

## ***C*-TOTALLY REAL WARPED PRODUCT SUBMANIFOLDS**

BY

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**Abstract.** We obtain a basic inequality involving the Laplacian of the warping function and the squared mean curvature of any warped product isometrically immersed in a Riemannian manifold (cf. Theorem 2.2). Applying this general theory, we obtain basic inequalities involving the Laplacian of the warping function and the squared mean curvature of *C*-totally real warped product submanifolds of  $(\kappa, \mu)$ -space forms, Sasakian space forms and non-Sasakian  $(\kappa, \mu)$ -manifolds. Then we obtain obstructions to the existence of minimal isometric immersions of *C*-totally real warped product submanifolds in  $(\kappa, \mu)$ -space forms, non-Sasakian  $(\kappa, \mu)$ -manifolds and Sasakian space forms. In the last, we obtain an example of a *C*-totally real warped product submanifold of a non-Sasakian  $(\kappa, \mu)$ -manifold, which satisfies the equality case of the basic inequality.

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### **1. Introduction**

The concept of warped product was first introduced by BISHOP and O'NEILL [1] to construct the examples of Riemannian manifolds with negative curvature. Let  $M_1$  and  $M_2$  be two Riemannian manifolds with Riemannian metrics  $g_1$  and  $g_2$  respectively. Let  $f > 0$  be a differentiable function on  $M_1$ . Consider the product manifold  $M_1 \times M_2$  with its canonical projections  $\pi_1 : M_1 \times M_2 \rightarrow M_1$  and  $\pi_2 : M_1 \times M_2 \rightarrow M_2$ . The warped product  $M = M_1 \times_f M_2$  is the product manifold  $M_1 \times M_2$  equipped with the Riemannian structure such that

$$(1.1) \quad \|X\|^2 = \|\pi_{1*}X\|^2 + (f^2 \circ \pi_1)(p) \|\pi_{2*}X\|^2$$

for each tangent vector  $X \in T_pM$ ,  $p \in M$ , or equivalently, the Riemannian metric  $g$  of  $M$  is given by

$$(1.2) \quad g = g_1 + f^2 g_2$$

with the usual meaning. The function  $f$  is called the warping function of the warped product. In particular, if the warping function is constant, then the manifold  $M$  is said to be *trivial*.

For a warped product  $M_1 \times_f M_2$  we denote by  $\mathcal{D}_1$  and  $\mathcal{D}_2$  the distributions given by the vectors tangent to leaves and fibres, respectively. Thus,  $\mathcal{D}_1$  is obtained from tangent vectors to  $M_1$  via the horizontal lift and  $\mathcal{D}_2$  is obtained by the tangent vectors of  $M_2$  via the vertical lift. It is well known that

$$(1.3) \quad \nabla_X Y = \nabla_Y X = \frac{1}{f}(Xf)Y, \quad X \in \mathcal{D}_1, \quad Y \in \mathcal{D}_2.$$

From (1.3) for unit vector fields  $X, Z$  tangent to  $M$ , horizontal and vertical, respectively; the sectional curvature  $K(X \wedge Z)$  of the plane section spanned by  $X$  and  $Z$  becomes

$$(1.4) \quad K(X \wedge Z) = g(\nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z) = \frac{1}{f} \{(\nabla_X X)f - X^2 f\}.$$

Choosing a local orthonormal frame  $\{e_1, \dots, e_n\}$  such that  $e_1, \dots, e_{n_1}$  are tangent to  $M_1$  and  $e_{n_1+1}, \dots, e_n$  are tangent to  $M_2$ , we obtain

$$(1.5) \quad \frac{\Delta f}{f} = \sum_{j=1}^{n_1} K(e_j \wedge e_s), \quad s \in \{n_1 + 1, \dots, n\}.$$

The notion of warped products plays some important roles in differential geometry as well as in physics. For instance, the best relativistic model of the Schwarzschild space-time that describes the outer space around a massive star or a black hole is given as a warped product [22, pp. 364-367]. For more details we refer to [1] and [22]. For a survey on warped products as Riemannian submanifolds we refer to [12].

In [10, Theorem 1.4], CHEN established a sharp relationship between the warping function  $f$  of a warped product  $M_1 \times_f M_2$  isometrically immersed in a real space form  $R^m(c)$  and the squared mean curvature  $\|H\|^2$  given by

$$(1.6) \quad \frac{\Delta f}{f} \leq \frac{(n_1 + n_2)^2}{4n_2} \|H\|^2 + n_1 c,$$

where  $n_i = \dim(M_i)$ ,  $i = 1, 2$  and  $\Delta$  is the Laplacian operator of  $M_1$ . The equality case of (1.6) holds identically if and only if  $x$  is a mixed totally geodesic immersion and  $n_1H_1 = n_2H_2$ , where  $H_i$ ,  $i = 1, 2$ , are the partial mean curvature vectors. As a refinement, in [15, Theorem 5], he proved that the equality case of (1.6) is true if and only if one of the following two cases occur: (1) The warping function  $f$  is an eigenfunction of the Laplacian operator  $\Delta$  with eigenvalue  $n_1c$  and  $x$  is a minimal immersion; (2)  $\Delta f \neq (n_1c)f$  and locally  $x$  is a non-minimal warped product immersion  $(x_1, x_2) : M_1 \times_f M_2 \rightarrow N_1 \times_\rho N_2$  of  $M_1 \times_f M_2$  into some warped product representation  $N_1 \times_\rho N_2$  of the real space form  $\bar{M}(c)$  such that  $x_2 : M_2 \rightarrow N_2$  is a minimal immersion and the mean curvature vector of  $x_1 : M_1 \rightarrow N_1$  is given by  $-(n_2/n_1)D \ln \rho$ , where  $D \ln \rho$  is the normal component of the gradient of  $\rho$  restricted on  $M_1$ .

The inequality (1.6) was noticed by several authors and they established similar inequalities for different submanifolds in ambient manifolds possessing different kind of structures (for example, see [11], [13], [14], [16], [18], [19], [20], [26], [27], [28]). Now, it is a natural motivation to find a basic inequality involving the warping function and the squared mean curvature of any warped product isometrically immersed in any Riemannian manifold without assuming any restriction on the Riemann curvature tensor of the ambient manifold. Using technique of B.-Y. Chen and the concept of the scalar curvature of  $k$ -plane sections, in section 2 we achieve this goal by obtaining a basic inequality involving the Laplacian of the warping function  $f$  and the squared mean curvature of a warped product  $M_1 \times_f M_2$  isometrically immersed in a Riemannian manifold (see Theorem 2.2). Section 3 contains some necessary background of contact geometry including the concepts of Sasakian manifolds,  $(\kappa, \mu)$ -manifolds,  $(\kappa, \mu)$ -space forms and non-Sasakian  $(\kappa, \mu)$ -manifolds. In section 4, we apply the general theory given by Theorem 2.2 to obtain corresponding results for *C*-totally real warped product submanifolds of  $(\kappa, \mu)$ -space forms, Sasakian space forms and non-Sasakian  $(\kappa, \mu)$ -manifolds. Then we obtain obstructions to the existence of minimal isometric immersions of *C*-totally real warped product submanifolds in these spaces. In the last, we also give an example of a warped product *C*-totally real submanifold of a non-Sasakian  $(\kappa, \mu)$ -manifold which satisfies the equality case of the corresponding basic inequality.

