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COMMUTATIVE MODULAR GROUP ALGEBRAS OF SPECIAL CLASSES OF ABELIAN GROUPS

BY

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Abstract. Structural results on the Direct Factor Problem, the Classification Problem and the Isomorphism Problem are proved for modular group algebras of certain sorts of abelian groups.

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1. Introduction

Suppose FG is the F-group algebra of an abelian multiplicative group G over a field F of $\mathrm{char}(F) = p \neq 0$. Traditionally, $V(FG)$ denotes the group of all normalized invertible elements (often called *normed units*) with p-primary component $V_p(FG) = S(FG)$, called also Sylow p-subgroup, and G_p denotes the p-torsion part of G.

A question of some importance in group ring theory is to study the behavior of commutative group algebras of special types of abelian groups. In abelian group theory there are many exotic classes of groups such as Fuchs 5-groups, Q-groups, starred and fully starred groups, thin groups, $\aleph_1$-separable groups, and their modifications as weakly and strongly $\aleph_1$-separable groups, $\aleph_1$-Σ-cyclic groups etcetera (a valuable background in the literature are [13], [17], [18], [27], [28], [29] and [30]).

To facilitate the terminology, the direct sum (= the coproduct) of cyclic (countable, respectively) groups will be abbreviated in the sequel as Σ-cyclic (Σ-countable, respectively) groups.

The first major investigation in this aspect starts recently by Göbel-May in [16] for group algebras of $\aleph_1$-separable p-groups, and by us in [6], [7] and [8] for group algebras of starred abelian p-groups.

Our aim in the current paper, that we pursue, is to continue the explorations for some of the above quoted group kinds. Our advantage is in proving a basic Direct Factor Theorem for $S(FG)$ and, with this in hand, we provide its complete characterization for some specific sorts of abelian groups (for more details see the second and third sections).

2. The direct factor theorem

Here we proceed by proving a central statement that is the omnibus for obtaining our next assertions. It strengthens the corresponding ones established in [16] and [5] (see [4] as well).

Theorem 2.1. *Let G be an abelian group whose subgroup G_p is separable, and F a field of char$(F) = p > 0$. Then $S(FG)/G_p$ is Σ-cyclic. In particular, the separable group G_p is a direct factor of $S(FG)$ with Σ-cyclic complement.*

Proof. Suppose for a moment F is perfect. Since G_p is separable, and G_p is nice in $S(FG)$ (see, for instance, [4] and [5]), it easily follows that so is $S(FG)/G_p$. Furthermore, we deal with the construction of special selected generating subgroups.

For every natural number n we define

$$\begin{aligned} S_n = \langle & f_{1n} + f_{2n}g_{2n} + \ldots + f_{s_nn}g_{s_nn} \in S(FG)[p^n] | 0 \neq f_{in} \in F, \\ & \sum_{i=1}^{s_n} f_i = 1; 1 \neq g_{in} \in G : \\ & g_{2n}^{\pm\varepsilon_2} \ldots g_{s_nn}^{\pm\varepsilon_{s_n}} \in (G \setminus G^{p^n}) \cup G^{p^\omega}, \forall 0 \leq \varepsilon_i < p^n, \forall 1 < i \leq s_n \in \mathbb{N} \rangle. \end{aligned}$$

Of course, if G is p-torsion, the condition $g_{2n}^{\pm\varepsilon_2} \ldots g_{s_nn}^{\pm\varepsilon_{s_n}} \in (G \setminus G^{p^n}) \cup G^{p^\omega}$ may be replaced by $\langle g_{2n}, \ldots, g_{s_nn} \rangle \subseteq (G \setminus G^{p^n}) \cup \{1\}$ that is $\langle g_{2n}, \ldots, g_{s_nn} \rangle \setminus \{1\} \subseteq G \setminus G^{p^n}$. The latter is also equivalent to $\langle g_{2n}, \ldots, g_{s_nn} \rangle \cap G^{p^n} = 1$.

The so-defined groups S_n are correctly defined because all posed height's restrictions are trivially satisfied owing to the finiteness of the support.

Now, given an element $x \in S(FG)/G_p$. That is why, $x = (r_1a_1 + \ldots + r_ka_k)G_p$ where $r_1a_1 + \ldots + r_ka_k \in S(FG)$; k a natural. Therefore, there is an index j with $1 \leq j \leq k$ such that $a_j \in G_p$ and $r_j \neq 0$. Thus

