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ABOUT SOME CHLODOVSKY-TYPE OPERATORS

BY

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Abstract. In this paper we consider some Chlodovsky-type operators. We obtain the Voronovskaja-type theorem, the convergence and the evaluation of the rate of convergence in terms of the first modulus of smoothness for these operators.

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1. Introduction

The aim of this paper is to consider the Chlodovsky operators in a more general condition.

We don't study the convergence of these operators with the well-known theorem of Bohman-Korovkin. In every case we prove the Voronovskaja type theorem for these operators. The evaluation theorems of the rate of convergence are different from the well-known theorem of Shisha-Mond.

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $(h_n)_{n\in\mathbb{N}}$ be an increasing sequence of strictly positive and real numbers. Suppose that

$$\lim_{n\to\infty} h_n = h \in \overline{\mathbb{R}}, \tag{1.1}$$

where $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. We note

$$\mathbb{R}_{+,h} = \begin{cases} [0,h], & \text{if } h \in \mathbb{R} \\ [0,\infty), & \text{if } h = \infty. \end{cases} \tag{1.2}$$

Definition 1.1. For any $n \in \mathbb{N}$, consider the operator $C_{n,h} : \mathcal{F}(\mathbb{R}_{+,h}) \to \mathcal{F}(\mathbb{R}_{+,h})$ defined for any function $f \in \mathcal{F}(\mathbb{R}_{+,h})$ by

$$(C_{n,h}f)(x) = \begin{cases} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{h_n}\right)^k \left(1 - \frac{x}{h_n}\right)^{n-k} f\left(\frac{h_n k}{n}\right), & \text{if } 0 \le x \le h_n \\ f(x), & \text{if } x \in \mathbb{R}_{+,h} \cap (h_n, \infty) \end{cases} \tag{1.3}$$

and any $x \in \mathbb{R}_{+,h}$, where $\mathcal{F}(\mathbb{R}_{+,h}) = \{f | f : \mathbb{R}_{+,h} \to \mathbb{R}\}$.

The sequence of the operators $(C_{n,h})_{n\in\mathbb{N}}$ are called Chlodovsky-type operators.

Remark 1.1. If $\lim_{n\to\infty} h_n = \infty$, then we obtain the Chlodovsky operators (see [1] or [2]).

It is known that these operators are linear and positive and the results contained in the following lemma (see [1]).

Lemma 1.1. *If* $n \in \mathbb{N}$, *then*

$$(C_{n,h}e_0)(x) = 1, \quad \forall x \in [0, h_n], \tag{1.4}$$

$$(C_{n,h}e_1)(x) = x, \quad \forall x \in [0, h_n], \tag{1.5}$$

$$(C_{n,h}e_2)(x) = x^2 - \frac{1}{n}x^2 + \frac{1}{n}h_n x, \quad \forall x \in [0, h_n]. \tag{1.6}$$

2. Preliminaries

For the given real interval I, we shall use the function sets: $B(I) = \{f | f : I \to \mathbb{R}, f \text{ bounded on } I\}$, $C(I) = \{f | f : I \to \mathbb{R}, f \text{ continuous on } I\}$ and $C_B(I) = B(I) \cap C(I)$. For any $x \in I$ consider the function $\psi_x : I \to \mathbb{R}$ given by $\psi_x(t) = t - x$, for any $t \in I$. For $f \in C_B(I)$, by the first order modulus of smoothness of f is meant the function $\omega : [0, \infty) \to \mathbb{R}$ defined for any $\delta \ge 0$ by $\omega(f; \delta) = \sup\{|f(x') - f(x'')| : x',\ x' \in I,\ |x' - x''| \le \delta\}$. The following construction and results are from [3].

Let I, J be real intervals with $I \cap J \ne \emptyset$. For any $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, n\}$ consider the functions $\varphi_{n,k} : J \to \mathbb{R}$ with the property that $\varphi_{n,k}(x) \ge 0$, $x \in J$, and the linear positive functionals $A_{n,k} : E(I) \to \mathbb{R}$, where $E(I)$ is subset of the set of real functions defined on I.