## 2. A basic inequality for warped product submanifolds

Let  $M$  be an  $n$ -dimensional Riemannian manifold equipped with a Riemannian metric  $g$ . The inner product of the metric  $g$  is denoted by  $\langle \cdot, \cdot \rangle$ . Let  $\{e_1, \dots, e_n\}$  be any orthonormal basis for  $T_p M$ . The *scalar curvature*  $\tau(p)$  of  $M$  at  $p$  is defined by

$$(2.1) \quad \tau(p) = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j),$$

where  $K(e_i \wedge e_j)$  is the *sectional curvature* of the plane section spanned by  $e_i$  and  $e_j$  at  $p \in M$ . Let  $\Pi_k$  be a  $k$ -plane section of  $T_p M$  and  $\{e_1, \dots, e_k\}$  any orthonormal basis of  $\Pi_k$ . The scalar curvature  $\tau(\Pi_k)$  of  $\Pi_k$  is given by [9]

$$(2.2) \quad \tau(\Pi_k) = \sum_{1 \leq i < j \leq k} K(e_i \wedge e_j).$$

The scalar curvature  $\tau(p)$  of  $M$  at  $p$  is identical with the scalar curvature of the tangent space  $T_p M$  of  $M$  at  $p$ , that is,  $\tau(p) = \tau(T_p M)$ . Geometrically,  $\tau(\Pi_k)$  is the scalar curvature of the image  $\exp_p(\Pi_k)$  of  $\Pi_k$  at  $p$  under the exponential map at  $p$ . If  $\Pi_2$  is a 2-plane section,  $\tau(\Pi_2)$  is simply the sectional curvature  $K(\Pi_2)$  of  $\Pi_2$ .

Let  $(M, g)$  be a submanifold of a Riemannian manifold  $\tilde{M}$  equipped with a Riemannian metric  $\tilde{g}$ . Covariant derivatives and curvatures with respect to  $(M, g)$  will be written in the usual manner, while those with respect to the ambient manifold  $(\tilde{M}, \tilde{g})$  will be written with “tildes” over them. We use the inner product notation  $\langle \cdot, \cdot \rangle$  for the metric  $\tilde{g}$  of  $\tilde{M}$  as well as for the induced metrics on the submanifold  $M$  and its normal bundle.

The Gauss and Weingarten formulas are given respectively by  $\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y)$  and  $\tilde{\nabla}_X N = -A_N X + \nabla_X^\perp N$ , for all  $X, Y \in TM$  and  $N \in T^\perp M$ , where  $\tilde{\nabla}$ ,  $\nabla$  and  $\nabla^\perp$  are respectively the Riemannian, induced Riemannian and induced normal connections in  $\tilde{M}$ ,  $M$  and the normal bundle  $T^\perp M$  of  $M$  respectively, and  $\sigma$  is the second fundamental form related to the shape operator  $A$  by  $\langle \sigma(X, Y), N \rangle = \langle A_N X, Y \rangle$ . The equation of Gauss is given by

$$(2.3) \quad \begin{aligned} R(X, Y, Z, W) &= \tilde{R}(X, Y, Z, W) + \langle \sigma(X, W), \sigma(Y, Z) \rangle \\ &\quad - \langle \sigma(X, Z), \sigma(Y, W) \rangle, \end{aligned}$$

for all  $X, Y, Z, W \in TM$ , where  $\tilde{R}$  and  $R$  are the curvature tensors of  $\tilde{M}$  and  $M$  respectively.

For any orthonormal basis  $\{e_1, \dots, e_n\}$  of the tangent space  $T_pM$ , the mean curvature vector  $H(p)$  is given by

$$(2.4) \quad H(p) = \frac{1}{n} \sum_{i=1}^n \sigma(e_i, e_i),$$

where  $n = \dim(M)$ . The submanifold  $M$  is *totally geodesic* in  $\tilde{M}$  if  $\sigma = 0$ , and *minimal* if  $H = 0$ . If  $\sigma(X, Y) = g(X, Y)H$  for all  $X, Y \in TM$ , then  $M$  is *totally umbilical*.

Now, let  $\{e_1, \dots, e_n\}$  be an orthonormal basis of the tangent space  $T_pM$  and  $e_r$  belongs to an orthonormal basis  $\{e_{n+1}, \dots, e_m\}$  of the normal space  $T_p^\perp M$ . We put  $\sigma_{ij}^r = \langle \sigma(e_i, e_j), e_r \rangle$  and  $\|\sigma\|^2 = \sum_{i,j=1}^n \langle \sigma(e_i, e_j), \sigma(e_i, e_j) \rangle$ . Let  $K(e_i \wedge e_j)$  and  $\tilde{K}(e_i \wedge e_j)$  denote the sectional curvature of the plane section spanned by  $e_i$  and  $e_j$  at  $p$  in the submanifold  $M$  and in the ambient manifold  $\tilde{M}$  respectively. In view of the equation (2.3) of Gauss, we have

$$(2.5) \quad K(e_i \wedge e_j) = \tilde{K}(e_i \wedge e_j) + \sum_{r=n+1}^m (\sigma_{ii}^r \sigma_{jj}^r - (\sigma_{ij}^r)^2).$$

From (2.5) it follows that

$$(2.6) \quad 2\tau(p) = 2\tilde{\tau}(T_pM) + n^2 \|H\|^2 - \|\sigma\|^2,$$

where

$$\tilde{\tau}(T_pM) = \sum_{1 \leq i < j \leq n} \tilde{K}(e_i \wedge e_j)$$

denotes the scalar curvature of the  $n$ -plane section  $T_pM$  in the ambient manifold  $\tilde{M}$ .

Now, let  $x : M_1 \times_f M_2 \rightarrow \tilde{M}$  be an isometric immersion of a warped product  $M_1 \times_f M_2$  into a Riemannian manifold  $\tilde{M}$ . We write  $n_i H_i = \text{trace}(\sigma_i|_{M_i})$ , where  $\text{trace}(\sigma_i|_{M_i})$  is the trace of the second fundamental form  $\sigma$  of  $x$  restricted to  $M_i$  and  $n_i = \dim M_i$  ( $i = 1, 2$ ). The immersion  $x$  is called *mixed totally geodesic* if  $\sigma(X, Y) = 0$  for  $X \in \mathcal{D}_1$  and  $Y \in \mathcal{D}_2$ .

We shall need the following Lemma.

**Lemma 2.1** ([8]). *Let  $\ell \geq 2$  and  $a_1, \dots, a_\ell, b$  be real numbers such that*

$$\left( \sum_{i=1}^{\ell} a_i \right)^2 = (\ell - 1) \left( \sum_{i=1}^{\ell} a_i^2 + b \right).$$

*Then  $2a_1a_2 \geq b$ , with equality holding if and only if  $a_1 + a_2 = a_3 = \dots = a_\ell$ .*

Now, we obtain a basic inequality involving the Laplacian of the warping function and the squared mean curvature of a warped product submanifold of a Riemannian manifold.

