : A \to B$ and $\psi \overset{h}{\sim} \psi' : C \to B$. The the double (relative) mapping cylinders $Z_{(\varphi,\psi)}$ (resp.$Z_\varphi(\psi)$) and $Z_{(\varphi',\psi')}$ (resp. $Z_{\varphi'}(\psi')$) are $*$-homotopically equivalent.

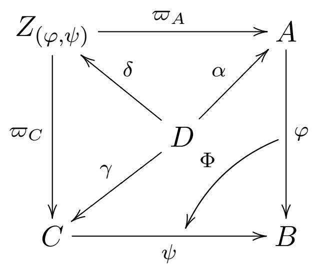
Remark 3.11. C*-algebras $Z_{(\varphi,\psi)}$ and $Z_{(\psi,\varphi)}$ are $*$-isomorphic.

Proposition 3.12. *For two $*$-homomorphisms $\varphi : A \to B$ and $\psi : C \to B$, the double mapping cylinder $Z_{(\varphi,\psi)}$ is a weak homotopy pull-back C*-algebra of B along of the pair of $*$-homomorphisms (φ, ψ).*

Proof. Denote the projections of $Z_{(\varphi,\psi)}$ on A, C and BI by ϖ_A, ϖ_C and ϖ_{BI} respectively. Then $(\rho_0 \circ \varpi_{BI})((a,c,\beta)) = \beta(0) = \varphi(a) = (\varphi \circ \varpi_A)((a,c,\beta)$, hence $\rho_0 \circ \varpi_{BI} = \varphi \circ \varpi_A$, and similarly $\rho_1 \circ \varpi_{BI} = \psi \circ \varpi_C$. Thus, we have the following homotopy commutative diagram of $*$-homomorphisms



To verify the weak homotopy pull-back universal property, let D be a C*-algebra and let $\alpha : D \to A$, $\gamma : D \to C$ be $*$-homomorphisms such that $\varphi \circ \alpha \overset{h}{\sim} \psi \circ \gamma$ by a homotopy $\Phi : D \to BI$. Then we define $\delta : D \to Z_{(\varphi,\psi)}$ by $\delta(d) = (\alpha(d), \gamma(d), \Phi(d))$. This is indeed well defined since $\Phi(d)(0) = (\rho_0 \circ \Phi)(d) = (\varphi \circ \alpha)(d) = \varphi(\alpha(d))$ and $\Phi(d)(1) = (\rho_1 \circ \Phi)(d) = (\psi \circ \gamma)(d) = \psi(\gamma(d))$. This is also a $*$-homomorphism and it verifies $\varpi_A \circ \delta = \alpha$ and $\varpi_C \circ \delta = \gamma$.



□

Remark 3.13. We have not proved the uniqueness of the $*$-homomorphism δ (up to homotopy), such that $Z_{(\varphi,\psi)}$ can be not just a homotopy pull-back (see [7]). The notion of weak pull-back appears, for example, in [4]. In [7], p. 226, the double mapping cylinder for a pair of continuous maps is called the standard homotopy pull-back.

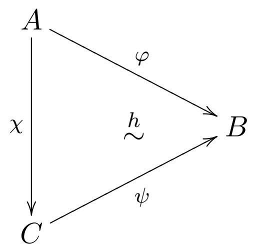
Definition 3.14. If $\varphi : A \to B$ and $\psi : C \to B$ are two $*$-homomorphisms with relative (resp. double) mapping cylinder $Z_\varphi(\psi)$ (resp. $Z_{(\varphi,\psi)}$, then A is called a canonical retract of $Z_\varphi(\psi)$ (resp. $Z_{(\varphi,\psi)}$) if there is a $*$-homomorphism $\theta : A \to Z_\varphi(\psi)$ (resp. $\kappa : A \to Z_{(\varphi,\psi)}$) such that $\pi_A \circ \theta = id_A$ (resp. $\varpi_A \circ \kappa = id_A$) for $\pi_A : A_\varphi(\psi) \to A$ (resp. $\varpi_A : Z_{(\varphi,\psi)} \to A$) the projection $\pi_A((a,\gamma) = a$ (resp. $\varpi_A((a,c,\beta)) = a$).