Definition 2.1. For $n \in \mathbb{N}$ define the operator $L_n : E(I) \to F(J)$ by

$$(L_n f)(x) = \sum_{k=0}^{n} \varphi_{n,k}(x) A_{n,k}(f), \tag{2.1}$$

$f \in E(I), x \in J$, where $F(J)$ is subset of the set of real functions defined on J.

Proposition 2.1. *The L_n, $n \in \mathbb{N}$, operators are linear and positive on $E(I \cap J)$.*

Definition 2.2. Let $L_n : E(I) \to F(J)$ be an operator defined in (2.1), $n \in \mathbb{N}$. For $i \in \mathbb{N}_0$ define $T^*_{n,i}$ by

$$\left(T^*_{n,i} L_n\right)(x) = n^i \left(L_n \psi_x^i\right)(x) = n^i \sum_{k=0}^{n} \varphi_{n,k}(x) A_{n,k}\left(\psi_x^i\right), \tag{2.2}$$

for any $x \in I \cap J$.

In what follows $s \in \mathbb{N}_0$ is even and we suppose that the operators $(L_n)_{n \in \mathbb{N}}$ verify the conditions: there exists, the smallest α_s, $\alpha_{s+2} \in [0, \infty)$ so that

$$\lim_{n \to \infty} \frac{\left(T^*_{n,j} L_n\right)(x)}{n^{\alpha j}} = B_j(x) \in \mathbb{R}, \tag{2.3}$$

$x \in I \cap J$, $j \in \{s, s+2\}$;

$$\alpha_{s+2} < \alpha_s + 2; \tag{2.4}$$

and $I \cap J$ is an interval.

Theorem 2.1. *Let $f : I \to \mathbb{R}$ be a function. If $x \in I \cap J$ and f is a s times differentiable function in x with $f^{(s)}$ continuous and bounded on a neighborhood V of the point x, then*

$$\lim_{n \to \infty} n^{s-\alpha_s} \left[(L_n f)(x) - \sum_{i=0}^{s} \frac{1}{n^i i!} \left(T^*_{n,i} L_n\right)(x) f^{(i)}(x) \right] = 0. \tag{2.5}$$

Assume that f is a s times differentiable function on I, with $f^{(s)}$ continuous and bounded on I and there exists an interval $K \subset I \cap J$ such that there *exist* $n(s) \in \mathbb{N}$ *and* $k_j \in \mathbb{R}$ *depending on* K, *so that for* $n \geq n(s)$ *and* $x \in K$ *we have*

$$\frac{(T^*_{n,j}L_n)(x)}{n^{\alpha_j}} \leq k_j, \tag{2.6}$$

where $j \in \{s, s+2\}$. *Then the convergence given in* (2.5) *is uniform on* K *and*

$$\begin{aligned} n^{s-\alpha_s}\left|(L_nf)(x) - \sum_{i=0}^{s} \frac{1}{n^i i!}\left(T^*_{n,i}L_n\right)(x)f^{(i)}(x)\right| \\ \leq \frac{1}{s!}(k_s + k_{s+2})\omega\left(f^{(s)}; \frac{1}{\sqrt{n^{2+\alpha_s-\alpha_{s+2}}}}\right), \end{aligned} \tag{2.7}$$

for $x \in K$ *and* $n \geq n(s)$.

Corollary 2.1. *Let* $f : I \to \mathbb{R}$ *be a* s *times differentiable function in* $x \in I \cap J$ *with* $f^{(s)}$ *continuous and bounded on a neighborhood* V *of the point* x. *Then*

$$\lim_{n\to\infty}(L_nf)(x) = B_0(x)f(x) \tag{2.8}$$

if $s = 0$ *and* $\alpha_0 = 0$. *If* $s \geq 2$, *then*

$$\lim_{n\to\infty} n^{s-\alpha_s}\left[(L_nf)(x) - \sum_{i=0}^{s-1}\frac{1}{n^i i!}\left(T^*_{n,i}L_n\right)(x)f^{(i)}(x)\right] = \frac{1}{s!}B_s(x)f^{(s)}(x). \tag{2.9}$$

If f *is a* s *times differentiable function on* $I \cap J$, *with* $f^{(s)}$ *continuous and bounded on* $I \cap J$ *and* (2.6) *takes place for an interval* $K \subset I \cap J$, *then the convergence in* (2.8) *and* (2.9) *are uniform on* K.