**Theorem 2.2.** *Let  $x$  be an isometric immersion of an  $n$ -dimensional warped product manifold  $M_1 \times_f M_2$  into an  $m$ -dimensional Riemannian manifold  $\tilde{M}$ . Then*

$$(2.7) \quad n_2 \frac{\Delta f}{f} \leq \frac{n^2}{4} \|H\|^2 + \tilde{\tau}(T_p M) - \tilde{\tau}(T_p M_1) - \tilde{\tau}(T_p M_2),$$

where  $n_i = \dim M_i$ ,  $i = 1, 2$ , and  $\Delta$  is the Laplacian operator of  $M_1$ .

Moreover, the equality case of (2.7) holds identically if and only if  $x$  is a mixed totally geodesic immersion and  $n_1 H_1 = n_2 H_2$ , where  $H_i$ ,  $i = 1, 2$ , are the partial mean curvature vectors.

**Proof.** Let  $M = M_1 \times_f M_2$  be a warped product submanifold of a Riemannian manifold  $\tilde{M}$ . We set

$$(2.8) \quad 2\delta = 4\tau(p) - 4\tilde{\tau}(T_p M) - n^2 \|H\|^2,$$

so that the equation (2.6) can be written as

$$(2.9) \quad n^2 \|H\|^2 = 2(\delta + \|\sigma\|^2).$$

Choose a local orthonormal frame  $\{e_1, \dots, e_n, e_{n+1}, \dots, e_m\}$ , such that  $e_1, \dots, e_{n_1}$  are tangent to  $M_1$ ,  $e_{n_1+1}, \dots, e_n$  are tangent to  $M_2$  and  $e_{n+1}$  is parallel to the mean curvature vector  $H$ . If we put

$$a_1 = \sigma_{11}^{n+1}, \quad a_2 = \sum_{i=2}^{n_1} \sigma_{ii}^{n+1}, \quad a_3 = \sum_{t=n_1+1}^n \sigma_{tt}^{n+1},$$

then with respect to the above orthonormal frame the equation (2.9) becomes

$$(2.10) \quad \left( \sum_{i=1}^3 a_i \right)^2 = 2 \left( \sum_{i=1}^3 a_i^2 + b \right),$$

where

$$(2.11) \quad b = \left\{ \delta + \sum_{1 \leq i \neq j \leq n} (\sigma_{ij}^{n+1})^2 + \sum_{r=n+2}^m \sum_{i,j=1}^n (\sigma_{ij}^r)^2 \right. \\ \left. - \sum_{2 \leq j \neq k \leq n_1} \sigma_{jj}^{n+1} \sigma_{kk}^{n+1} - \sum_{n_1+1 \leq s \neq t \leq n} \sigma_{ss}^{n+1} \sigma_{tt}^{n+1} \right\}.$$

Applying Lemma 2.1 to (2.10), we get  $2a_1a_2 \geq b$ , with equality holding if and only if  $a_1 + a_2 = a_3$ . Equivalently, we get

$$(2.12) \quad \sum_{1 \leq j < k \leq n_1} \sigma_{jj}^{n+1} \sigma_{kk}^{n+1} + \sum_{n_1+1 \leq s < t \leq n} \sigma_{ss}^{n+1} \sigma_{tt}^{n+1} \\ \geq \frac{\delta}{2} + \sum_{1 \leq \alpha < \beta \leq n} (\sigma_{\alpha\beta}^{n+1})^2 + \frac{1}{2} \sum_{r=n+2}^m \sum_{\alpha,\beta=1}^n (\sigma_{\alpha\beta}^r)^2$$

with equality holding if and only if

$$(2.13) \quad \sum_{i=1}^{n_1} \sigma_{ii}^{n+1} = \sum_{s=n_1+1}^n \sigma_{ss}^{n+1}.$$

From equation (1.5) we get

$$n_2 \frac{\Delta f}{f} = \sum_{j=1}^{n_1} \sum_{s=n_1+1}^n K(e_j \wedge e_s) = \tau(p) - \tau(T_p M_1) - \tau(T_p M_2),$$

which in view of (2.3) gives

$$(2.14) \quad n_2 \frac{\Delta f}{f} = \tau(p) - \tilde{\tau}(T_p M_1) - \sum_{r=n+1}^m \sum_{1 \leq j < k \leq n_1} (\sigma_{jj}^r \sigma_{kk}^r - (\sigma_{jk}^r)^2) \\ - \tilde{\tau}(T_p M_2) \sum_{r=n+1}^m \sum_{n_1+1 \leq s < t \leq n} (\sigma_{ss}^r \sigma_{tt}^r - (\sigma_{st}^r)^2).$$

In view of (2.12), (2.14) and (2.8) we get

$$n_2 \frac{\Delta f}{f} \leq \frac{n^2}{4} \|H\|^2 + \tilde{\tau}(T_p M) - \tilde{\tau}(T_p M_1) - \tilde{\tau}(T_p M_2)$$

$$\begin{aligned}
& - \sum_{j=1}^{n_1} \sum_{t=n_1+1}^n (\sigma_{jt}^{n+1})^2 - \frac{1}{2} \sum_{r=n+2}^m \sum_{\alpha,\beta=1}^n (\sigma_{\alpha\beta}^r)^2 \\
& + \sum_{r=n+2}^m \sum_{1 \leq j < k \leq n_1} ((\sigma_{jk}^r)^2 - \sigma_{jj}^r \sigma_{kk}^r) + \sum_{r=n+2}^m \sum_{n_1+1 \leq s < t \leq n} ((\sigma_{st}^r)^2 - \sigma_{ss}^r \sigma_{tt}^r)
\end{aligned}$$

or

$$\begin{aligned}
(2.15) \quad n_2 \frac{\Delta f}{f} & \leq \frac{n^2}{4} \|H\|^2 + \tilde{\tau}(T_p M) - \tilde{\tau}(T_p M_1) \\
& - \tilde{\tau}(T_p M_2) - \sum_{r=n+1}^m \sum_{j=1}^{n_1} \sum_{t=n_1+1}^n (\sigma_{jt}^r)^2 \\
& - \frac{1}{2} \sum_{r=n+2}^m \left( \sum_{j=1}^{n_1} \sigma_{jj}^r \right)^2 - \frac{1}{2} \sum_{r=n+2}^m \left( \sum_{t=n_1+1}^n \sigma_{tt}^r \right)^2,
\end{aligned}$$

which implies the inequality (2.7).

The equality sign of (2.15) is true if and only if

$$(2.16) \quad \sigma_{jt}^r = 0, \quad 1 \leq j \leq n_1, \quad n_1 + 1 \leq t \leq n, \quad n + 1 \leq r \leq m,$$

and

$$(2.17) \quad \sum_{i=1}^{n_1} \sigma_{ii}^r = 0 = \sum_{t=n_1+1}^n \sigma_{tt}^r, \quad n + 2 \leq r \leq m.$$

Obviously (2.16) is true if and only if the warped product  $M_1 \times_f M_2$  is mixed totally geodesic. From the equations (2.13) and (2.17) it follows that  $n_1 H_1 = n_2 H_2$ .