$x = r_1a_1a_j^{-1} + \ldots + r_j + \ldots + r_ka_ka_j^{-1})G_p$, where the sum obviously lies in S_m for some natural number m. Consequently, $x \in (S_mG_p)/G_p$ whence $S(FG)/G_p = \cup_{n<\omega_0}((S_nG_p)/G_p)$. For the successful application of the classical Kulikov's criterion for Σ-cyclic groups, what suffices to prove is that $((S_nG_p)/G_p) \cap (S(FG)/G_p)^{p^n} = 1$ for all $n < \omega_0$. The last is equivalent via the modular law to $(S_nG_p) \cap S^{p^n}(FG) = (S_nG_p) \cap S(F^{p^n}G^{p^n}) \subseteq G_p$. To materialize this inclusion, first and foremost, we shall compute that $S_n \cap S(F^{p^n}G^{p^n}) = 1$. In order to do that, take an element $y \in S_n$. Thus, $y = u_1^{(n)^{\varepsilon_1}}u_2^{(n)^{\varepsilon_2}}\ldots u_{t_n}^{(n)^{\varepsilon_{t_n}}}$, where $u_1^{(n)}, u_2^{(n)}, \ldots, u_{t_n}^{(n)}$ with $t_n \in \mathbb{N}$ are generators described as above, and $0 \leq \varepsilon_l \leq p^n - 1$ for all $l \in [1, t_n]$. We can restrict our attention on two generators both of degree one, i.e., $t_n = 2$ and $\varepsilon_1 = \varepsilon_2 = 1$, since the general case follows by the same token as the particular one, presented below.

So, write $u_1^{(n)} = f_{1n} + f_{2n}g_{2n} + \ldots + f_{s_nn}g_{s_nn}$ and $u_2^{(n)} = r_{1n} + r_{2n}a_{2n} + \ldots + r_{s_nn}a_{s_nn}$ with elements of the group basis possessing the foregoing stated properties on heights. The multiplication of $u_1^{(n)}$ and $u_2^{(n)}$ gives

$$\begin{aligned} u_1^{(n)}u_2^{(n)} &= f_{1n}r_{1n} + f_{1n}r_{2n}a_{2n} + \ldots + f_{1n}r_{s_nn}a_{s_nn} + f_{2n}r_{1n}g_{2n} \\ &+ f_{2n}r_{2n}g_{2n}a_{2n} + \ldots + f_{2n}r_{s_nn}g_{2n}a_{s_nn} + \ldots + f_{s_nn}r_{1n}g_{s_nn} \\ &+ f_{s_nn}r_{2n}g_{s_nn}a_{2n} + \ldots + f_{s_nn}r_{s_nn}g_{s_nn}a_{s_nn}. \end{aligned}$$

Since the p-height of $u_1^{(n)}u_2^{(n)}$ is equal to the minimal p-height of some group element from the support, we will be finished provided that at least one of the different elements $a_{2n}, \ldots, a_{s_nn}; g_{2n}, \ldots, g_{s_nn}$ belongs to this support. Otherwise, the estimation of p-height$(u_1^{(n)}u_2^{(n)})$ is more difficult and needs a more precise tactic. We emphasize that there are no zero divisors in the coefficient field, so $f_{in}r_{jn} \neq 0$ $\forall 1 \leq i, j \leq s_n$. On that tactic it is a crucial moment, because of the fact that the group elements $a_{2n}, \ldots, a_{s_nn}$ and $g_{2n}, \ldots, g_{s_nn}$ are no members of the canonical form of $u_1^{(n)}u_2^{(n)}$ is equivalent to this that all of them must be in a relationship with some elements from the above record of $u_1^{(n)}u_2^{(n)}$ such that their joint coefficient is zero. In fact, without loss of generality, we will assume that the following relations are fulfilled: $a_{2n} = a_{i_2n}g_{j_2n}, \ldots, a_{s_n} = a_{i_{s_n}n}g_{j_{s_n}n}; g_{2n} = a_{k_2n}g_{l_2n}, \ldots, g_{s_n} = a_{k_{s_n}n}g_{l_{s_n}n}$, where $(i_2, \ldots, i_{s_n})$, $(j_2, \ldots, j_{s_n})$ and $(k_2, \ldots, k_{s_n})$, $(l_2, \ldots, l_{s_n})$ are arbitrary permutations of the indices $(2, \ldots, s_n)$. All other possible situations have a similar treatment following the previous paradigm. Thereby,

we obtain that

$$\begin{aligned} u_1^{(n)}u_2^{(n)}=&f_{1n}r_{1n}+f_{2n}r_{2n}a_{i_2n}a_{k_2n}g_{j_2n}g_{l_2n}+\ldots+f_{2n}r_{s_nn}a_{i_{s_n}n}a_{k_2n}g_{j_{s_n}n}g_{l_2n}\\ &+\ldots+f_{s_nn}r_{2n}a_{i_2n}a_{k_{s_n}n}g_{j_2n}g_{l_{s_n}n}\\ &+\ldots+f_{s_nn}r_{s_nn}a_{i_{s_n}n}a_{k_{s_n}n}g_{j_{s_n}n}g_{l_{s_n}n}. \end{aligned}$$

An essential observation, originated via the foregoing ratios, is that there do exist pairs of joint equal indexes which realize in the given above element's identities. This claim is rather natural and follows by standard combinatorial arguments, because the indices run on a finite number of finite sets. Therefore, although we can have other extra dependencies between the remaining group members, in the canonical form of $u_1^{(n)}u_2^{(n)}$ there is at least one product of group elements that depends only on a_{mn}'s and g_{mn}'s, including their individual products, for $1 \leq m \leq s_n$; $n \in \mathbb{N}$. So, we are done.