Remark 3.15. The results of this will be used in the following section. But some of these, and especially Proposition 3.5, have been used in the papers [8] and [11].

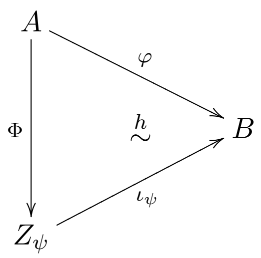
4. Main result

Theorem 4.1. *For any two $*$-homomorphisms $\varphi : A \to B$, $\psi : C \to B$, the following statements are equivalent:*

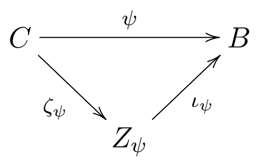
(*i*) *There is a $*$-homomorphism $\chi : A \to C$, such that $\psi \circ \chi \overset{h}{\sim} \varphi$*



(*ii*) *There is a $*$-homomorphism $\Phi : A \to Z_\psi$ such that $\iota_\psi \circ \Phi \overset{h}{\sim} \varphi$*



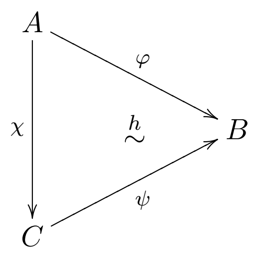
Proof. (i)⇒ (ii). By Proposition 3.5, we have the commutative diagram



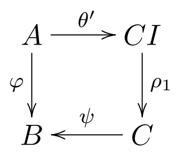
Then the relation $\psi \circ \chi \overset{h}{\sim} \varphi$ implies $(\iota_\psi \circ \zeta_\psi) \circ \chi \overset{h}{\sim} \varphi$, and we can define $\Phi : A \to Z_\psi$ by $\Phi = \zeta_\psi \circ \chi$.

(ii)⇒ (i). Consider the morphism $\rho_\psi : Z_\psi \to C$ from the proof of Proposition 3.5, and take the composition $\chi = \rho_\psi \circ \Phi$. Then we have $\psi \circ \chi = \psi \circ (\rho_\psi \circ \Phi) = (\psi \circ \rho_\psi) \circ \Phi$. But $\psi = \iota_\psi \circ \zeta_\psi$ and $\zeta_\psi \circ \rho_\psi \overset{h}{\sim} id_{Z_\psi}$. Therefore we obtain: $\psi \circ \chi = \iota_\psi \circ \zeta_\psi \circ \rho_\psi \circ \Phi \overset{h}{\sim} \iota_\psi \circ \Phi \overset{h}{\sim} \varphi$. □

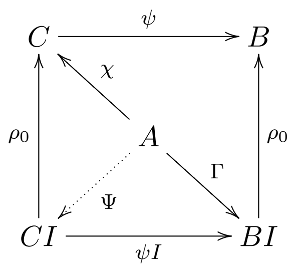
Theorem 4.2. *Let be given a $*$- homotopy commutative diagram*



with ψ a cofibration. Then A is a canonical retract of the relative mapping cylinder $Z_\varphi(\psi)$. This also implies the existence of a commutative diagram



Proof. Denote by $\Gamma : A \to BI$ a homotopy $*$-homomorphism between $\psi \circ \chi$ and φ. Hence, $\Gamma \circ \rho_0 = \psi\chi$ and $\Gamma \circ \rho_1 = \varphi$. We include these morphisms in the following diagram of cofibration property of ψ.