The converse statement is straightforward.  $\square$

### 3. $(\kappa, \mu)$ -manifolds

A  $(2m + 1)$ -dimensional differentiable manifold  $\tilde{M}$  is called an almost contact metric manifold if there is an almost contact metric structure  $(\varphi, \xi, \eta, g)$  consisting of a  $(1, 1)$  tensor field  $\varphi$ , a vector field  $\xi$ , a 1-form  $\eta$  and a compatible Riemannian metric  $g$  satisfying

$$(3.1) \quad \varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0,$$

$$(3.2) \quad \langle X, Y \rangle = \langle \varphi X, \varphi Y \rangle + \eta(X)\eta(Y),$$

$$(3.3) \quad \langle X, \varphi Y \rangle = -\langle \varphi X, Y \rangle, \quad \langle X, \xi \rangle = \eta(X)$$



for all  $X, Y \in T\tilde{M}$ , where  $\langle \cdot, \cdot \rangle$  is the inner product as in previous section. An almost contact metric structure becomes a *contact metric structure* if  $\langle X, \varphi Y \rangle = d\eta(X, Y)$ ,  $X, Y \in T\tilde{M}$ . A contact metric structure is a *Sasakian structure* if and only if the Riemann curvature tensor  $\tilde{R}$  satisfies

$$(3.4) \quad \tilde{R}(X, Y)\xi = \eta(Y)X - \eta(X)Y, \quad X, Y \in T\tilde{M}.$$

In a contact metric manifold  $\tilde{M}$ , the  $(1, 1)$ -tensor field  $h$  defined by  $2h = \mathcal{L}_\xi\varphi$ , which is the Lie derivative of  $\varphi$  in the characteristic direction  $\xi$ , is symmetric and satisfies

$$(3.5) \quad h\xi = 0, \quad h\varphi + \varphi h = 0, \quad \tilde{\nabla}\xi = -\varphi - \varphi h, \quad \text{trace}(h) = \text{trace}(\varphi h) = 0,$$

where  $\tilde{\nabla}$  is Levi-Civita connection. In [3], BLAIR, KOUFOGIORGOS and PAPANTONIOU introduced the class of contact metric manifolds  $\tilde{M}$  with contact metric structures  $(\varphi, \xi, \eta, g)$ , which satisfy

$$(3.6) \quad \tilde{R}(X, Y)\xi = (\kappa I + \mu h)(\eta(Y)X - \eta(X)Y), \quad X, Y \in T\tilde{M},$$

where  $\kappa, \mu$  are real constants. A contact metric manifold belonging to this class is called a  $(\kappa, \mu)$ -manifold. In a  $(\kappa, \mu)$ -manifold we have  $h^2 = (\kappa - 1)\varphi^2$  and  $\kappa \leq 1$ . For a  $(\kappa, \mu)$ -manifold, the conditions of being a Sasakian manifold,  $\kappa = 1$  and  $h = 0$  are all equivalent. If  $\kappa < 1$ , and the eigenvalues of  $h$  are  $0, \lambda$  and  $-\lambda$ , where  $\lambda = \sqrt{1 - \kappa}$ . The eigenspace relative to the eigenvalue  $0$  is  $\{\xi\}$ . Moreover, for  $\kappa \neq 1$ , the subbundle  $\mathcal{D} = \ker(\eta)$  can be decomposed in the eigenspace distributions  $\mathcal{D}_+$  and  $\mathcal{D}_-$  relative to the eigenvalues  $\lambda$  and  $-\lambda$ , respectively. These distributions are orthogonal to each other and have dimension  $n$ . There are many motivations for studying  $(\kappa, \mu)$ -manifolds: the first is that, in the non-Sasakian case the condition (3.6) determines the curvature completely; moreover, while the values of  $\kappa$  and  $\mu$  change, the form of (3.6) is invariant under  $D$ -homothetic deformations ([3]); finally, there is a complete classification of these manifolds, given in [5] by BOECKX, who proved also that any non-Sasakian  $(\kappa, \mu)$ -manifold is locally homogeneous and strongly locally  $\varphi$ -symmetric ([4], [6]). There are also non-trivial examples of  $(\kappa, \mu)$ -manifolds, the most important being the unit tangent sphere bundle  $T_1\tilde{M}$  of a Riemannian manifold  $\tilde{M}$  of constant sectional curvature with the usual contact metric structure. In particular, if  $\tilde{M}$  has constant curvature  $c$ , then  $\kappa = c(2 - c)$  and  $\mu = -2c$ . Characteristic examples of non-Sasakian  $(\kappa, \mu)$ -manifolds are the tangent sphere bundles

of Riemannian manifolds of constant sectional curvature not equal to one and certain Lie groups [5]. For more details we refer to [2], [3] and [17].

Like complex space forms in Hermitian geometry, in contact geometry we have the notion of manifolds with constant  $\varphi$ -sectional curvature. In an almost contact metric manifold, for a unit vector  $X$  orthogonal to  $\xi$ , the sectional curvature  $\tilde{K}(X \wedge \varphi X)$  is called a  $\varphi$ -sectional curvature. In [17], KOUFOGIORGOS showed that in a  $(\kappa, \mu)$ -manifold  $(\tilde{M}, \varphi, \xi, \eta, g)$  of dimension  $> 3$  if the  $\varphi$ -sectional curvature at a point  $p$  is independent of the  $\varphi$ -section at  $p$ , then it is constant. If the  $(\kappa, \mu)$ -manifold  $\tilde{M}$  has constant  $\varphi$ -sectional curvature  $c$  then it is called a  $(\kappa, \mu)$ -space form and is denoted by  $\tilde{M}(c)$ . The Riemann curvature tensor  $\tilde{R}$  of  $\tilde{M}(c)$  is given explicitly in its  $(0, 4)$ -form by [17]

$$\begin{aligned}
 \tilde{R}(X, Y, Z, W) &= \frac{c+3}{4} \{ \langle Y, Z \rangle \langle X, W \rangle - \langle X, Z \rangle \langle Y, W \rangle \} \\
 &+ \frac{c-1}{4} \{ 2 \langle X, \varphi Y \rangle \langle \varphi Z, W \rangle + \langle X, \varphi Z \rangle \langle \varphi Y, W \rangle - \langle Y, \varphi Z \rangle \langle \varphi X, W \rangle \} \\
 &+ \frac{c+3-4\kappa}{4} \{ \eta(X) \eta(Z) \langle Y, W \rangle - \eta(Y) \eta(Z) \langle X, W \rangle \\
 &+ \langle X, Z \rangle \eta(Y) \eta(W) - \langle Y, Z \rangle \eta(X) \eta(W) \} \\
 &+ \frac{1}{2} \{ \langle hY, Z \rangle \langle hX, W \rangle - \langle hX, Z \rangle \langle hY, W \rangle \\
 (3.7) \quad &+ \langle \varphi hX, Z \rangle \langle \varphi hY, W \rangle - \langle \varphi hY, Z \rangle \langle \varphi hX, W \rangle \} \\
 &+ \langle \varphi Y, \varphi Z \rangle \langle hX, W \rangle - \langle \varphi X, \varphi Z \rangle \langle hY, W \rangle \\
 &+ \langle hX, Z \rangle \langle \varphi^2 Y, W \rangle - \langle hY, Z \rangle \langle \varphi^2 X, W \rangle \\
 &+ \mu \{ \eta(Y) \eta(Z) \langle hX, W \rangle - \eta(X) \eta(Z) \langle hY, W \rangle \\
 &+ \langle hY, Z \rangle \eta(X) \eta(W) - \langle hX, Z \rangle \eta(Y) \eta(W) \},
 \end{aligned}$$