After this, we turn to to the question of whether $1 \in G$ can be chosen to lie in the support of $u_1^{(n)}u_2^{(n)}$. To this purpose, let us assume the contrary that this unity is not directly from the canonical record. Letting $1 = a_{i_2n}a_{k_2n}g_{j_2n}g_{l_2n}$ with $f_{1n}r_{1n} + f_{2n}r_{2n} = 0$, we consequently have $g_{j_2n}g_{l_2n} = a_{i_2n}^{-1}a_{k_2n}^{-1}$. Now, we shall modify the product $u_1^{(n)}u_2^{(n)}$ such that $u_1^{(n)}u_2^{(n)}G_p = v_nG_p$ where v_n is another element (eventually from S_n) such that p-height$(v_n) < p^n$ and such that $1 \in$supp(v_n); we note that $u_1^{(n)}u_2^{(n)}$ as well as v_n may not be generators of S_n. To this end, we first write

$$\begin{aligned} u_1^{(n)}u_2^{(n)} =&\ldots+f_{2n}r_{s_nn}a_{i_{s_n}n}a_{k_2n}g_{j_{s_n}n}g_{l_2n}+\ldots+f_{s_nn}r_{2n}a_{i_2n}a_{k_{s_n}n}g_{j_2n}g_{l_{s_n}n}\\ &+\ldots+f_{s_nn}r_{s_nn}a_{i_{s_n}n}a_{k_{s_n}n}g_{j_{s_n}n}g_{l_{s_n}n}, \end{aligned}$$

and, without loss of generality, assume (eventually after a finite number of steps) that this is the canonical form of $u_1^{(n)}u_2^{(n)}$. Furthermore, under the same combinatorial arguments already illustrated above, there exist appropriate pair of indexes, say for instance (i_2, i_{s_n}), (j_2, j_{s_n}), (k_2, k_{s_n}) and (l_2, l_{s_n}), such that $a_{i_{s_n}n}a_{i_2n}^{-1}a_{k_2n}a_{k_{s_n}n}^{-1}g_{j_{s_n}n}g_{j_2n}^{-1}g_{l_2n}g_{l_{s_n}n}^{-1} \in G_p$ and such that they have the property that the products of all other group elements with the element $(a_{i_{s_n}n}a_{i_2n}^{-1}a_{k_2n}a_{k_{s_n}n}^{-1}g_{j_{s_n}n}g_{j_2n}^{-1}g_{l_2n}g_{l_{s_n}n}^{-1})^{-1}$ are a function only of the elements a_{mn}'s and/or g_{mn}'s. This substantiates our claim. The further conclusions on p-heights follow in the same manner.

Suppose now $x \in (S_nG_p) \cap S(F^{p^n}G^{p^n})$, hence $x - v_ng_p \in S(F^{p^n}G^{p^n})$ where $v_n \in S_n$ and $g_p \in G_p$. A key moment, that we have previously shown

above, is that in the canonical form of v_n there is the basis element 1. Whence it is immediate that $g_p \in G_p^{p^n}$ and thus, again in conjunction with the preceding computations, $v_n \in S_n \cap S(F^{p^n}G^{p^n}) = 1$. Finally, $x = g_p \in G_p$, so that the desired embedding holds as expected.

If now F is arbitrary, then $F \subseteq \overline{F}$ and so $S(FG) \subseteq S(\overline{F}G)$ plus $S(FG)/G_p \subseteq S(\overline{F}G)/G_p$. By what we have already shown above, $S(\overline{F}G)/G_p$ is Σ-cyclic, and therefore we deduce that so is its subgroup $S(FG)/G_p$. The theorem is proved in all generality. □

Remark 2.2. It is a possible variant to have $t_n > p^n \geq \max\{\text{order}(u_1^{(n)}), \ldots, \text{order}(u_{t_n}^{(n)})\}$ but, however, as early demonstrated, this does not affect on the final conclusions.

3. Modular group algebras of abelian groups

Hereafter, even if it is not stated explicitly, all groups are abelian, written multiplicatively as is the custom when regarding group rings.

I. *Fuchs 5 and Q-groups*

First of all, for the sake of completeness and for the convenience of the readers, we begin with some more definitions.

