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CONFORMALLY FLAT SEMI-PSEUDO SYMMETRIC MANIFOLDS

BY

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Abstract. In this paper we study a conformally flat semi pseudo symmetric manifold. Finally a special conformally flat semi pseudo symmetric manifold is studied.

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1. Introduction

In a paper [5], one of the authors, introduced a semi pseudo symmetric manifold. A non-flat Riemannian manifold (M^n, g), $n > 3$, whose curvature tensor R satisfies the condition,

$$(1)\qquad (\nabla_X R)(Y,Z)W = 2A(X)R(Y,Z)W + A(Y)R(X,Z)W + A(Z)R(Y,X)W + A(W)R(Y,Z)X,$$

where A is a non-zero 1-form, with associated vector field ρ, that is

$$(2)\qquad g(X,\rho) = A(X)$$

for every vector field X and ∇ denotes the operator of covariant differentiation with respect to the metric g, is called a semi-pseudo symmetric manifold and the 1-form A is called its associated 1-form. An n-dimensional semi-pseudo symmetric manifold is denoted by $(SPS)_n$.

In the year 1995, Tarafdar [6] had studied conformally at pseudo-symmetric manifolds. Motivated by this, the authors in the present paper dealt with a conformally flat $(SPS)_n$, $n > 3$. It is shown that, in such a manifold, the vector field ρ defined by (2) is a torse forming vector field [4]. Considering a conformally flat $(SPS)_n$, $n > 3$, of constant scalar curvature, it is shown that if in such a manifold, the associated 1-form A is not closed, then the vector field ρ is a proper torse forming vector field and if A is closed, then ρ is a proper concircular vector field. Further it is shown that if in a conformally flat $(SPS)_n$, $n > 3$, of constant scalar curvature, the associated 1-form is closed, then the manifold is a subprojective manifold in the sense of Kagan [1].

The notion of a special conformally flat manifold was introduced by Chen and Yano [2]. The last section deals with a type of conformally flat $(SPS)_n$, $n > 3$, in which a certain scalar is constant. It is shown that such an $(SPS)_n$ is a special conformally flat manifold and by theorem [2], it is shown that such an $(SPS)_n$, $n > 3$, can be isometrically immersed in Euclidean space $\mathbb{R}^{n+1}$ as a hypersurface.

2. Preliminaries

We now consider some formulae [3], [5] which will be required in the study of conformally flat $(SPS)_n$. Let S and r denote respectively the Ricci tensor of type (0, 2) and the scalar curvature and L denote the symmetric endomorphism of the tangent space at each point of the manifold, corresponding to the Ricci tensor S i.e.,

$$S(X, Y) = g(LX, Y) \tag{3}$$

for any vector fields X, Y in M. Let B be the 1-form defined by,

$$B(X) = A(LX) = g(LX, \rho) \tag{4}$$

for any vector field X in M. From (1) we get,

$$\begin{aligned}(\nabla_X S)(Y, Z) &= 2A(X)S(Y, Z) + A(Y)S(X, Z)\\ &\quad + A(Z)S(Y, X) + A(R(X, Y)Z).\end{aligned} \tag{5}$$

Contracting (5) we have,

$$dr(X) = 2A(X)r + 3A(LX) = 2A(X)r + 3B(X). \tag{6}$$

Also,

$$\begin{aligned}(\nabla_X S)(Y, Z) - (\nabla_Z S)(Y, X) &= A(X)S(Y, Z) - A(Z)S(X, Y)\\ &\quad + A(R(X, Z)Y).\end{aligned} \tag{7}$$

We shall use these formulae later on.

Let us recall:

Definition 1. A vector field ρ in a Riemannian manifold (M^n, g) is said to be torse forming [4] if it satisfies the equation $\nabla_X \rho = a\,X + \omega(X)\rho$, for every vector field X on M, where a is a non-zero scalar and ω is a 1-form.

Definition 2. A vector field ρ is said to be concircular if ω is a gradient vector field. If $d\omega = 0$, then ρ is said to be a proper concircular vector field [4].