3. Main results

For obtaining the Chlodovsky-type operators, we consider $I = J = \mathbb{R}_{+,h}$ and for $n \in \mathbb{N}$ we sets $\varphi_{n,k}(x) = \binom{n}{k}(\frac{x}{h_n})^k(1-\frac{x}{h_n})^{n-k}$ and $A_{n,k}(f) = f(\frac{h_n k}{n})$, $k \in \{0, 1, \ldots, n\}$, $f \in \mathcal{F}(\mathbb{R}_{+,h})$ and $x \in \mathbb{R}_{+,h}$.

Lemma 3.1. *The relations*

$$(C_{n,h}e_3)(x) = \frac{(n-1)(n-2)}{n^2}x^3 + \frac{3(n-1)}{n^2}h_nx^2 + \frac{1}{n^2}h_n^2x \tag{3.1}$$

and

$$(3.2)\qquad (C_{n,h}e_4)(x) = \frac{(n-1)(n-2)(n-3)}{n^3}\,x^4 + \frac{6(n-1)(n-2)}{n^3}\,h_n x^3 + \frac{7(n-1)}{n^3}\,h_n^2 x^2 + \frac{1}{n^3}\,h_n^3 x$$

hold for any $n \in \mathbb{N}$, *any* $x \in [0, h_n]$.

Proof. Taking $k^3 = k(k-1)(k-2)+3k(k-1)+k$, $k \in \mathbb{N}_0$, into account, we have that

$$(C_{n,h}e_3)(x) = \frac{(n-1)(n-2)}{n^2}\,x^3(C_{n-3,h}e_0)(x) + \frac{3(n-1)}{n^2}\,h_n x^2(C_{n-2,h}e_0)(x) + \frac{1}{n^2}\,h_n^2 x(C_{n-1,h}e_0)(x),$$

from where (3.1) results. With similar calculus, starting from $k^4 = k(k-1)(k-2)(k-3) + 6k(k-1)(k-2) + 7k(k-1) + k$, $k \in \mathbb{N}_0$, we obtain the relation (3.2). $\square$

Lemma 3.2. *We have*

$$(3.3)\qquad \left(T^*_{n,0}C_{n,h}\right)(x) = 1,$$

$$(3.4)\qquad \left(T^*_{n,1}C_{n,h}\right)(x) = 0,$$

$$(3.5)\qquad \left(T^*_{n,2}C_n h\right)(x) = nx(h_n - x),$$

$$(3.6)\qquad \frac{\left(T^*_{n,2}C_{n,h}\right)(x)}{n^{\alpha_2}} = n^{1-\alpha_2}x(h_n - x)$$

and

$$(3.7)\qquad \frac{\left(T^*_{n,4}C_{n,h}\right)(x)}{n^{\alpha_4}} = n^{2-\alpha_4}\left[\frac{3(n-2)}{n}\,x^4 - \frac{6(n-2)}{n}\,h_n x^3 + \frac{3n-7}{n}\,h_n^2 x^2 + \frac{1}{n}\,h_n^3 x\right]$$

for any $n \in \mathbb{N}$, *any* $x \in [0, h_n]$.

Proof. We have

$$\left(T^*_{n,0}C_{n,h}\right)(x) = (C_{n,h}e_0)(x),$$

$$\left(T^*_{n,1}C_{n,h}\right)(x) = n(C_{n,h}\psi_x)(x) = n\big[(C_{n,h}e_1)(x) - (C_{n,h}e_0)(x)\big],$$

$$\begin{aligned}\left(T^*_{n,2}C_{n,h}\right)(x) &= n^2\left(C_{n,h}\psi_x^2\right)(x)\\ &= n^2\big[(C_{n,h}e_2)(x) - 2x(C_{n,h}e_1)(x) + x^2(C_{n,h}e_0)(x)\big],\\ \left(T^*_{n,4}C_{n,h}\right)(x) &= n^4\left(C_{n,h}\psi_x^4\right)(x) = n^4\big[(C_{n,h}e_4)(x) - 4x(C_{n,h}e_3)(x)\\ &\quad + 6x^2(C_{n,h}e_2)(x) - 4x^3(C_{n,h}e_1)(x) + x^4(C_{n,h}e_0)(x)\big]\end{aligned}$$

and taking Lemma 1.1 and Lemma 3.1 into account. □

Case I. In this case, we suppose that $h \in \mathbb{R}$. Then, $\alpha_0 = 0$, $\alpha_2 = 1$ and $\alpha_4 = 2$.

