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ON MULTIPLIERS OF THE GROUP $L_{p,w}(G)$-ALGEBRAS

BY

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Abstract. Let G be a locally compact abelian group (non-compact, non-discrete) with Haar measure and $1 \leq p < \infty$. The purpose of this paper is to study the space of multipliers on $L_{p,w}(G)$ and characterize it as the algebra of all multipliers of the closely related Banach algebra of tempered elements in $L_{p,w}(G)$.

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1. Introduction and preliminaries

The multipliers of type (p,p) and multipliers of the group L_p-algebras $\left(L_p^t(G)\right)$ were studied and developed by McKennon [10,11,12] and Griffin [8] where the multipliers are identified as the operators commuting with the translation operators. In [3], Feichtinger showed that Banach algebra of all continuous linear operators on B which commute with left translations can be identified with the algebra of all right multipliers of the algebra of its tempered elements where B is a translation invariant Banach space of functions on locally compact groups. In this paper, the ideas and the methods in [2,3,8-12,14,16] are used frequently for the generalization of the results of McKennon and Griffin concerning multipliers of type (p,p) to weighted Lebesgue spaces $L_{p,w}(G)$ for $1 \leq p < \infty$.

Let G be locally compact abelian group with Haar measure λ. For a Beurling weight w on G, i.e. a continuous function w satisfying $w(x) \geq 1$, and $w(x+y) \leq w(x)\, w(y)$, for all $x, y \in G$, it was set for $1 \leq p < \infty$ such

that $L_{p,w}(G) = \{f | fw \in L_p(G)\}$ ([15]). It is a Banach space under the natural norm

$$\|f\|_{p,w} = \left(\int_G |f(x)|^p w^p(x)\, d\lambda(x) \right)^{\frac{1}{p}}$$

and an essential Banach $L_{1,w}(G)$-module under convolution ([5]). It is known that $C_c(G)$, the space of complex-valued continuous functions with compact support, is dense in $L_{p,w}(G)$ ([13]).

Throughout the paper, the symbols $B_{p,w}$ and $M_{p,w}$ will stand for Banach algebra of all bounded linear operators on $L_{p,w}(G)$ and the subset of $B_{p,w}$ consisting of these operators which commute with all translation operators, respectively. Let $Hom_{L_{1,w}(G)}(L_{p,w}(G), L_{p,w}(G))$ denote the space of all module homomorphisms of $L_{1,w}(G)$-module $L_{p,w}(G)$, that is, an operator $T \in B_{p,w}$ satisfies $T(f * g) = f * T(g)$, for all $f \in L_{1,w}(G)$ and $g \in L_{p,w}(G)$. The module homomorphisms space, called the *multipliers space*

$$Hom_{L_{1,w}(G)}(L_{p,w}(G), L_{p,w}(G)) = Hom_{L_{1,w}(G)}(L_{p,w}(G))$$

is a Banach $L_{1,w}(G)$-module by $(f \circ T)(g) = f * T(g) = T(f * g)$, for all $f \in L_{1,w}(G)$ and $g \in L_{p,w}(G)$.

Now, let us briefly describe the content of this paper. In section 2, we will construct the p-temperate (tempered) functions space for weighted Lebesgue spaces. In section 3, we will characterize the multipliers space of $L_{p,w}(G)$ as a certain Banach algebra and generalize the results of McKennon to $L_{p,w}(G)$. Finally in section 4, we will exhibit an isometric isomorphism between the Banach algebra $M_{p,w}$ and the normed algebra $M^t_{p,w}$ of all multipliers on the group $L_{p,w}$-algebra, $L^t_{p,w}(G)$.

2. The space $L^t_{p,w}(G)$ and its basic properties

Definition 1. A function $f \in L_{p,w}(G)$ is said to be $p-temperate$ (tempered) if $(h * f)(x) = \int_G h(t) f\left(t^{-1}x\right) d\lambda(t)$ exists for λ-almost all $x \in G$ whenever $h \in C_c(G)$ and it satisfies the following:

$$\sup \left\{ \|h * f\|_{p,w} : \ h \in C_c(G), \|h\|_{p,w} \leq 1 \right\} < \infty. \tag{2.1}$$

The space of all such functions is denoted by $L^t_{p,w}(G)$ and the number given by (2.1) will be written as $\|f\|^t_{p,w}$. It is easy to see that the function $\|\cdot\|^t_{p,w}$ is a norm and $L^t_{p,w}(G)$ is a normed space under this norm.