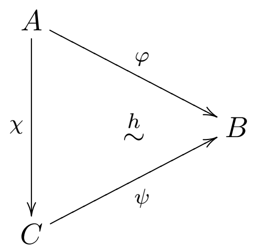


Hence $\psi(\Psi(a)(t)) = \Gamma(a)(t)$ and $\Psi(a)(0) = \chi(a), \forall a \in A, \forall t \in I$. Then we can define a $*$-homomorphism $\theta : A \to Z_\varphi(\psi)$, by $\theta(a) = (a, \Psi(a))$, $\forall a \in A$, since $\gamma(\Psi(a)(1)) = \Gamma(a)(1) = \varphi(a)$. This implies the equality $\pi_A \circ \theta = id_A$, which proves that A is a canonical retract of the relative mapping cylinder $Z_\varphi(\psi)$.

The commutative square from the enunciation is obtained if we take $\theta' = \pi_{CI} \circ \theta$. □

Now we will prove the main result of the paper.

Theorem 4.3. *Let $\varphi : A \to B$, $\psi : C \to B$ be two arbitrary $*$-homomorphisms. Then there is a homotopy commutative diagram of $*$-homomorphisms*



if and only if A is a canonical retract of the double mapping cylinder $Z_{(\varphi,\psi)}$. Moreover, if $\kappa : A \to Z_{(\varphi,\psi)}$ defines a canonical retraction, i.e., satisfies $\varpi_A \circ \kappa = id_A$, then we can take $\chi = \varpi_C \circ \kappa$.

Proof. If we suppose that there exists a $*$-homomorphism $\chi : A \to C$ such that $\psi \circ \chi \overset{h}{\sim} \varphi$ then, by Theorem 4.1, there is a $*$-homomorphism $\Phi : A \to Z_\psi$ such that $\iota_\psi \circ \Phi \overset{h}{\sim} \varphi$. And since ι_ψ is a cofibration (by Proposition 3.5), we can apply Theorem 4.2. It follows that A is a canonical retract of the relative mapping cylinder $Z_\varphi(\iota_\psi) = \{(a, u) \in A \oplus Z_\psi I : \iota_\psi((u(1)) = \varphi(a)\}$.

Let (a, u) be an element of $Z_\varphi(\iota_p si)$ and $u(t) = (c_t, v_t) \in C \oplus BI$, such that $v_t(1) = \psi(c_t)$. We have $\iota_\psi(u(1)) = \iota_\psi((c_1, v_1)) = v_1(0)$, so that we obtain $v_1(0) = \varphi(a)$. Thus, $(a, c_1, v_1) \in Z_{(\varphi,\psi)}$. and we obtain a $*$-homomorphism $\Upsilon : Z_\varphi(\iota_\psi) \to Z_{(\varphi,\psi)}$, $\Upsilon((a, u)) = (a, c_1, v_1)$. Then, if $\theta : A \to Z_\varphi(\iota_\psi)$ is a $*$-homomorphism such that $\pi_A \circ \theta = id_A$, this means that we have $\theta(a) = (a, u_a)$, with $\iota_\psi(u_a(1)) = \varphi(a)$. If we define $\kappa : A \to Z_{(\varphi,\psi)}$ by $\kappa = \Upsilon \circ \theta$ and if $\varpi_A : Z_{(\varphi,\psi)} \to A$ is the projection $\varpi_A((a, c, u)) = a$, then it follows that $\varpi_A \circ \kappa = id_A$. Therefore, A is a canonical retract of the double mapping cylinder $Z_{(\varphi,\psi)}$ of the pair (φ, ψ).