3. Conformally flat $(SPS)_n, n > 3$

It is known that in a conformally flat $(SPS)_n$, $n > 3$, (on writing $Y = Z$; $Z = Y$ in the equation (5.5) of [5])

$$\begin{aligned} 0 &= (n-1)A(X)S(Y,Z) - (n-1)A(Y)S(X,Z) \\ &\quad -rA(X)g(Y,Z) + rA(Y)g(X,Z). \end{aligned} \tag{8}$$

Putting $Z = \rho$ in (8) and using (3), (4) and (2), we get,

$$\begin{aligned} (n-1)A(X)B(Y) - (n-1)A(Y)B(X) &= 0, \\ \text{or, } A(X)B(Y) - A(Y)B(X) &= 0, \quad \text{as n} > 3. \end{aligned} \tag{9}$$

Hence,

$$B(X) = tA(X) \tag{10}$$

holds for any vector field X in M where t is a scalar. Using (2), (3), (4), (10) we find from (8), $(n-1)A(\rho)S(Y,Z) - (nt-t-r)A(Y)A(Z) - rA(\rho)g(Y,Z) = 0$, i.e.,

$$S(Y,Z) = \frac{r}{n-1}\, g(Y,Z) + \frac{nt-t-r}{(n-1)A(\rho)} A(Y)A(Z). \tag{11}$$

From (11) we get,

$$LY = \frac{r}{n-1}\, Y + \frac{nt-t-r}{n-1}\, \frac{A(Y)}{A(\rho)}\, \rho. \tag{12}$$

Again in a conformally flat space,

$$\begin{aligned} R(X,Y)Z &= \frac{1}{n-2}\,[S(Y,Z)X - S(X,Z)Y + g(Y,Z)LX - g(X,Z)LY] \\ &\quad + \frac{r}{(n-1)(n-2)}\,[g(X,Z)Y - g(Y,Z)X]. \end{aligned}$$

Again,

$$A(R(X,Y)\rho) = g(R(X,Y)\rho,\rho) = 0 \tag{13}$$

and

$$A(R(X,\rho)Y) = \frac{t}{n-2}\,A(X)A(Y) - \frac{t}{n-2}\,g(X,Y)A(\rho). \tag{14}$$

Using (13) we get from (5) on using (3),(4),(10)

$$\begin{aligned} (\nabla_X S)(Y,\rho) &= 2A(X)tA(Y) + tA(Y)A(X) + A(\rho)S(X,Y) \\ &= A(\rho)S(X,Y) + 3tA(X)A(Y). \end{aligned} \tag{15}$$

Again, $(\nabla_X B)(Y) = \nabla_X B(Y) - B(\nabla_X Y) = A(Y)\nabla_X t + t\ \nabla_X a(Y) - t\,g(\nabla_X Y,\rho)$, by (4) and (5) $= A(Y)(X\cdot t) + t\nabla_X g(Y,\rho) - tg(\nabla_X Y,\rho)$. Therefore,

$$(\nabla_X B)Y = (X\cdot t)A(Y) + tg(Y,\nabla_X \rho). \tag{16}$$

Again, $(\nabla_X S)(Y,\rho) = \nabla_X S(Y,\rho) - S(\nabla_X Y,\rho) - S(Y,\nabla_X \rho) = (\nabla_X B)(Y) - S(Y,\nabla_X \rho)$, by (3), (4). Therefore,

$$(\nabla_X S)(Y,\rho) = (X\cdot t)\,A(Y) + t\,g(Y,\nabla_X\rho) - S(Y,\nabla_X\rho),\ \text{by (16)}. \tag{17}$$

Hence from (15) and (17), it follows that,

$$A(\rho)S(X,Y) + 3tA(X)A(Y) = (X\cdot t)A(Y) + tg(Y,\nabla_X\rho) - S(Y,\nabla_X\rho), \tag{18}$$

that is,

$$A(\rho)LX + 3tA(X)\rho = (X\cdot t)\rho + t(\nabla_X\rho) - L(\nabla_X\rho). \tag{19}$$