Lemma 3.3. *We have*

$$\lim_{n\to\infty} \frac{\left(T^*_{n,2}C_{n,h}\right)(x)}{n} = x(h-x), \tag{3.8}$$

$$\lim_{n\to\infty} \frac{\left(T^*_{n,4}C_{n,h}\right)(x)}{n^2} = 3[x(h-x)]^2, \tag{3.9}$$

for any $x \in [0, h]$ *and*

$$\left(T^*_{n,0}C_{n,h}\right)(x) = 1 = k_0, \tag{3.10}$$

$$\frac{\left(T^*_{n,2}C_{n,h}\right)(x)}{n} \leq \left(\frac{h}{2}\right)^2 = k_2, \tag{3.11}$$

for any $n \in \mathbb{N}$, *any* $x \in [0, h]$ *and exists* $n(4) \in \mathbb{N}$ *such that*

$$\frac{\left(T^*_{n,4}C_{n,h}\right)(x)}{n^2} \leq 1 + 3\left(\frac{h}{2}\right)^4 = k_4, \tag{3.12}$$

for any $n \in \mathbb{N}$, $n \geq n(4)$, *any* $x \in [0, h]$.

Proof. The relations (3.8) and (3.9) result from Lemma 3.2. Taking $x(h_n - x) \leq (\frac{h_n}{2})^2 \leq (\frac{h}{2})^2$ for any $x \in [0, h]$ into account, (3.11) result. From definition of the limit, from (3.9) and because $x(h-x) \leq (\frac{h}{2})^2$ for any $x \in [0, h]$, we obtain (3.10). □

Theorem 3.1. *Let* $f : [0, h] \to \mathbb{R}$ *be a function. If* $x \in [0, h]$ *and* f *is continuous and bounded on a neighborhood of the point* x, *then*

$$\lim_{n\to\infty} (C_{n,h}f)(x) = f(x). \tag{3.13}$$

If f is continuous on $[0,h]$, then the convergence given in (3.13) *is uniform on* $[0,h]$ *and*

$$|(C_{n,h}f)(x)-f(x)| \leq \left(1+\frac{h^2}{4}\right)\omega\left(f;\frac{1}{\sqrt{n}}\right) \tag{3.14}$$

for any $n \in \mathbb{N}$, any $x \in [0,h]$.

Proof. It results from Theorem 2.1 for $s=0$ and Lemma 3.3. □

Theorem 3.2. *Let $f:[0,h]\to\mathbb{R}$ be a function. If $x\in[0,h]$ and f is two times differentiable in x, the function $f^{(2)}$ is continuous and bounded on a neighborhood of the point x, then*

$$\lim_{n\to\infty} n\left[(C_{n,h}f)(x)-f(x)\right]=\frac{1}{2}\left[x(h-x)\right]f^{(2)}(x). \tag{3.15}$$

If f is two times differentiable on $[0,h]$, the function $f^{(2)}$ is continuous on $[0,h]$, then the convergence given in (3.15) *is uniform on* $[0,h]$ *and exists $n(4)\in\mathbb{N}$ such that*

$$\begin{aligned} & n\left|(C_{n,h}f)(x)-f(x)-\frac{1}{2n}x(h_n-x)f^{(2)}(x)\right| \\ & \leq \frac{1}{2}\left(1+\left(\frac{h}{2}\right)^2+3\left(\frac{h}{2}\right)^4\right)\omega\left(f;\frac{1}{\sqrt{n}}\right) \end{aligned} \tag{3.16}$$

for any $n\in\mathbb{N}$, $n\geq n(4)$, any $x\in[0,h]$.