In [3, Section 2], he considered Banach function spaces $B \subseteq L^1_{loc}(G)$ satisfying the following conditions:

(I) $(B, \|\|)$ is an essential two-sided Banach module over a suitable Beurling algebra $L^1_w(G)$ with respect to convolution,

(II) $B \cap L^1_w(G)$ is dense in B as well as in $L^1_w(G)$,

(III) $B * B \subseteq L^1_{loc}(G)$ in the sense that for all $f,\ g \in B,\ (f*g)(x) = \int_G f(y)\, g\left(y^{-1}x\right) dy$ exists l.a.e. and $f*g$ is in $L^1_{loc}(G)$

and he stated the following theorem.

Theorem 1 ([3, Theorem 2.5])**.** *Let B be a solid BF-space satisfying (III). Then $f \in B$ is tempered iff $g*f$ is in B for all $g \in B$. Moreover the extended multiplication on B^t is given by ordinary convolution.*

At the end of Section 2 in [3], he mentioned that if $B \subseteq L^1(G) + L^2(G)$ and G is unimodular, then B satisfies (III) and as a result $B = L_p(G)$ on a unimodular group G proves Theorem 2.5 for $1 \leq p \leq 2$. Therefore, we can write an equivalent definition to Definition 1: a function $f \in L_{p,w}(G)$ is said to be $p-temperate$ (tempered) if $(h*f)(x) = \int_G h(t)\, f\left(t^{-1}x\right) d\lambda(t)$ exists for λ-almost all $x \in G$ whenever $h \in L_{p,w}(G)$ and it satisfies the following:

$$\sup\{\|h*f\|_{p,w} :\ h \in L_{p,w}(G), \|h\|_{p,w} \leq 1\} < \infty,$$

for $1 \leq p \leq 2$ and $\frac{1}{w} \in L_r(G)$ where $r \in [q, \infty]$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{2} \geq 0$.

Proposition 1. *$L^t_{p,w}(G)$ is a dense subspace of $L_{p,w}(G)$.*

Proof. Let us take any $g \in C_c(G)$. Since $C_c(G) * L_{p,w}(G) \subset L_{p,w}(G)$ and $\sup_{\|h\|_{p,w}\leq 1} \|h*g\|_{p,w} < \infty$, we get $C_c(G) \subset L^t_{p,w}(G)$. Therefore, by using the inclusion $C_c(G) \subset L^t_{p,w}(G) \subset L_{p,w}(G)$ and $\overline{C_c(G)} = L_{p,w}(G)$, we have $\overline{L^t_{p,w}(G)} = L_{p,w}(G)$. Also each element of $L_{1,w}(G) \cap L_{p,w}(G)$ is a tempered function. □

Also, note that $f \in L_{p,w}(G)$ is tempered if and only if the map $g \to g*f$ (convolution operator) from $C_c(G)$ into $L_{p,w}(G)$ can be extended for all $g \in C_c(G)$ to a bounded operator on $L_{p,w}(G)$ whose norm is just $\|f\|^t_{p,w}$. Now, this situation shall be considered.

Proposition 2. *For each $f \in L_{p,w}^{t}(G)$, there is precisely one bounded linear operator on $L_{p,w}(G)$ defined as $W_f(g) = g * f$, for all $g \in L_{p,w}(G)$.*

Proof. For each $f \in L_{p,w}^{t}(G)$, if we use the equality $W_f(g) = g * f$, for all $g \in L_{p,w}(G)$, then W_f is well defined, bounded and linear by the density of $C_c(G)$ in $L_{p,w}(G)$ and the definition of $L_{p,w}^{t}(G)$ spaces. Also

$$\|W_f\| = \sup_{\|g\|_{p,w}\leq 1} \|W_f(g)\|_{p,w} = \sup_{\|g\|_{p,w}\leq 1} \|g * f\|_{p,w} = \|f\|_{p,w}^{t}. \tag{2.2}$$

Proposition 3. *$(L_{p,w}^{t}(G), \|\cdot\|_{p,w}^{t})$ is a normed algebra under the convolution.*

Proof. The proof is a special case of the proof of [3, Theorem 1.1]. □

Since $(L_{p,w}^{t}(G), \|\cdot\|_{p,w}^{t})$ is a normed algebra under the convolution, $L_{p,w}^{t}(G)$ spaces are also called as group$-L_{p,w}(G)$ algebras. We know that $L_{p,w}(G)$ is an essential Banach $L_{1,w}(G)$-module under convolution. Therefore, the next proposition can be given without its proof. Or, an alternative proof can be derived from the proof of [3, Theorem 1.1].