Definition 3.1 ([20])**.** A group G is said to be *Fuchs 5* if for each $C \leq G$ with $|C| \geq \aleph_0$ there is $A \leq G$ such that $C \subseteq A$, such that A is a direct factor of G and such that $|C| = |A|$.

Apparently, the divisible groups are Fuchs 5.

Definition 3.2 ([15])**.** A group G is termed *separable* if it is without elements of infinite height, that is, $G^1 = \bigcap_{n<\omega_0} G^n = \bigcap_p G^{p^{\omega_0}} = \bigcap_p \bigcap_{n<\omega_0} G^{p^n} = 1$.

Definition 3.3 ([20])**.** A separable group is called *a Q-group* if for every $C \leq G$ with $|C| \geq \aleph_0$ it is fulfilled that $|(G/C)^1| \leq |C|$.

We emphasize that the last inequality may be equivalently replaced by the equality $|\bigcap_{n<\omega_0}(G^nC)| = |C|$.

Definition 3.4 ([20])**.** A group G is named *starred* if whenever for $H \leq G$ we have that G/H is divisible, then $|G| = |H|$. A group G is named *fully starred* if every its subgroup is starred.

It is worthwhile noticing that if G is starred, then it has no finite subgroups H with the property that G/H is divisible; thus G/H being divisible implies $|G/H| \leq |H|$ whence, in an equivalent manner, $|G| = |H| \cdot |G/H| = |H|$ since $|G/H| \geq \aleph_0$ when $G \neq H$. Consequently, the nonidentity divisible groups are not starred.

Now, we quote some elementary but useful properties concerning these groups:

(a) the fully starred groups are starred;

(b) the Σ-countable groups and their subgroups are Fuchs 5;

(c) any reduced Fuchs 5-group is fully starred;

(d) any Q-group is fully starred;

(e) any separable Fuchs 5-group is a Q-group (see [20]), while the opposite claim is wrong (e.g. [21]).

Without any further concrete referring, we shall use two times the following not hard but helpful set-theoretical fact, namely if M is an infinite set then $|\langle M\rangle| = |M|$, where $\langle M\rangle$ designates the group generated by the elements of M (see [23], p.44). Note that if M is finite, then it is clear that $|\langle M\rangle| > |M|$.

Also, it is well known that (see, for instance, [17], p.768, proof of Theorem 4; respectively [20]) direct sums of Fuchs 5-groups, respectively Q-groups, is again a Fuchs 5-group, respectively a Q-group. Nevertheless, for our further application, we shall verify the validity of the following weaker but helpful observation. Specifically, we state:

Proposition 3.5. *If A is a Fuchs 5-group and B is a Σ-countable group, then $A \times B$ is a Fuchs 5-group.*

Proof. Given $C \times C' \leq A \times B$ with $C \leq A, C' \leq B$ and $|C \times C'| \geq \aleph_0$. Hence $|C| \geq \aleph_0$ and/or $|C'| \geq \aleph_0$. Of both of these groups are infinite, there is nothing to prove.

Assume now that $|C| \geq \aleph_0$ and $|C'| < \aleph_0$; by symmetry the case when $|C| < \aleph_0$ and $|C'| \geq \aleph_0$ can be treated similarly, so that it is omitted voluntarily. Thus $C \subseteq D$ with D a direct factor of A such that $|C| = |D|$. Moreover, since B is Σ-countable and C' is finite, we derive that C' can be expanded in some pure countable subgroup of B, and even more $C' \subseteq E$ with $|E| \leq \aleph_0$ and E is a direct factor of B. Finally, $C \times C' \subseteq D \times E$ such that $D \times E$ is a direct factor of $A \times B$ and $|C \times C'| = |C| = |D| = |D \times E|$. This substantiates our claim.

Let us now consider an arbitrary subgroup $U \leq A \times B$ with $|U| \geq \aleph_0$. Hence, there exist $C \leq A$ and $C' \leq B$ such that $U \leq C \times C'$ and $|U| = |C \times C'| = |C| \times |C'| = \max(|C|, |C'|)$. Furthermore, the previous point works. □

So, we are now able to state the following.

Theorem 3.6. *Suppose G is an abelian group, and F is a field of $char(F) = p \neq 0$. Then $S(FG)$ is a Q-group if and only if G_p is a Q-group.*

In particular, $FG \cong FH$ as F-algebras for some arbitrary group H and G_p as a Q-group imply that H_p is a Q-group.

Proof. If $S(FG)$ is a Q-group, by virtue of [20] it follows at once that G_p inherits the same property as being a subgroup.

Let now G_p be a Q-group. Therefore, by definition, G_p is separable and thus Theorem 2.1 is applicable to obtain that $S(FG) \cong G_p \times (S(FG)/G_p)$, where the complementary quotient is Σ-cyclic. Consequently, we wish to employ again [20] to conclude that $S(FG)$ is a Q-group, as claimed.