for all  $X, Y, Z, W \in T\tilde{M}$ . In particular, if  $\kappa = 1$  then a  $(\kappa, \mu)$ -space form  $\tilde{M}(c)$  reduces to a Sasakian space form  $\tilde{M}(c)$  and (3.7) reduces to

$$\begin{aligned}
 \tilde{R}(X, Y, Z, W) &= \frac{c+3}{4} \{ \langle Y, Z \rangle \langle X, W \rangle - \langle X, Z \rangle \langle Y, W \rangle \} \\
 &+ \frac{c-1}{4} \{ 2 \langle X, \varphi Y \rangle \langle \varphi Z, W \rangle + \langle X, \varphi Z \rangle \langle \varphi Y, W \rangle - \langle Y, \varphi Z \rangle \langle \varphi X, W \rangle \\
 (3.8) \quad &+ \eta(X) \eta(Z) \langle Y, W \rangle - \eta(Y) \eta(Z) \langle X, W \rangle \\
 &+ \langle X, Z \rangle \eta(Y) \eta(W) - \langle Y, Z \rangle \eta(X) \eta(W) \}.
 \end{aligned}$$

For a non-Sasakian  $(\kappa, \mu)$ -manifold  $\tilde{M}$ , its Riemann curvature tensor  $\tilde{R}$  is given explicitly in its  $(0, 4)$ -form by ([4],[5])

$$\begin{aligned}
 \tilde{R}(X, Y, Z, W) = & \left(1 - \frac{\mu}{2}\right) \{ \langle Y, Z \rangle \langle X, W \rangle - \langle X, Z \rangle \langle Y, W \rangle \} \\
 & - \frac{\mu}{2} \{ 2 \langle X, \varphi Y \rangle \langle \varphi Z, W \rangle + \langle X, \varphi Z \rangle \langle \varphi Y, W \rangle - \langle Y, \varphi Z \rangle \langle \varphi X, W \rangle \} \\
 & + \langle Y, Z \rangle \langle hX, W \rangle - \langle X, Z \rangle \langle hY, W \rangle \\
 & - \langle Y, W \rangle \langle hX, Z \rangle + \langle X, W \rangle \langle hY, Z \rangle \\
 (3.9) \quad & + \frac{1 - (\mu/2)}{1 - \kappa} \{ \langle hY, Z \rangle \langle hX, W \rangle - \langle hX, Z \rangle \langle hY, W \rangle \} \\
 & + \frac{\kappa - (\mu/2)}{1 - \kappa} \{ \langle \varphi hY, Z \rangle \langle \varphi hX, W \rangle - \langle \varphi hX, Z \rangle \langle \varphi hY, W \rangle \} \\
 & + \eta(X) \eta(W) \{ (\kappa - 1 + (\mu/2)) \langle Y, Z \rangle + (\mu - 1) \langle hY, Z \rangle \} \\
 & - \eta(X) \eta(Z) \{ (\kappa - 1 + (\mu/2)) \langle Y, W \rangle + (\mu - 1) \langle hY, W \rangle \} \\
 & + \eta(Y) \eta(Z) \{ (\kappa - 1 + (\mu/2)) \langle X, W \rangle + (\mu - 1) \langle hX, W \rangle \} \\
 & - \eta(Y) \eta(W) \{ (\kappa - 1 + (\mu/2)) \langle X, Z \rangle + (\mu - 1) \langle hX, Z \rangle \},
 \end{aligned}$$

for all vector fields  $X, Y, Z, W$  on  $\tilde{M}$ .

In [23], TANNO asked whether there exists a non-Sasakian contact metric manifold of constant  $\varphi$ -sectional curvature. In [17], KOUFOGIORGOS answered this problem in affirmative. A 3-dimensional non-Sasakian  $(\kappa, \mu)$ -manifold has a constant  $\varphi$ -sectional curvature, but for higher dimension this is not in general true. In fact, he proved that a non-Sasakian  $(\kappa, \mu)$ -manifold is of constant  $\varphi$ -sectional curvature  $c = -\kappa - \mu$  if and only if  $\mu = \kappa + 1$ . In particular, he derived that the tangent sphere bundle of a manifold of constant curvature  $c \neq 1$  has constant  $\varphi$ -sectional curvature  $c^2 = 9 \pm 4\sqrt{5}$  if and only if  $c = 2 \pm \sqrt{5}$ . For more details we refer to [2], [3] and [17].

**Remark 3.1.** In [17], KOUFOGIORGOS also presented a method to construct non-Sasakian  $(\kappa, \mu)$ -manifolds of constant  $\varphi$ -sectional curvature. However, we remark that if a non-Sasakian  $(\kappa, \mu)$ -manifold is of constant  $\varphi$ -sectional curvature  $c$ , then  $c$  cannot be arbitrary. In fact, in view of  $\mu = \kappa + 1$ ,  $c = -\kappa - \mu$  and  $\kappa < 1$ , it follows that  $c \in (-3, \infty)$ . Now, it would be interesting to find a non-Sasakian  $(\kappa, \mu)$ -manifold of constant  $\varphi$ -sectional curvature  $c$  for every  $c \in (-3, \infty)$ .

#### 4. $C$ -totally real warped product submanifolds

A submanifold  $M$  in a contact manifold is called a  $C$ -totally real submanifold [24] if every tangent vector of  $M$  belongs to the contact distribution. Thus, a submanifold  $M$  in a contact metric manifold is a  $C$ -totally real submanifold if  $\xi$  is normal to  $M$ . A submanifold  $M$  in an almost contact metric manifold is called *anti-invariant* [25] if  $\varphi(TM) \subset T^\perp M$ . If a submanifold  $M$  in a contact metric manifold is normal to the structure vector field  $\xi$ , then it is anti-invariant. Thus  $C$ -totally real submanifolds in a contact metric manifold are anti-invariant, as they are normal to  $\xi$ .

**Example 4.1.** Given a point  $p$  of a non-Sasakian  $(\kappa, \mu)$ -manifold, there are at least two  $C$ -totally real submanifolds, passing through  $p$ . To see this, consider the foliations given by the eigendistributions of  $h$ ; then their leaves are totally geodesic  $C$ -totally real submanifolds of the given non-Sasakian  $(\kappa, \mu)$ -manifold.

For a  $C$ -totally real submanifold in a contact metric manifold we have  $\langle A_\xi X, Y \rangle = \langle -\tilde{\nabla}_X \xi, Y \rangle = \langle \varphi X + \varphi hX, Y \rangle$ , which implies that

$$(4.1) \quad A_\xi = (\varphi h)^T,$$

where  $(\varphi h)^T X$  is the tangential part of  $\varphi hX$  for all  $X \in TM$ .

Now, we obtain a basic inequality involving the Laplacian of the warping function and the squared mean curvature of a  $C$ -totally real warped product submanifold of a  $(\kappa, \mu)$ -space form.