Conversely, suppose that A is a canonical retract of the double mapping cylinder $Z_{(\varphi,\psi)}$, with $\kappa : A \to Z_{(\varphi,\psi)}$ a $*$-homomorphism such that $\varpi_A \circ \kappa = id_A$. Let $\varpi_C : Z_{(\varphi,\psi)} \to C$ be the projection $\varpi_C((a, c, u)) = c$. Define $\chi : A \to C$ by $\chi = \varpi_C \circ \kappa$. Then we can prove that $\psi \circ \chi \overset{h}{\sim} \varphi$. For this we define $\Phi : A \to BI$ by $\Phi = \varpi_{BI} \circ \kappa$, where $\varpi_{BI} : Z_{(\varphi,\psi)} \to BI$ is the projection $\varpi_{BI}((a, c, u)) = u$. If $\kappa(a) = (a, c, u)$, we have $u(0) = \varphi(a)$ and $u(1) = \psi(c)$. Then , $\chi(a) = c$, and therefore $(\psi \circ \chi)(a) = u(1)$. Thus, it is clear that $\rho_0 \circ \Phi = \varphi$ and $\rho_1 \circ \Phi = \psi \circ \chi$. Therefore $\Phi : \varphi \overset{h}{\sim} \psi \circ \chi$, which finishes the proof. □

5. Some corollaries

Applying Theorems 4.2 and 4.3, we can obtain some interesting corollaries.

Corollary 5.1. *Let*

$$\begin{array}{ccc} P & \xrightarrow{p_1} & A_1 \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \varphi_1} \\ A_2 & \xrightarrow[\varphi_2]{} & B \end{array}$$

be a weak homotopy pull-back of C^*-algebras. Id D is a C^*-algebra with $\delta_i : D \to A_i, i = 1, 2$, two $*$-homomorphisms satisfying $\varphi_1 \circ \delta_1 \overset{h}{\sim} \varphi_2 \circ \delta_2$, then D is a canonical retract of the double mapping cylinders $Z_{(\delta_1,p_1)}$ and $Z_{(\delta_2,p_2)}$.

Corollary 5.2. *A $*$-homomorphism $\varphi : A \to B$ admits a homotopy right inverse if and only if B is a canonical retract of the double mapping cylinder $Z_{(1_B,\varphi)}$, or equivalent, if and only if there exists a $*$-homomorphism $k' : B \to Z_\varphi$ satisfying $(\rho_0 \circ \pi_{BI}) \circ k' = id_B$, for $\pi_{BI} : Z_\varphi \to BI$ the canonical projection.*

Besides, if φ is a cofibration $*$-homomorphism, then this is also equivalent to the fact that B is a canonical retract of the relative mapping cylinder

$Z_{1_B}(\varphi)$, or with the fact that there exists a $*$-homomorphism $k'' : B \to AI$ satisfying $(\varphi \circ \rho_1) \circ k'' = id_B$.

Corollary 5.3. *Let $\varphi : A \to B$, $\chi : C \to B$ and $\psi : D \to B$ be three $*$-homomorphisms. If A is a canonical retract of $Z_{(\varphi,\chi)}$ and C is a canonical retract of $Z_{(\chi,\psi)}$, then A is also a canonical retract of $Z_{(\varphi,\psi)}$.*

Corollary 5.4. *Let $\varphi_i : A_i \to B$, $i = 1, 2, 3$, be three $*$-homomorphisms. Define the triple mapping cylinder $Z_{(\varphi_1,\varphi_2,\varphi_3)} := \{(a_1, a_2, a_3, \beta) \in A_1 \oplus A_2 \oplus A_3 \oplus BI : \beta(0) = \varphi_1(a_1), \beta(\frac{1}{2}) = \varphi_2(a_2), \beta(1) = \varphi_3(a_3)\}$, and suppose that A_i, $i \in \{1, 2, 3\}$, are canonical retracts of $Z_{(\varphi_1,\varphi_2,\varphi_3)}$. Then there exist $*$-homomorphisms $\chi_1 : A_1 \to A_2$, $\chi_2 : A_2 \to A_3$ and $\chi_3 : A_3 \to A_1$, such that $\varphi_2 \circ \chi_1 \overset{h}{\sim} \varphi_1$, $\varphi_2 \circ \chi_2 \overset{h}{\sim} \varphi_2$ and $\varphi_1 \circ \chi_3 \overset{h}{\sim} \varphi_3$.*