The latter half-part follows by the routine back-and-forth argument because $FG \cong FH$ obviously yields that $S(FG) \cong S(FH)$. The proof is complete. □

Theorem 3.7. *Suppose G is an abelian group for which G_p is of power $\aleph_1$, and F is a field of $char(F) = p > 0$. Then $S(FG)$ is a separable Fuchs 5-group if and only if G_p is a separable Fuchs 5-group.*

In particular, $FG \cong FH$ for some other group H and G_p being a separable Fuchs 5-group of cardinality $\aleph_1$ imply that H_p is a separable Fuchs 5-group of cardinality $\aleph_1$.

Proof. Since G_p is separable, Theorem 2.1 insures that $S(FG) \cong G_p \times (S(FG)/G_p)$, where the complementary factor-group is Σ-cyclic.

Given now that $S(FG)$ is a Fuchs 5-group, we utilize [21] (see [22]) along with the foregoing direct decomposition to deduce that so is G_p. Conversely, let G_p be a Fuchs 5-group. Invoking Proposition 3.5 together with the preceding isomorphism decomposition, we obtain that the same holds of $S(FG)$.

The last half-part follows by the usual back-and-forth method as in the previous theorem. □

II. *Strongly $\aleph_1$-Σ-cyclic groups and weakly $\aleph_1$-separable groups*

We continue the definitions of some special groups.

Definition 3.8. A group G is termed *an $\aleph_1$-separable group* (in other terms *an ω_1-separable group*) if every countable subset of G is contained in a countable Σ-cyclic direct factor of G.

For torsion groups, because of the validity of the second Prüfer theorem, G is $\aleph_1$-separable if it is separable (= without elements of infinite height = $\aleph_0$-separable) and each countable subgroup is contained in a countable direct factor of G.

Evidently, the second Prüfer's theorem forces that all primary separable Fuchs 5-groups are $\aleph_1$-separable, while the reverse claim demonstrably fails; however, it is worth noting that, all $\aleph_1$-separable groups of power $\aleph_1$ are Fuchs 5-groups. Thus, as a consequence, all Σ-cyclic groups are $\aleph_1$-separable. Besides, it is not difficult to see that the $\aleph_1$-separable groups of cardinality $\aleph_1$ are Q-groups.

Definition 3.9. An *$\aleph_1$-Σ-cyclic* group is a group such that each countable subgroup is Σ-cyclic.

Certainly, according again to the second Prüfer's theorem, every separable p-group is $\aleph_1$-Σ-cyclic, whereas the converse claim is manifestly false, and thus each $\aleph_1$-separable group is $\aleph_1$-Σ-cyclic. Moreover, subgroups of an $\aleph_1$-Σ-cyclic group are themselves $\aleph_1$-Σ-cyclic as well.

With the aid of this definition, one can state the following.

Theorem 3.10. *Suppose that G is an abelian group, and F is a field of $char(F) = p$. Then $S(FG)$ is $\aleph_1$-Σ-cyclic if and only if G_p is $\aleph_1$-Σ-cyclic.*

In particular, if FG and FH are F-isomorphic for some other group H and G_p is $\aleph_1$-Σ-cyclic then H_p is $\aleph_1$-Σ-cyclic.

Proof. Since $G_p \subseteq S(FG)$ and each subgroup of an $\aleph_1$-Σ-cyclic group is again $\aleph_1$-Σ-cyclic, we deduce that the necessity holds.

In order to establish the sufficiency, let $V \leq S(FG)$ be a countable subgroup. That is why, since the support of every element is finite, it is a straightforward argument that there exist subgroups $P \leq F$ and $C \leq G$ with countable powers such that $V \leq S(PC)$. By hypothesis C_p being countable yields that it is Σ-cyclic. Henceforth, we appeal to [5] getting that $S(PC)$ is Σ-cyclic. Therefore, so is V as a subgroup. Finally, we infer that $S(FG)$ is really $\aleph_1$-Σ-cyclic, as wanted.

The invariant property of the p-component of the group basis plainly follows in virtue of the first part. □

Definition 3.11. *Strongly* $\aleph_1$-Σ-*cyclic*group is a group A such that it is $\aleph_1$-Σ-cyclic and every countable subgroup of A is contained in a countable pure subgroup B of A such that A/B is $\aleph_1$-Σ-cyclic. Such a countable subgroup B is known as $\aleph_1$-*pure* in A.

In [26], MEKLER had proved in the torsion-free case that under (MA + ¬ CH) the strongly $\aleph_1$-Σ-cyclic groups of cardinality $\aleph_1$ are precisely the $\aleph_1$-separable groups. A question of challenging interest is of whether or not this result has an analogue for p-groups.