**Theorem 4.2.** *Let  $x$  be a  $C$ -totally real immersion of an  $n$ -dimensional warped product manifold  $M_1 \times_f M_2$  into a  $(2m + 1)$ -dimensional  $(\kappa, \mu)$ -space form  $\tilde{M}(c)$ . Then*

$$(4.2) \quad \begin{aligned} \frac{\Delta f}{f} \leq & \frac{n^2}{4n_2} \|H\|^2 + \frac{1}{4} n_1 (c + 3) + \text{trace}(h^T|_{M_1}) + \frac{n_1}{n_2} \text{trace}(h^T|_{M_2}) \\ & + \frac{1}{4n_2} \{ (\text{trace}(h^T))^2 - (\text{trace}(h^T|_{M_1}))^2 - (\text{trace}(h^T|_{M_2}))^2 \\ & - (\text{trace}(A_\xi))^2 + (\text{trace}(A_\xi|_{M_1}))^2 + (\text{trace}(A_\xi|_{M_2}))^2 \\ & - \|h^T\|^2 + \|h^T|_{M_1}\|^2 + \|h^T|_{M_2}\|^2 + \|A_\xi\|^2 - \|A_\xi|_{M_1}\|^2 - \|A_\xi|_{M_2}\|^2 \}, \end{aligned}$$

where  $n_i = \dim M_i$ ,  $i = 1, 2$ , and  $\Delta$  is the Laplacian operator of  $M_1$ . The equality case of (4.2) holds identically if and only if  $x$  is a mixed totally

geodesic immersion and  $n_1 H_1 = n_2 H_2$ , where  $H_i$ ,  $i = 1, 2$ , are the partial mean curvature vectors.

**Proof.** Let  $M_1 \times_f M_2$  be a  $C$ -totally real warped product submanifold in a  $(\kappa, \mu)$ -space form  $\tilde{M}(c)$  of constant  $\varphi$ -sectional curvature  $c$ . We choose a local orthonormal frame  $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2m+1}\}$  such that  $e_1, \dots, e_{n_1}$  are tangent to  $M_1$ ,  $e_{n_1+1}, \dots, e_n$  are tangent to  $M_2$ . Then from (3.7) and (4.1) we have

$$(4.3) \quad \begin{aligned} \tilde{K}(e_i \wedge e_j) &= \frac{c+3}{4} + \langle h^T e_i, e_i \rangle + \langle h^T e_j, e_j \rangle + \frac{1}{2} \{ \langle h^T e_i, e_i \rangle \langle h^T e_j, e_j \rangle \\ &\quad - \langle h^T e_i, e_j \rangle^2 - \langle A_\xi e_i, e_i \rangle \langle A_\xi e_j, e_j \rangle + \langle A_\xi e_i, e_j \rangle^2 \}, \end{aligned}$$

where  $h^T X$  is the tangential part of  $hX$  for  $X \in TM$ .

For a  $k$ -plane section  $\Pi_k$ , from (4.3) it follows that

$$(4.4) \quad \begin{aligned} \tilde{\tau}(\Pi_k) &= \frac{1}{8} k(k-1)(c+3) + (k-1) \text{trace}(h^T|_{\Pi_k}) \\ &\quad + \frac{1}{4} \{ (\text{trace}(h^T|_{\Pi_k}))^2 - \|h^T|_{\Pi_k}\|^2 - (\text{trace}(A_\xi|_{\Pi_k}))^2 + \|A_\xi|_{\Pi_k}\|^2 \}. \end{aligned}$$

Consequently, we have

$$(4.5) \quad \begin{aligned} \tilde{\tau}(T_p M) &= \frac{1}{8} n(n-1)(c+3) + (n-1) \text{trace}(h^T) \\ &\quad + \frac{1}{4} \{ (\text{trace}(h^T))^2 - \|h^T\|^2 - (\text{trace}(A_\xi))^2 + \|A_\xi\|^2 \}, \end{aligned}$$

$$(4.6) \quad \begin{aligned} \tilde{\tau}(T_p M_i) &= \frac{1}{8} n_i(n_i-1)(c+3) + (n_i-1) \text{trace}(h^T|_{M_i}) \\ &\quad + \frac{1}{4} \{ (\text{trace}(h^T|_{M_i}))^2 - \|h^T|_{M_i}\|^2 - (\text{trace}(A_\xi|_{M_i}))^2 + \|A_\xi|_{M_i}\|^2 \}, \end{aligned}$$

where  $i = 1, 2$ . Using (4.5) and (4.6) in (2.7) we get (4.2).  $\square$

Putting  $h = 0$  in (4.2), we have the following

**Corollary 4.3** (Theorem 3.1, [18]). *Let  $x$  be a  $C$ -totally real immersion of an  $n$ -dimensional warped product manifold  $M_1 \times_f M_2$  into a Sasakian space form. Then*

$$(4.7) \quad \frac{\Delta f}{f} \leq \frac{n^2}{4n_2} \|H\|^2 + \frac{1}{4} n_1 (c+3),$$

where  $n_i = \dim M_i$ ,  $i = 1, 2$ , and  $\Delta$  is the Laplacian operator of  $M_1$ . The equality case of (4.7) holds identically if and only if  $x$  is a mixed totally geodesic immersion and  $n_1 H_1 = n_2 H_2$ , where  $H_i$ ,  $i = 1, 2$ , are the partial mean curvature vectors.

Next, we establish a sharp relationship between the warped function  $f$  of a  $C$ -totally real warped product submanifold  $M_1 \times_f M_2$  isometrically immersed in a non-Sasakian  $(\kappa, \mu)$ -manifold  $\tilde{M}$  and the squared mean curvature  $\|H\|^2$  in the following

**Theorem 4.4.** *Let  $x$  be a  $C$ -totally real immersion of an  $n$ -dimensional warped product manifold  $M_1 \times_f M_2$  into a  $(2m + 1)$ -dimensional non-Sasakian  $(\kappa, \mu)$ -manifold  $\tilde{M}$ . Then*

$$\begin{aligned}
 \frac{\Delta f}{f} &\leq \frac{n^2}{4n_2} \|H\|^2 + n_1 \left(1 - \frac{\mu}{2}\right) + \text{trace}(h^T|_{M_1}) + \frac{n_1}{n_2} \text{trace}(h^T|_{M_2}) \\
 &+ \frac{1}{2n_2} \left(\frac{1 - (\mu/2)}{1 - \kappa}\right) \{(\text{trace}(h^T))^2 \\
 &- (\text{trace}(h^T|_{M_1}))^2 - (\text{trace}(h^T|_{M_2}))^2\} \\
 (4.8) \quad &- \frac{1}{2n_2} \left(\frac{\kappa - (\mu/2)}{1 - \kappa}\right) \{(\text{trace}(A_\xi))^2 \\
 &- (\text{trace}(A_\xi|_{M_1}))^2 - (\text{trace}(A_\xi|_{M_2}))^2\} \\
 &- \frac{1}{2n_2} \left(\frac{1 - (\mu/2)}{1 - \kappa}\right) \{\|h^T\|^2 - \|h^T|_{M_1}\|^2 - \|h^T|_{M_2}\|^2\} \\
 &+ \frac{1}{2n_2} \left(\frac{\kappa - (\mu/2)}{1 - \kappa}\right) \{\|A_\xi\|^2 - \|A_\xi|_{M_1}\|^2 - \|A_\xi|_{M_2}\|^2\},
 \end{aligned}$$

where  $n_i = \dim M_i$ ,  $i = 1, 2$ , and  $\Delta$  is the Laplacian operator of  $M_1$ . The equality case of (4.8) holds identically if and only if  $x$  is a mixed totally geodesic immersion and  $n_1 H_1 = n_2 H_2$ , where  $H_i$ ,  $i = 1, 2$ , are the partial mean curvature vectors.