Proof. Let $\kappa_i : A_i \to Z_{(\varphi_1,\varphi_2,\varphi_3)}$, $i \in \{1, 2, 3\}$, be canonical retracts, i.e., if $\pi_i : Z_{(\varphi_1,\varphi_2,\varphi_3)} \to A_i$,$i \in \{1, 2, 3\}$, are canonical projections, $\pi_i \circ \kappa_i = id_{A_i}$. Then we can define the projections $\sigma_{12} : Z_{(\varphi_1,\varphi_2,\varphi_3)} \to Z_{(\varphi_1,\varphi_2)}$, by $\sigma_{12}((a_1, a_2, a_3, \beta)) = (a_1, a_2, \beta')$, with $\beta'(t) = \beta(\frac{t}{2})$; $\sigma_{23} : Z_{(\varphi_1,\varphi_2,\varphi_3)} \to Z_{(\varphi_2,\varphi_3)}$, by $\sigma_{23}((a_1, a_2, a_3, \beta)) = (a_2, a_3, \beta'')$, with $\beta''(t) = \beta(\frac{t+1}{2})$; $\sigma_{3,1} : Z_{(\varphi_1,\varphi_2,\varphi_3)} \to Z_{(\varphi_3,\varphi_1)}$, by $\sigma_{31}((a_1, a_2, a_3, \beta) = (a_3, a_1, \beta''')$, with $\beta'''(t) = \beta(1 - t)$.

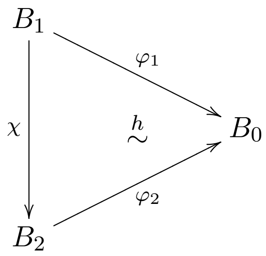
Now if we consider the composition $\sigma_{12} \circ \kappa_1 : A_1 \to Z_{(\varphi_1,\varphi_2)}$, $\sigma_{23} \circ \kappa_2 : A_2 \to Z_{(\varphi_2,\varphi_3)}$, $\sigma_{31} \circ \kappa_3 : A_3 \to Z_{(\varphi_3,\varphi_1)}$, these $*$-homomorphisms are canonical retracts, and then we can apply Theorem 4.3. $\square$

For the following corollary we need the homotopy groups of C*-algebras in sense of Čerin [1]. For a pair (A, B) of C*-algebras, denote by $\pi_n(A; B)$, $n \geq 1$, Čerin's n-th homotopy group of B over A. This is a functor contravariant in the firs argument A and covariant in the second argument B and these homotopy groups of C*-algebras are invariants of homotopy type.

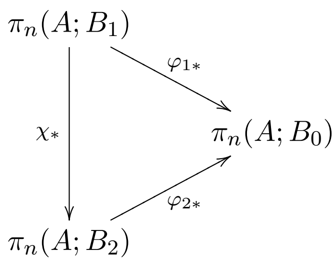
Corollary 5.5. *a) Let $\varphi_1 : B_1 \to B_0, \varphi_2 : B_2 \to B_0$ be $*$-homomorphisms. Consider the group homomorphisms $\varphi_{1*} : \pi_n(A; B_1) \to \pi_n(A; B_0)$ and $\varphi_{2*} : \pi_n(A; B_2) \to \pi_n(A; B_0)$, $n \geq 1$, for an arbitrary C*-algebra A. If B_1 is a canonical retract of $Z_{(\varphi_1,\varphi_2)}$, then $Im\varphi_{1*} \subseteq Im\varphi 2*$.*

b) Let $\psi_1 : A_1 \to A_0$, $\psi_2 : A_2 \to A_0$ be $*$-homomorphisms. Consider the group homomorphisms $\psi_{1*} : \pi_n(A_0; B) \to \pi_n(A_1; B)$ and $\psi_{2*} : \pi_n(A_0; B) \to \pi_n(A_2; B)$, $n \geq 1$, for an arbitrary C-algebra B. If A_1 is a canonical retract of $Z_{(\psi_1,\psi_2)}$, then $Ker\psi_{2*} \subseteq Ker\psi_{1*}$.*