Also, in [24], MEGIBBEN had showed under (MA+ ¬ CH) that any $\aleph_1$-separable p-group of power $\aleph_1$ is a Crawley group. Moreover, in [30] was obtained that under (CH) any Crawley group with a countable basic subgroup is Σ-cyclic.

We emphasize that a group G is strongly Crawley if for all Σ-cyclic groups C, the direct sum $G \oplus C$ remains a Crawley group (see [13]).

Now, we concentrate on a new notion, introduced by MEGIBBEN in [25], namely on the so-called weakly $\aleph_1$-separable groups.

Definition 3.12. A is a *weakly* $\aleph_1$-*separable* group if every countable subset is contained in a countable $\aleph_1$-pure Σ-cyclic subgroup (see Definition 7).

In other words, A is weakly $\aleph_1$-separable if each countable subset of A is contained in a pure subgroup B of A such that B is Σ-cyclic and for all $A \supseteq C \supseteq B$ if C/B is countable then C/B is Σ-cyclic.

In [26] it was shown that, under the assumption of (MA(ω_1)), a weakly $\aleph_1$-separable torsion-free group of power $\aleph_1$ is $\aleph_1$-separable, but whether or not this holds true for primary groups is still unknown. (See Theorem 3.20 below).

A good reference for weakly $\aleph_1$-separable p-groups is the remarkable paper of MEGIBBEN [25]. Here we list some important contributions from there.

Definition 3.13. Given a separable p-group G of cardinality $\aleph_1$. A subgroup A is $\aleph_1$-*pure* (in other terms is ω_1-*pure*) in G if and only if A is pure and closed in G, that is, A is pure in G and G/A is separable (see Definition 7).

So, we come to an equivalent reformulation to the stated above definition for p-primary groups.

Definition 3.14. A separable p-group G is termed *weakly $\aleph_1$-separable* if every subset of G of countable cardinality is contained in a countable pure subgroup of G which is closed in G. In other words, a separable p-group G of power $\aleph_1$ is said to be *weakly $\aleph_1$-separable* if each countable subset of G is contained in a countable $\aleph_1$-pure subgroup of G (see Definitions 8 and 9).

The following necessary and sufficient condition, argued in [25], is the central tool.

Criterion. A separable p-group G is weakly $\aleph_1$-separable if and only if for every countable subgroup A of G the factor-group $(G/A)^{p^\omega}$ is also at most countable.

We note that this is equivalent to $\forall\ A \leq G : |A| = \aleph_0$ forces that $|\bigcap_{n<\omega_0}(AG^{p^n})| = \aleph_0$. According to Definition 3, it is a routine exercise to see that any p-torsion Q-group is a weakly $\aleph_1$-separable p-group.

This characterization immediately yields the following (see [25]).

Theorem 3.15. *A weakly $\aleph_1$-separable p-group can be simultaneously recognized by its socle, that is, if G is a weakly $\aleph_1$-separable p-group and H is an arbitrary group such that $G[p]$ and $H[p]$ are isometric ($=$ isomorphic as valuated vector spaces) then H is weakly $\aleph_1$-separable too.*

Remark 3.16. It is undecidable in (ZFC) whether an $\aleph_1$-separable p-group is recognizable from its socle as a filtered vector space.

At the moment, there is no criterion demonstrated when two (weakly) $\aleph_1$-separable (p-) groups are isomorphic, but however an equivalence theorem for countable subgroups of those primary groups has been established in [18].

It is elementary to see that the $\aleph_1$-separable p-groups are weakly $\aleph_1$-separable, but the converse is invalid. The next attainments give a satisfactory connection between the studied group classes and also guarantee some extra properties.

Theorem 3.17 ([25])**.** *A separable p-group of cardinality $\aleph_1$ is a Q-group if and only if it is weakly $\aleph_1$-separable.*

Theorem 3.18 ([14, 25]). (*MA+¬CH*) *Every pure subgroup of an* $\aleph_1$*-separable p-group of power* $\aleph_1$ *is itself* $\aleph_1$*-separable.*

For direct factors, the last may be done more precise like this:

Theorem 3.19. *A direct factor of an* $\aleph_1$*-separable p-group is itself* $\aleph_1$*-separable.*

Theorem 3.20 ([25]). (*MA+¬CH*) *Every weakly* $\aleph_1$*-separable p-group of cardinality less than* $2^{\aleph_0}$ (*in particular, of cardinality* $\aleph_1 < c = 2^{\aleph_0}$) *is* $\aleph_1$*-separable.*

On the other hand, assuming $2^{\aleph_0} < 2^{\aleph_1}$, there are weakly $\aleph_1$-separable p-groups of cardinality $\aleph_1$ which are not $\aleph_1$-separable, thus contrasting the aforementioned Megibben's affirmation. Even more, notice that in [12], it was proved that if $2^{\aleph_0} < 2^{\aleph_1}$ there is a $p^{\omega+1}$-projective p-group of cardinality $\aleph_1$ which is weakly $\aleph_1$-separable but not $\aleph_1$-separable.