**Proof.** Let  $M_1 \times_f M_2$  be a  $C$ -totally real warped product submanifold of a non-Sasakian  $(\kappa, \mu)$ -manifold  $\tilde{M}$ . We choose a local orthonormal frame  $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2m+1}\}$  such that  $e_1, \dots, e_{n_1}$  are tangent to  $M_1$ ,

$e_{n_1+1}, \dots, e_n$  are tangent to  $M_2$ . Then from (3.9) we have

$$(4.9) \quad \begin{aligned} \tilde{K}(e_i \wedge e_j) &= \left(1 - \frac{\mu}{2}\right) + \langle h^T e_i, e_i \rangle + \langle h^T e_j, e_j \rangle \\ &+ \frac{1 - (\mu/2)}{1 - \kappa} \left\{ \langle h^T e_i, e_i \rangle \langle h^T e_j, e_j \rangle - \langle h^T e_i, e_j \rangle^2 \right\} \\ &+ \frac{\kappa - (\mu/2)}{1 - \kappa} \left\{ \langle A_\xi e_i, e_i \rangle \langle A_\xi e_j, e_j \rangle - \langle A_\xi e_i, e_j \rangle^2 \right\}. \end{aligned}$$

For a  $k$ -plane section  $\Pi_k$ , from (4.9) we obtain

$$(4.10) \quad \begin{aligned} \tilde{\tau}(\Pi_k) &= \frac{1}{2} k(k-1) \left(1 - \frac{\mu}{2}\right) + (k-1) \operatorname{trace}(h^T|_{\Pi_k}) \\ &+ \frac{1}{2} \left( \frac{1 - (\mu/2)}{1 - \kappa} \right) \left\{ (\operatorname{trace}(h^T|_{\Pi_k}))^2 - \|h^T|_{\Pi_k}\|^2 \right\} \\ &- \frac{1}{2} \left( \frac{\kappa - (\mu/2)}{1 - \kappa} \right) \left\{ (\operatorname{trace}(A_\xi|_{\Pi_k}))^2 - \|A_\xi|_{\Pi_k}\|^2 \right\}. \end{aligned}$$

Consequently, we have

$$(4.11) \quad \begin{aligned} \tilde{\tau}(T_p M) &= \frac{1}{2} n(n-1) \left(1 - \frac{\mu}{2}\right) + (n-1) \operatorname{trace}(h^T) \\ &+ \frac{1}{2} \left( \frac{1 - (\mu/2)}{1 - \kappa} \right) \left\{ (\operatorname{trace}(h^T))^2 - \|h^T\|^2 \right\} \\ &- \frac{1}{2} \left( \frac{\kappa - (\mu/2)}{1 - \kappa} \right) \left\{ (\operatorname{trace}(A_\xi))^2 - \|A_\xi\|^2 \right\}, \end{aligned}$$

$$(4.12) \quad \begin{aligned} \tilde{\tau}(T_p M_i) &= \frac{1}{2} n_i(n_i-1) \left(1 - \frac{\mu}{2}\right) + (n_i-1) \operatorname{trace}(h^T|_{M_i}) \\ &+ \frac{1}{2} \left( \frac{1 - (\mu/2)}{1 - \kappa} \right) \left\{ (\operatorname{trace}(h^T|_{M_i}))^2 - \|h^T|_{M_i}\|^2 \right\} \\ &- \frac{1}{2} \left( \frac{\kappa - (\mu/2)}{1 - \kappa} \right) \left\{ (\operatorname{trace}(A_\xi|_{M_i}))^2 - \|A_\xi|_{M_i}\|^2 \right\}, \end{aligned}$$

where  $i = 1, 2$ . Putting values from (4.11) and (4.12) in (2.7) we obtain (4.8).  $\square$

As applications, we derive certain obstructions to the existence of minimal  $C$ -totally real warped product submanifolds in  $(\kappa, \mu)$ -space forms, non-Sasakian  $(\kappa, \mu)$ -manifolds and Sasakian space forms.

**Corollary 4.5.** *Let  $M_1 \times_f M_2$  be a warped product manifold, whose warping function  $f$  is harmonic. Then:*

(a)  $M_1 \times_f M_2$  cannot admit minimal  $C$ -totally real immersion into a  $(\kappa, \mu)$ -space form  $\tilde{M}(c)$  with

$$(4.13) \quad \begin{aligned} & 0 > \frac{1}{4} n_1 (c + 3) + \text{trace} (h^T|_{M_1}) + \frac{n_1}{n_2} \text{trace} (h^T|_{M_2}) \\ & + \frac{1}{4n_2} \left\{ (\text{trace} (h^T))^2 - (\text{trace} (h^T|_{M_1}))^2 - (\text{trace} (h^T|_{M_2}))^2 \right. \\ & \quad - (\text{trace}(A_\xi))^2 + (\text{trace}(A_\xi|_{M_1}))^2 + (\text{trace}(A_\xi|_{M_2}))^2 \\ & \quad \left. - \|h^T\|^2 + \|h^T|_{M_1}\|^2 + \|h^T|_{M_2}\|^2 + \|A_\xi\|^2 - \|A_\xi|_{M_1}\|^2 - \|A_\xi|_{M_2}\|^2 \right\}. \end{aligned}$$

(b) Every minimal  $C$ -totally real immersion of  $M_1 \times_f M_2$  in a  $(\kappa, \mu)$ -space form  $\tilde{M}(c)$  with

$$(4.14) \quad \begin{aligned} & 0 = \frac{1}{4} n_1 (c + 3) + \text{trace} (h^T|_{M_1}) + \frac{n_1}{n_2} \text{trace} (h^T|_{M_2}) \\ & + \frac{1}{4n_2} \left\{ (\text{trace} (h^T))^2 - (\text{trace} (h^T|_{M_1}))^2 - (\text{trace} (h^T|_{M_2}))^2 \right. \\ & \quad - (\text{trace}(A_\xi))^2 + (\text{trace}(A_\xi|_{M_1}))^2 + (\text{trace}(A_\xi|_{M_2}))^2 \\ & \quad \left. - \|h^T\|^2 + \|h^T|_{M_1}\|^2 + \|h^T|_{M_2}\|^2 + \|A_\xi\|^2 - \|A_\xi|_{M_1}\|^2 - \|A_\xi|_{M_2}\|^2 \right\} \end{aligned}$$

is a warped product immersion.