Proof. a) By Theorem 4.3 , there exists a homotopy commutative diagram

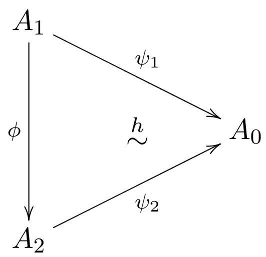


which implies the following commutative of group homomorphisms

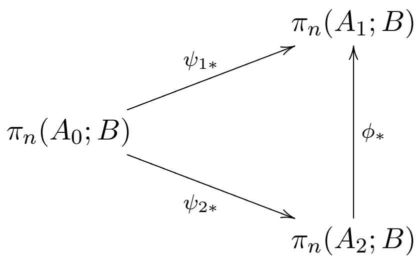


Then the relation $\varphi_{2*} \circ \chi_* = \varphi_{1*}$ implies $Im\varphi_{1*} \subseteq Im\varphi_{2*}$

b) Similarly, by Theorem 4.3 , there exists a homotopy commutative diagram



which implies the following commutative of group homomorphisms



Then the relation $\phi_* \circ \psi_{2*} = \psi_{1*}$ implies $Ker\psi_{2*} \subseteq Ker\psi_{1*}$. □

Remark 5.6. A similar result to that of Corollary 5.5 can be proved by replacing the homotopy groups by the shape groups of C*-algebras in the Čerin's sense [2].

6. The unital case

Now we will consider the problems of previous sections in the case of unital C*-algebras and unital $*$-homomorphisms.

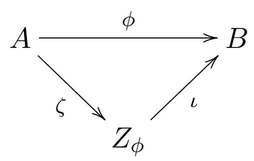
If A is an unital C*-algebra, the AI is also an unital C*-algebra with the unity the constant path $t \to 1$. If A, B are unital C*-algebras and $\eta, \phi : A \to B$ are unital $*$-homomorphisms, then we say that η is unital homotopic to ϕ, written $\eta \overset{h}{\sim}_1 \phi$, if there exists an unital $*$-homomorphism $\Psi : A \to BI$ such that $\rho_0 \circ \Psi = \eta$, $\rho_1 \circ \Psi = \phi$. Therefore, in this case Ψ satisfies also the condition $\Phi(1_A)(t) = 1_B$, $\forall t \in I$, where 1_A and 1_B are the units of A and respectively B.

If $\phi_1 : A_1 \to B$ and $\phi_2 : A_2 \to B$ are unital $*$-homomorphisms, then the pull-back $A(B; \phi_1, \phi_2)$ is an unital C*-algebra with the unit $(1_{A_1}, 1_{A_2})$. Particularly, if $\phi : A \to B$ is an unital C*-algebra, then the mapping cylinder C*-algebra Z_ϕ is unital, with the unit $(1_A, \beta_1)$, where β_1 is the unit of BI, i.e., the constant path $\beta_1(t) = 1_B, \forall t \in I$.

An unital $*$-homomorphism $\phi : A \to B$ is an unital cofibration if the condition from Definition 3.4 is verified with the following restriction: if D is an unital C*-algebra, $\psi : D \to A$ is an unital $*$-homomorphism and $\Phi : D \to BI$ is an unital $*$-homotopy, then there exists an unital homotopy $\Psi : D \to AI$ such that $\rho_0 \circ \Psi = \psi$ and $\phi I \circ \Psi = \Phi$.

Most results of previous sections remain valid in the case unital also. The proofs remain the same and it is only necessary to verify that the constructed $*$-homomorphisms and $*$-homotopies are unital.

Proposition 6.1. *For any unital $*$-homomorphism $\phi : A \to B$ there exists a commutative diagram of unital $*$-homomorphism*



with ζ a strong deformation unital $*$- homomorphism and ι a cofibration unital $*$- homomorphism.