Proof. It results from Theorem 2.1, Corollary 2.1 for $s=2$ and Lemma 3.3. □

Remark 3.1. The Theorem 3.2 is a Voronovskaja's type theorem (see [1],[4] or [5]).

Remark 3.2. For $h_n=1$, $n\in\mathbb{N}$, the Chlodovsky operators become the Bernstein operators. The results from Theorem 3.1 and Theorem 3.2 for the Bernstein operators are known.

Example 3.1. Let $(h_n)_{n\in\mathbb{N}}$ be a sequence defined by $h_n=\frac{n}{2(n+2)}$, for any $n\in\mathbb{N}$. Then $\alpha_2=1$, $\alpha_4=2$, $h=\frac{1}{2}$, $k_2=\frac{1}{16}$, $k_4=\frac{259}{256}$ and the Theorem 3.1 and Theorem 3.2 hold.

Case II. In the following we suppose that $h = +\infty$ and

$$\lim_{n\to\infty} n^{1-\alpha_2} h_n = l_2 \in \mathbb{R}. \tag{3.17}$$

Because $\alpha_0 = 0$, from (2.1) and (3.17) results

$$1 < \alpha_2 < 2. \tag{3.18}$$

Lemma 3.4. *We have*

$$\lim_{n\to\infty} \frac{\left(T^*_{n,2} C_{n,\infty}\right)(x)}{n^{\alpha_2}} = l_2 x \tag{3.19}$$

for any $x \in [0,\infty)$ *and exists* $n(2) \in \mathbb{N}$ *such that*

$$\frac{\left(T^*_{n,2} C_{n,\infty}\right)(x)}{n^{\alpha_2}} \le 1 + a l_2 = k_2 \tag{3.20}$$

for any $n \in \mathbb{N}$, $n \ge n(2)$ *and any* $x \in [0,a]$, *where* $a > 0$.

Proof. The (3.19) relation results from (3.6) and (3.17). From (3.19), we obtain relation (3.20). □

Theorem 3.3. *Let* $f : [0,\infty) \to \mathbb{R}$ *be a function. If* $x \in [0,\infty)$ *and* f *is continuous and bounded on a neighborhood of the point* x, *then*

$$\lim_{n\to\infty} (C_{n,\infty} f)(x) = f(x). \tag{3.21}$$

If f *is continuous and bounded on* $[0,\infty)$, *then the convergence given in* (3.21) *is uniform on any interval* $[0,a]$, *where* $a > 0$, *and exists* $n(2) \in \mathbb{N}$ *such that*

$$|(C_{n,\infty} f)(x) - f(x)| \le (2 + a l_2)\omega\left(f; \frac{1}{\sqrt{n^{2-\alpha_2}}}\right) \tag{3.22}$$

for any $n \in \mathbb{N}$, $n \ge n(2)$, *any* $x \in [0,a]$.

Proof. It results from Theorem 2.1 and Lemma 3.4.

In the following, suppose that

$$\lim_{n\to\infty} n^{\frac{2-\alpha_4}{2}} h_n = l_4 \in \mathbb{R}. \tag{3.23}$$

Because α_2, α_4 are the smallest such that (3.17) and (3.23) take place, it results that $1 - \alpha_2 = \frac{2-\alpha_4}{2}$, from where

$$\alpha_4 = 2\alpha_2 \quad \text{and} \quad 2 < \alpha_4 < 4. \tag{3.24}$$

Then, from (3.17) and (3.23) it results that $l_2 = l_4$. □

Lemma 3.5. *We have*

$$\lim_{n\to\infty}\frac{\left(T^*_{n,4}C_{n,\infty}\right)(x)}{n^{\alpha_4}}=3l_4^2x^2 \tag{3.25}$$

for any $x\in[0,\infty)$ *and* $n(4)\in\mathbb{N}$ *exists such that*

$$\frac{\left(T^*_{n,4}C_{n,\infty}\right)(x)}{n^{\alpha_4}}\le 1+3l_4^2a^2 \tag{3.26}$$

for any $n\in\mathbb{N}$, $n\ge n(4)$ *and any* $x\in[0,a]$, *where* $a>0$.