**Proof.** We assume that the warping function  $f$  is a harmonic function on  $M_1$  and  $M_1 \times_f M_2$  admits a minimal  $C$ -totally real immersion in a  $(\kappa, \mu)$ -space form  $\tilde{M}(c)$ . Then, the inequality (4.2) becomes

$$\begin{aligned} 0 & \leq \frac{1}{4} n_1 (c + 3) + \text{trace} (h^T|_{M_1}) + \frac{n_1}{n_2} \text{trace} (h^T|_{M_2}) \\ & + \frac{1}{4n_2} \left\{ (\text{trace} (h^T))^2 - (\text{trace} (h^T|_{M_1}))^2 - (\text{trace} (h^T|_{M_2}))^2 \right. \\ & \quad - (\text{trace}(A_\xi))^2 + (\text{trace}(A_\xi|_{M_1}))^2 + (\text{trace}(A_\xi|_{M_2}))^2 \\ & \quad \left. - \|h^T\|^2 + \|h^T|_{M_1}\|^2 + \|h^T|_{M_2}\|^2 + \|A_\xi\|^2 - \|A_\xi|_{M_1}\|^2 - \|A_\xi|_{M_2}\|^2 \right\}. \end{aligned}$$

This proves (a).

Now, we prove (b). If (4.14) is true then the equality case of (4.2) is true and by Theorem 4.2, it follows that  $M_1 \times_f M_2$  is mixed totally geodesic



and  $H_1 = H_2 = 0$ . Then a well-known result of NÖLKER [21] implies that the immersion is a warped product immersion.  $\square$

Similar to the above Corollary, we have

**Corollary 4.6.** *Let  $M_1 \times_f M_2$  be a warped product manifold, whose warping function  $f$  is harmonic. Then:*

(a)  *$M_1 \times_f M_2$  cannot admit minimal C-totally real immersion into a non-Sasakian  $(\kappa, \mu)$ -manifold with*

$$\begin{aligned}
 (4.15) \quad & 0 > n_1 \left(1 - \frac{\mu}{2}\right) + \text{trace}(h^T|_{M_1}) + \frac{n_1}{n_2} \text{trace}(h^T|_{M_2}) \\
 & + \frac{1}{2n_2} \left(\frac{1 - (\mu/2)}{1 - \kappa}\right) \{(\text{trace}(h^T))^2 \\
 & - (\text{trace}(h^T|_{M_1}))^2 - (\text{trace}(h^T|_{M_2}))^2\} \\
 & - \frac{1}{2n_2} \left(\frac{\kappa - (\mu/2)}{1 - \kappa}\right) \{(\text{trace}(A_\xi))^2 \\
 & - (\text{trace}(A_\xi|_{M_1}))^2 - (\text{trace}(A_\xi|_{M_2}))^2\} \\
 & - \frac{1}{2n_2} \left(\frac{1 - (\mu/2)}{1 - \kappa}\right) \{\|h^T\|^2 - \|h^T|_{M_1}\|^2 - \|h^T|_{M_2}\|^2\} \\
 & + \frac{1}{2n_2} \left(\frac{\kappa - (\mu/2)}{1 - \kappa}\right) \{\|A_\xi\|^2 - \|A_\xi|_{M_1}\|^2 - \|A_\xi|_{M_2}\|^2\}.
 \end{aligned}$$

(b) *Every minimal C-totally real immersion of  $M_1 \times_f M_2$  in a non-Sasakian  $(\kappa, \mu)$ -manifold with*

$$\begin{aligned}
 (4.16) \quad & 0 = n_1 \left(1 - \frac{\mu}{2}\right) + \text{trace}(h^T|_{M_1}) + \frac{n_1}{n_2} \text{trace}(h^T|_{M_2}) \\
 & + \frac{1}{2n_2} \left(\frac{1 - (\mu/2)}{1 - \kappa}\right) \{(\text{trace}(h^T))^2 \\
 & - (\text{trace}(h^T|_{M_1}))^2 - (\text{trace}(h^T|_{M_2}))^2\} \\
 & - \frac{1}{2n_2} \left(\frac{\kappa - (\mu/2)}{1 - \kappa}\right) \{(\text{trace}(A_\xi))^2 \\
 & - (\text{trace}(A_\xi|_{M_1}))^2 - (\text{trace}(A_\xi|_{M_2}))^2\} \\
 & - \frac{1}{2n_2} \left(\frac{1 - (\mu/2)}{1 - \kappa}\right) \{\|h^T\|^2 - \|h^T|_{M_1}\|^2 - \|h^T|_{M_2}\|^2\} \\
 & + \frac{1}{2n_2} \left(\frac{\kappa - (\mu/2)}{1 - \kappa}\right) \{\|A_\xi\|^2 - \|A_\xi|_{M_1}\|^2 - \|A_\xi|_{M_2}\|^2\}
 \end{aligned}$$

is a warped product immersion.

Using  $h = 0$  in Corollary 4.5, we immediately get the following

**Corollary 4.7** ([18, Corollary 3.2]). *Let  $M_1 \times_f M_2$  be a warped product manifold, whose warping function  $f$  is harmonic. Then:*

- (a)  $M_1 \times_f M_2$  cannot admit minimal  $C$ -totally real immersion into a Sasakian space form  $\tilde{M}(c)$  with  $c < -3$ .
- (b) Every minimal  $C$ -totally real immersion of  $M_1 \times_f M_2$  in the standard Sasakian space form  $\mathbb{R}^{2m+1}(-3)$  is a warped product immersion.

We also have the following

**Corollary 4.8** ([18, Corollary 3.3]). *If the warping function  $f$  of a warped product  $M_1 \times_f M_2$  is an eigenfunction of the Laplacian on  $M_1$  with corresponding eigenvalue  $\lambda > 0$ , then  $M_1 \times_f M_2$  does not admit a minimal  $C$ -totally real immersion into a Sasakian space form  $\tilde{M}(c)$  with  $c \leq -3$ .*

In the following we present an example of a warped product  $C$ -totally real submanifold of a non-Sasakian  $(\kappa, \mu)$ -manifold which satisfies the equality case of (4.8).

**Example 4.9.** Let  $(\tilde{M}, \tilde{g})$  be a 4-dimensional Riemannian manifold of constant sectional curvature  $c \neq 1$ . Then its tangent sphere bundle  $T_1\tilde{M}$  with the standard contact metric structure is a non-Sasakian  $(\kappa, \mu)$ -manifold with  $\kappa = c(2 - c)$ ,  $\mu = 2c$  [3]. Let  $M$  be a totally geodesic hypersurface of  $\tilde{M}$ . Then  $M$  equipped with the induced Riemannian metric  $g$  has constant sectional curvature  $c$  and also its tangent sphere bundle  $T_1M$  is a  $(\kappa, \mu)$ -manifold with  $\kappa = c(2 - c)$ ,  $\mu = 2c$ . From Example 3.3 of [7] it follows that  $T_1M$  is an invariant submanifold of  $T_1\tilde{M}$ . Let  $S$  be a minimal  $C$ -totally real surface of  $T_1M$  (which is always possible in view of Example 4.1). Define the warped product manifold  $(-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos t} S$ . Then the immersion  $x : (-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos t} S \rightarrow T_1\tilde{M}$  defined by  $x(t, p) = (\sin t)N + (\cos t)p$ , where  $N$  is a unit vector orthogonal to the linear subspace containing  $T_1M$ , is a  $C$ -totally real isometric immersion and satisfies the equality case of (4.8).

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