Using (12), we get

$$\begin{aligned} &\frac{r}{n-1}\,A(\rho)X + \left(\frac{nt-t-r}{n-1}\right)\,A(X)\rho \\ &\quad = (X\cdot t)\rho + \left(\frac{nt-t-r}{n-1}\right)\,\nabla_X\rho - \left(\frac{nt-t-r}{n-1}\right)\left(\frac{A(\nabla_X\rho)}{A(\rho)}\right)\,\rho. \end{aligned}$$

We note that, $A(\nabla_X\rho) = g(\nabla_X\rho, \rho)$. As, $(\nabla_X g)(\rho, \rho) = 0$, we get $A(\nabla_X\rho) = \frac{1}{2}\, X \cdot g(\rho, \rho) = \frac{1}{2}\, X \cdot A(\rho)$.

Thus the above result reduces to

$$\begin{aligned} \nabla_X\rho &= \left(\frac{rA(\rho)}{nt-t-r}\right) X + \left(A(X) - \frac{(n-1)(X\cdot t)}{nt-t-r} + \frac{X\cdot A(\rho)}{2A(\rho)}\right)\rho \\ &= \sigma X + \omega(X)\rho, \qquad (20) \end{aligned}$$

where

$$\sigma = \frac{r}{nt-t-r}\, A(\rho) \qquad (21)$$

and

$$\omega(X) = A(X) - \frac{n-1}{nt-t-r}\,(X\cdot t) + \frac{X\cdot A(\rho)}{2A(\rho)}. \qquad (22)$$

If $\sigma = 0$, then from (21) we get $r = 0$. Consequently from (6) we conclude that $B(X) = 0$. Using (10) this yields $tA(X) = 0$. Hence $A(X) = 0$, which is inadmissible by definition.

Hence, $\sigma \neq 0$ and we can state:

Theorem 3.1. *In a conformally flat $(SPS)_n, n > 3$, the vector field ρ given by* (2) *is a torse forming vector field.*

4. Conformally flat $(SPS)_n, n > 3$, with constant scalar curvature

If r is a non-zero constant, then from (6) we get $2A(X)r + 3B(X) = 0$, that is,

$$B(X) = -\frac{2}{3}\, rA(X). \qquad (23)$$

In this case, taking $t = -\frac{2}{3}r$, from (21) and (22) we get $\sigma = -\frac{3}{2n+1}\, A(\rho)$ and $\omega(X) = A(X) + \frac{X\cdot A(\rho)}{2A(\rho)}$. Thus,

$$\nabla_X\rho = \sigma X + \omega(X)\rho, \qquad (24)$$

where

$$\sigma = -\frac{3}{2n+1}\, A(\rho) \qquad (25)$$

and

$$\omega(X) = A(X) + \frac{X \cdot A(\rho)}{A(\rho)}. \tag{26}$$

From (25), it follows that σ cannot be zero. Again from (26), it follows that ω is closed if A is closed and ω is not closed if A is not so. Hence we can state:

Theorem 4.1. *In a conformally flat $(SPS)_n (n > 3)$ with constant scalar curvature, the vector field ρ defined by (2) is a proper torse forming vector field if the associated 1-form is not closed and ρ is a proper concircular vector field if A is closed.*

It is known [1] that if a conformally flat manifold admits a proper concircular vector field, then, it is a subprojective manifold in the sense of Kagan.

Thus we can state

Theorem 4.2. *If in a conformally flat $(SPS)_n, n > 3$, with constant scalar curvature, the associated 1-form is closed, the manifold is a subprojective manifold in the sense of Kagan.*

5. Conformally flat $(SPS)_n, n > 3$, with constant scalar t

It is to be noted that if in a conformally flat $(SPS)_n, n > 3$, the scalar curvature r is constant, then the scalar t defined by (10) is equal to $-\frac{2}{3}r$ and is thus a constant. On the other hand, if the scalar t is a constant, then the scalar curvature r is not necessarily a constant. In this section, we consider a conformally flat $(SPS)_n, n > 3$, with constant scalar t.