Proof. The proof is the same as that of Proposition 3.5 being only necessary to verify that all used $*$-homomorphisms are unital. For $\zeta : A \to Z_\phi$ defined by $\zeta(a) = (a, \beta_{\phi(a)})$, we have $\zeta(1_A) = (1_A, \beta_1) = 1_{BI}$. For $\iota : Z_\phi \to B$ defined by $\iota((a,\beta)) = \beta(0)$, it follows $\iota((1_A, \beta_1)) = \beta_1(0) = 1_B$. Therefore ζ and ι are unital $*$-homomorphisms. Also the projection $\rho : Z_\phi \to A$, defined by $\rho((a,\beta)) = a$ is obviously an unital $*$-homomorphism. Next, for the morphism $\Upsilon : Z_\phi \to Z_\phi I$ defined by $\Upsilon((a,\beta)(t) = (a, \beta_t)$, we have $\Upsilon((1_A,\beta_1))(t) = (1_A, (\beta_1)_1)$ and $(\beta_1)_1(\tau) = \beta_1(t\tau - t + 1) = 1_B, \forall \tau \in I$, so that $\Upsilon((1_A,\beta_1))(t) = (1_B, \beta_1) = 1_{Z_\phi}, \forall t \in I$. This means that Υ is unital. Finally, suppose that the C*-algebra D is unital and that $\Phi : D \to BI$ is an unital $*$-homomorphism. We need to verify that the $*$-homomorphism $\Psi : D \to Z_\phi I$, defined by $\Psi(d)(t) = (a, u_t)$ is unital. For $d = 1_D$, we have $\psi(d) = \psi(1_D) = 1_{Z_\phi} = (1_A, \beta_1)$. Therefore $\Psi(1_D)(t) = (1_A, u_t)$. Moreover, in this case, for $0 \le \tau \le \frac{t}{2}$, we have $u_t(\tau) = \Phi(1_D)((t - 2\tau)) = 1_{BI}((t-2\tau)) = 1_B$, and for $\frac{t}{2} \le \tau \le 1$, $u_t(\tau)) = \beta_1(\frac{2\tau - t}{2-t}) = 1_B$. So, $\Psi(1_D)(t) = 1_{Z_\phi}$, that is, $\Psi(1_D) = 1_{Z_\phi I}$. Therefore $\iota : Z_\phi \to B$ is an unital $*$-homomorphism cofibration. □

Corollary 6.2. *If ϕ is an unital homotopy equivalence of C*-algebras, then ι is also an unital homotopy equivalence of C*-algebras.*

If A, B, C are unital C*-algebras and $\varphi : A \to B$, $\psi : C \to B$ are unital $*$-homomorphisms, then the double mapping cylinder $Z_{(\varphi,\psi)}$ is an unital C*-algebras with the unit $(1_A, 1_C, \beta_1)$, for $\beta_1 \in BI$ the constant path $\_1(t) = 1_B, \forall t \in I$.

Proposition 6.3. *If $\varphi : A \to B$, $\psi : C \to B$ are unital $*$-homomorphisms then the double mapping cylinder $Z_{(\varphi,\psi)}$ is a weak unital homotopy pull-back C*-algebra of B along the pair of unital $*$-homomorphisms (φ, ψ).*

Proof. The proof is the same as that of Proposition 3.12 being only necessary to verify that all used $*$-homomorphisms are unital.

The projections ϖ_A, ϖ_C and ϖ_{BI} are obviously unital $*$-homomorphisms. Then, if D is an unital C*-algebra, $\alpha : D \to A$, $\gamma : D \to C$ are unital $*$-homomorphisms and $\Phi : D \to BI$ is an unital homotopy, $\Phi : \varphi \circ \alpha \overset{h}{\sim}_1 \psi \circ \gamma$, the $*$-homomorphism $\delta : D \to Z_{(\varphi,\psi)}$ defined by $\delta(d) = (\alpha(d), \gamma(d), \Phi(d))$ is obviously an unital $*$-homomorphism. Thus we have satisfied the definition of weak pull-back in the unital homotopy category of unital C*-algebras.□

The proof of the following theorem is the same as that of Theorem 4.1 since all $*$-morphisms and $*$-homotopies are unital being compositions of unital $*$-homomorphisms or unital $*$-homotopies.