In this aspect, we indicate the following.

Remark 3.21 ([11, 12]). Assume $2^{\aleph_0} < 2^{\aleph_1}$. Then every Crawley group of cardinality $\aleph_1$ is weakly $\aleph_1$-separable (see [24], Theorems 3.1 and 3.2).

Combining Theorems 3.17 and 3.20, we infer that under (MA+¬CH) the classes of Q-p-groups of power $\aleph_1$, weakly $\aleph_1$-separable p-groups of power $\aleph_1$ and $\aleph_1$-separable p-groups of power $\aleph_1$ do coincide.

Furthermore, we concentrate on major results from [19]. Firstly, we state a new description of weakly $\aleph_1$-separability. (Compare with the foregoing quoted facts.)

Criterion (Huber). A separable p-group G of cardinality $\aleph_1$ is weakly $\aleph_1$-separable if and only if it has a good $\aleph_1$-filtration, i.e., $G = \cup_{\nu<\omega_1} G_\nu$ is a smooth union such that for each $\nu < \omega_1$ we have $|G_\nu| \leq \aleph_0$ and $G_{\nu+1}$ is $\aleph_1$-pure in G (that is, $G_{\nu+1}$ is pure and closed in G).

Definition 3.22. Let G be an abelian p-group of cardinality ν and suppose that ν is a regular cardinal.

(1) A *ν-filtration* of G is an increasing chain $\{G_\alpha\}$ indexed by all the ordinals $< \nu$ satisfying:

(a) For a limit ordinal δ, $G_\delta = \cup_{\alpha<\delta} G_\alpha$;

(b) $G = \cup_{\alpha<\nu} G_\alpha$;

(c) $|G_\alpha| < \nu \ \forall \alpha < \nu$.

(2) A subgroup H of G is *ν-pure* in G if whenever K is a subgroup of G with power $< \nu$ containing H, then H is a direct summand of K.

(3) A subset $C \subset \nu$ is a *club* if it is closed in the order topology and is unbounded.

(4) Define an equivalence relation $\sim$ on the power set of ν by $X \sim Y \iff \exists$ a club $C : C \cap X = C \cap Y$.

A crucial notion is that of $\Gamma(G)$ for any abelian p-group G (see, for example, [19]). In fact, $\Gamma(G)$ means the following invariant of G. Given any two ν-filtrations $(G_\alpha)_{\alpha<\nu}$ and $(G'_\alpha)_{\alpha<\nu}$ of a group G of cardinality ν, the set $C = \{\alpha : G_\alpha = G'_\alpha\}$ is a club. Given a ν-filtration $(G_\alpha)_{\alpha<\nu}$ of G, the $\sim$-equivalence class of the set E defined by $E = \{\alpha \mid G_\alpha \text{ is not } \nu - \text{pure in } G\}$ is then a numerical invariant of G. We denote this invariant by $\Gamma(G)$.

Theorem 3.23 (Huber)**.** *Let G be a group of regular uncountable cardinality. Then $\Gamma(G) = 0$ if and only if G is a direct sum of groups of smaller cardinality.*

Corollary 3.24. *If G is a p-primary group of cardinality $\aleph_1$ we have $\Gamma(G) = 0$ if and only if G is Σ-countable. In particular, $\Gamma(G) = 0$ if and only if G is Σ-cyclic, provided G is separable p-torsion with $|G| = \aleph_1$.*

Theorem 3.25. *If G is a separable p-group of cardinality $\aleph_1$ such that $\Gamma(G) \neq 1$, then G is a Q-group (i.e., G is weakly $\aleph_1$-separable by Theorem* 3.17 *listed above).*

The regarding of function Γ in $V(FG)$ is accomplished by the following:

Conjecture. $\Gamma(V(FG)) = \Gamma(G) + \Gamma(V(FG)/G)$.

In some partial cases we may have $\Gamma(V(FG)) = \Gamma(G)$. In order to very this, we take a p-mixed group G with $|G| = \aleph_1$, length$(G) = \Omega$ and F a field of char$(F) = p > 0$ with $|F| = \aleph_1$. Hence, appealing to [9] (see [4]), $V(FG)/G$ is a Σ-countable p-group of length Ω and power $\aleph_1$. Thus, by what we have already stated above in the Huber's result, $\Gamma(V(FG)/G) = 0$, and so we are done.