From (11) we have, $S(Y,Z) = \frac{r}{n-1}\, g(Y,Z) + \frac{nt-t-r}{n-1}\, T(Y)T(Z)$, where

$$T(X) = \frac{A(X)}{\sqrt{A(\rho)}}. \tag{27}$$

Let,

$$H(X,Y) = -\frac{1}{n-2}\, S(X,Y) + \frac{r}{2(n-1)(n-2)}\, g(X,Y). \tag{28}$$

On using (26), we find from above,

$$H(X,Y) = -\frac{r}{2(n-1)(n-2)}\, g(X,Y) - \frac{nt-t-r}{(n-1)(n-2)}\, T(X)T(Y). \tag{29}$$

We now put

$$\alpha^2 = \frac{r}{(n-1)(n-2)}. \tag{30}$$

Then $2\alpha(X \cdot \alpha) = \frac{X \cdot r}{(n-1)(n-2)}$. Again from (6) we have $X \cdot r = dr(X) = 2A(X)r + 3B(X) = A(X)(2r + 3t)$. Thus from above

$$\alpha(X \cdot \alpha) = \frac{A(X)(2r + 3t)}{2(n-1)(n-2)}. \tag{31}$$

Hence $\alpha(X \cdot \alpha)\alpha(Y \cdot \alpha) = \frac{A(X)A(Y)(2r+3t)^2}{4(n-1)^2(n-2)^2}$ or, $(X \cdot \alpha)(Y \cdot \alpha)\ \frac{r}{(n-1)(n-2)} = \frac{A(X)A(Y)}{4(n-1)^2(n-2)^2}\ (2r + 3t)^2$ by (29). Thus

$$(X \cdot \alpha)(Y \cdot \alpha) = \frac{A(\rho)(2r + 3t)^2}{4r(n-1)(n-2)}\ T(X)T(Y). \tag{32}$$

Thus (28) takes the form,

$$H(X, Y) = -\frac{\alpha^2}{2}\ g(X, Y) + \beta(X \cdot \alpha)(Y \cdot \alpha) \tag{33}$$

where

$$\beta = \frac{4r(r + t - nt)}{A(\rho)(2r + 3t)^2}. \tag{34}$$

If α=0, then r=0 and consequently from (6), $A(X) = 0$, which is inadmissible. Thus α cannot be zero and hence we may take α as positive in (32).

According to Chen and Yano [2], if in a conformally flat manifold, a (0,2) tensor H defined by

$$H(X, Y) = -\frac{1}{n-2}S(X, Y) + \frac{r}{2(n-1)(n-2)}g(X, Y) \tag{35}$$

is expressible in the form

$$H(X, Y) = -\frac{\alpha^2}{2}g(X, Y) + \beta(X \cdot \alpha)(Y \cdot \alpha) \tag{36}$$

where α, β are two scalars such that α is positive, then the manifold is said to be a special conformally flat manifold. Hence from (33), it follows that

$(SPS)_n$ under consideration is a special conformally flat manifold. It is known from a theorem [2] that every simply connected special conformally flat manifold can be isometrically immersed in a Euclidean space $\mathbb{R}^{n+1}$ as a hypersurface. Thus we state,

Theorem 5.1. *Every simply connected conformally flat* $(SPS)_n, (n > 3)$ *with constant scalar t can be isometrically immersed in a Euclidean space* $\mathbb{R}^{n+1}$ *as a hypersurface.*

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REFERENCES

1. Adati, T. – *On Subprojective Spaces I, II, III*, Tôhoku Math. J., Vol. 3, 1951, 159–173, 330–342, 343–358.
2. Chen, B.Y.; Yano, K. – *Special conformally flat spaces and canal hypersurfaces*, Collection of articles dedicated to Shigeo Sasaki on his sixtieth birthday, Tôhoku Math. J., 25 (1973), 177–184.
3. Eisenhart, L.P. – *Riemannian Geometry*, Princeton University Press, Princeton, N.J., 1949.
4. Schouten, J.A. – *Ricci-Calculus. An Introduction to Tensor Analysis and its Geometrical Applications*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1954.
5. Tarafdar, M.; Jawarneh, Musa A.A. – *Semi-pseudosymmetric manifold*, An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi, Mat., 41 (1995), 145–152.
6. Tarafdar, M. – *On conformally flat pseudo-symmetric manifolds*, An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi, Mat., 41 (1995), 237–241.

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