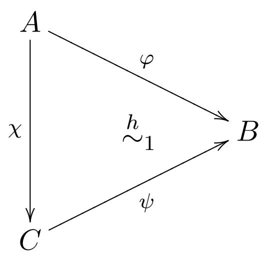
Theorem 6.4. *For any two unital $*$-homomorphisms $\varphi : A \to B$, $\psi : C \to B$, the following two statements are equivalent:*

(*i*) *There is an unital $*$-homomorphism $\chi : A \to C$, such that $\psi \circ \chi \overset{h}{\sim}_1 \varphi$,*

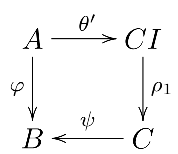
(*ii*) *There is an unital $*$-homomorphism $\Phi : A \to Z_\psi$, such that $\iota_\psi \circ \Phi \overset{h}{\sim}_1 \varphi$.*

The proof of the following theorem is the same as that of Theorem 4.2 using only the fact that χ and Ψ are unital $*$-homorphisms and ψ an unital $*$-cofibration the $*$-homotopy Ψ is unital.

Theorem 6.5. *Let be given an unital $*$- homotopy commutative diagram*

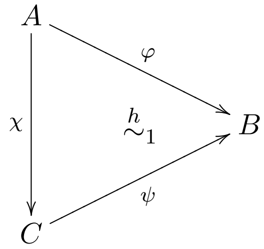


with ψ an unital $*$-cofibration. Then A is an unital canonical retract of the relative mapping cylinder $Z_\varphi(\psi)$. This also implies the existence of a commutative diagram of unital $*$-homorphisms.



The proof of the following theorem is the same as that of Theorem 4.3 since all $*$-morphisms and $*$-homotopies are unital being compositions of unital $*$-homomorphisms or unital $*$-homotopies.

Theorem 6.6. *Let $\varphi : A \to B$, $\psi : C \to B$ be two arbitrary unital $*$-homomorphisms. Then there is an unital homotopy commutative diagram of $*$-homomorphisms*



if and only if A is a unital canonical retract of the double mapping cylinder $Z_{(\varphi,\psi)}$. Moreover, if $\kappa : A \to Z_{(\varphi,\psi)}$ defines a canonical retraction, i.e., satisfies $\varpi_A \circ \kappa = id_A$, then we can take $\chi = \varpi_C \circ \kappa$.

Corollaries 5.1, 5.2, 5.3 and 5.4 can be formulated and proved in the unital case. For example, Corollary 5.4 becomes

Corollary 6.7. *Let $\varphi_i : A_i \to B$, $i = 1, 2, 3$, be three unital $*$-homomorphisms. Consider the triple mapping cylinder $Z_{(\varphi_1,\varphi_2,\varphi_3)}$ with the unit $(1_{A_1}, 1_{A_2}, 1_{A_3}, \beta_1)$, and suppose that A_i, $i \in \{1, 2, 3\}$, are unital canonical retracts of $Z_{(\varphi_1,\varphi_2,\varphi_3)}$. Then there exist unital $*$-homomorphisms $\chi_1 : A_1 \to A_2$, $\chi_2 : A_2 \to A_3$ and $\chi_3 : A_3 \to A_1$, such that $\varphi_2 \circ \chi_1 \overset{h}{\sim}_1 \varphi_1$, $\varphi_2 \circ \chi_2 \overset{h}{\sim}_1 \varphi_2$ and $\varphi_1 \circ \chi_3 \overset{h}{\sim}_1 \varphi_3$.*

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