In the next lines, we improve in some way an isomorphism theorem due to Göbel-May in [16] pertaining to the modular group algebras of $\aleph_1$-separable abelian p-groups of cardinality $\aleph_1$. But, first of all, we list for completeness of the exposition some statements documented in [16]:

Claim 1 [16, Proposition 1]**.** *Let G be a p-group of cardinality $\aleph_1$ and H a group such that $FG \cong FH$. Then G and H are quotient equivalent.*

Claim 2 [16, Corollary 2]**.** *If G and H are $\aleph_1$-separable p-groups of cardinality $\aleph_1$ such that $FG \cong FH$. Then $\Gamma(G) = \Gamma(H)$.*

Specifically, we now prove the following extension of the latter claim.

Proposition 3.26. *Suppose G is a separable p-primary group of power $\aleph_1$ and $FG \cong FH$ for an arbitrary group H. Then $\Gamma(G) = \Gamma(H)$.*

First proof. It follows at once by Claim 1 above and [19]. □

Second proof. In accordance with [11], $\Gamma(G) = \Gamma(G[p])$. But it is well known that $FG \cong FH$ implies $G[p] \cong H[p]$, and hence the assertion. □

The following plays a significant role for the isomorphism question of group algebras of abelian groups with $\aleph_1$-separable p-components.

Theorem 3.27. *For an abelian group G and a field F of characteristic $p \neq 0$, $S(FG)$ is (weakly) $\aleph_1$-separable if and only if G_p is (weakly) $\aleph_1$-separable.*

In particular, $FG \cong FH$ as F-algebras for any group H and G_p being (weakly) $\aleph_1$-separable yield that H_p is (weakly) $\aleph_1$-separable.

Proof. Since G_p is separable in both situations, by exploiting Theorem 2.1 we have that $S(FG) \cong G_p \times (S(FG)/G_p)$, where the latter factor is Σ-cyclic.

First, we concern the $\aleph_1$-separable groups. Thus $S(FG)$ being $\aleph_1$-separable jointly with Theorem 3.19 ensure that G_p is $\aleph_1$-separable too. The reverse follows trivially in virtue of the same arguments presented.

Second, we treat the case for weakly $\aleph_1$-separable groups. The asserted property holds again by consulting with the preceding decomposing isomorphism formula combined with [25].

The final half-part is true in conjunction with the aforementioned standard back-and-forth procedure. This completes the proof. □

The *cancellation property* says the following: An abelian (p-) group G is said to have the cancellation property if, for any abelian (p-) groups K and L, the following implication is fulfilled: $G \times K \cong G \times L \Rightarrow K \cong L$. It is worth noting that the Σ-cyclic p-groups do not possess in general this property (see, e.g., [1], [2] and [3]).

Nevertheless, the weaker dual variant of that property is of own interest. We shall say that the abelian $\aleph_1$-separable p-groups K and L satisfy the

Kaplansky test problem (see [10]) if $K \times G \cong L \times G$ for each nontrivial Σ-cyclic p-group G implies that $K \cong L$.

Now we become to the following affirmation, which possesses a conditional character.

Theorem 3.28. *If the Kaplansky test problem holds true for the classes of $\aleph_1$-separable p-groups and weakly $\aleph_1$-separable p-groups (both of cardinality $\aleph_1$) and if $FG \cong FH$ are F-isomorphic for any group H, then $G_p \cong H_p$ provided G_p is a (weakly) $\aleph_1$-separable group. In particular, $G \cong H$ provided G is a (weakly) $\aleph_1$-separable p-group.*

Proof. Since G_p is separable, we deduce that so is H_p and, by applying Theorem 2.1, we obtain that $G_p \times (S(FG)/G_p) \cong H_p \times (S(FH)/H_p)$ where the two quotients are Σ-cyclic. After the selection of a Σ-cyclic p-group T such that $(S(FG)/G_p) \times T \cong T \cong (S(FH)/H_p) \times T$ (this can be easily done by choosing T with Ulm-Kaplansky invariants greater than those of $S(FG)/G_p$ and $S(FH)/H_p$ - it is obviously seen that the construction is possible and so we omit the details), we infer that $G_p \times T \cong H_p \times T$, hence by hypothesis $G_p \cong H_p$ as claimed. □

Remark 3.29. Göbel and May have established in [16] by the usage of some original arguments (similar to the Kaplansky test problem) that under (MA+¬CH) the F-group algebra FG determines the isomorphism class of the $\aleph_1$-separable abelian p-group G of power $\aleph_1$.

4. Concluding discussion and remarks

In closing, we comment some more details concerning the investigated theme. In fact, we ask whether $\Gamma(G) = \Gamma(G[p])$ provided G is a p-group with $|G| = \aleph_1$ and length$(G) > \omega$. If yes, Proposition 3.26 can be strengthened to inseparable primary groups. In this aspect, we state the following problem:

Question. Does it follow that the restriction in Theorems 3.7 and 3.28 on G to have power $\aleph_1$ can be ignored?

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