Proof. Taking (3.23) and (3.24) into account, we have

$$\begin{aligned}\lim_{n\to\infty}n^{2-\alpha_4}h_n&=\lim_{n\to\infty}n^{\frac{2-\alpha_4}{2}}\left(n^{\frac{2-\alpha_4}{4}}h_n\right)=l_4\lim_{n\to\infty}n^{\frac{2-\alpha_4}{2}}=0,\\ \lim_{n\to\infty}n^{\frac{1-\alpha_4}{3}}h_n&=\lim_{n\to\infty}n^{\frac{\alpha_4-4}{6}}\left(n^{\frac{2-\alpha_4}{2}}h_n\right)=l_4\lim_{n\to\infty}n^{\frac{\alpha_4-4}{6}}=0.\end{aligned}$$

From (3.7) and these limits, we have

$$\begin{aligned}\lim_{n\to\infty}\frac{\left(T^*_{n,4}C_{n,\infty}\right)(x)}{n^{\alpha_4}}=\lim_{n\to\infty}&\left[\frac{3(n-2)}{n}n^{2-\alpha_4}x^4-\frac{6(n-2)}{n}n^{\frac{2-\alpha_4}{2}}\left(n^{\frac{2-\alpha_4}{2}}h_n\right)x^3\right.\\&\left.+\frac{3n-7}{n}\left(n^{\frac{2-\alpha_4}{2}}h_n\right)^2x^2+\left(n^{\frac{1-\alpha_4}{3}}h_n\right)^3x\right],\end{aligned}$$

from where (3.25) results. From (3.25), we obtain (3.26). □

Theorem 3.4. *Let* $f:[0,\infty)\to\mathbb{R}$ *be a function. If* $x\in[0,\infty)$ *and* f *is two times differentiable in* x, *the function* $f^{(2)}$ *is continuous and bounded on a neighborhood of the point* x, *then*

$$\lim_{n\to\infty}n^{2-\alpha_2}\left[(C_{n,\infty}f)(x)-f(x)\right]=\frac{1}{2}l_2xf^{(2)}(x). \tag{3.27}$$

If f *is two times differentiable on* $[0,\infty)$, *the function* $f^{(2)}$ *is continuous and bounded on* $[0,\infty)$, *then the convergence given in* (3.27) *is uniform on any interval* $[0,a]\subset[0,\infty)$.

Proof. It results from Corollary 2.1 and Lemma 3.5. □

Remark 3.3. The Theorem 3.4 is a Voronovskaja's type theorem (see [1],[4] or [5]).

Remark 3.4. Taking (3.17) and (3.18) into account, we have $\lim_{n\to\infty}\frac{h_n}{n} = \lim_{n\to\infty} n^{1-\alpha_2} h_n n^{\alpha_2-2} = l_2 \lim_{n\to\infty} n^{\alpha_2-2} = 0$, so these operators from Case II are more general operators to Chlodovsky operators.

Example 3.2. Let $h_n = 2\sqrt[3]{n^2} + 1$, $n \in \mathbb{N}$. Then $\alpha_2 = \frac{5}{3}$, $\alpha_4 = \frac{10}{3}$, $l_2 = l_4 = 2$ and Theorem 3.3 and Theorem 3.4 hold.

Remark 3.5. If $\lim_{n\to\infty}\frac{h_n}{n} \neq 0$, it results that $\lim_{n\to\infty}\frac{h_n}{n} = \alpha$, where $\alpha > 0$ or $\lim_{n\to\infty}\frac{h_n}{n} = +\infty$. Then $\lim_{n\to\infty} n^{1-\alpha_2} h_n = \lim_{n\to\infty}\frac{h_n}{n} n^{2-\alpha_2} = +\infty$, so the relation (3.17) doesn't take place. In this situation, taking (1.6) into account, we have $\lim_{n\to\infty}(C_{n,\infty}e_2)(x) \neq e_2(x)$, $x \in [0,\infty)$, so the sequence of the operators $(C_{n,\infty}f)_{n\in\mathbb{N}}$ doesn't converge to f, for any function f, continuous and bounded on $[0,\infty)$.

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