Proposition 4. *$L_{p,w}^{t}(G)$ is a normed module over $L_{1,w}(G)$ under convolution and invariant under translation.*

In [3,8], it is shown that $L_{p}^{t}(G)$ is not a Banach space with the norm $\|\cdot\|_{p}^{t}$ unless $p = 1$. Similarly, $L_{p,w}^{t}(G)$ is not a Banach space with $\|\cdot\|_{p,w}^{t}$ in most cases unless $p = 1$. If $p = 1$, we have already $L_{1,w}^{t}(G) = L_{1,w}(G)$ as $L_{1,w}(G)$ is a Banach algebra and also $L_{2,w}^{t}(G) = \{f \in L_{2,w}(G) : \widehat{f} \in L_{\infty,w}(\widehat{G})\}$. Several particular cases can be seen in [3,10].

Proposition 5. *$L_{p,w}^{t}(G)$ is a Banach algebra with respect to the norm $|||\cdot|||_{p,w}^{t} = \|\cdot\|_{p,w} + \|\cdot\|_{p,w}^{t}$.*

Proof. Let us take a Cauchy sequence $\{f_n\} \subset L_{p,w}^{t}(G)$. Then for each $\varepsilon > 0$, there exists a positive integer n_0 such that $|||f_n - f_m|||_{p,w}^{t} < \varepsilon$, for all $n, m \geq n_0$. Since $|||\cdot|||_{p,w}^{t} \geq \|\cdot\|_{p,w}$, we have $\|f_n - f_m\|_{p,w} < \varepsilon$ for the same ε and n_0. By this way, we have a Cauchy sequence in $L_{p,w}(G)$. Since $L_{p,w}(G)$ is a Banach space, there exists an element of $L_{p,w}(G)$, say f, such that $\lim_n \|f_n - f\|_{p,w} = 0$. Also, for each positive integer n, we can form a Cauchy sequence $\{W_{f_n}\} \subset B_{p,w}$ due to Proposition 2. Completeness of $B_{p,w}$ gives rise to an operator $W \in B_{p,w}$ such that $\lim_n \|W_{f_n} - W\| = 0$. Let us

take $g \in C_c(G)$ with $\|g\|_{p,w} \leq 1$. Since $L_{p,w}(G)$ is a $L_{1,w}(G)$-module, we can write $\lim_n \|f_n * g - f * g\|_{p,w} = 0$. Therefore we get

$$\|g * f\|_{p,w} = \lim_n \|g * f_n\|_{p,w} \leq \lim_n \|f_n\|^t_{p,w} \leq \lim_n \|W_{f_n}\| = \|W\| < \infty \tag{2.3}$$

and $\|f\|^t_{p,w} = \sup_{\|g\|_{p,w}\leq 1} \|g * f\|_{p,w} \leq \|W\| < \infty$ by (2.3). This implies that $f \in L^t_{p,w}(G)$. Also, for any $h \in C_c(G)$, we have

$$W(h) = \lim_n W_{f_n}(h) = \lim_n f_n * h = \lim_n f * h = W_f(h). \tag{2.4}$$

Since $C_c(G)$ is a dense subspace of $L_{p,w}(G)$, we get $W = W_f$. Besides this,

$$\begin{aligned} \lim_n |||f_n - f|||^t_{p,w} &= \lim_n \left(\|f_n - f\|_{p,w} + \|f_n - f\|^t_{p,w} \right) \\ &= \lim_n \left(\|f_n - f\|_{p,w} + \|W_{f_n} - W_f\| \right) = 0 \end{aligned} \tag{2.5}$$

can be written. Hence $L^t_{p,w}(G)$ is a Banach space.

On the other hand, by using the Propositions 2 and 3, we get

$$\begin{aligned} |||f * g|||^t_{p,w} &= \|f * g\|_{p,w} + \|f * g\|^t_{p,w} \leq \|W_g(f)\|_{p,w} + \|f\|^t_{p,w} \|g\|^t_{p,w} \\ &\leq \|W_g\| \|f\|_{p,w} + \|f\|^t_{p,w} \|g\|^t_{p,w} = \|g\|^t_{p,w} \|f\|_{p,w} + \|f\|^t_{p,w} \|g\|^t_{p,w} \\ &= \|g\|^t_{p,w} \left(\|f\|_{p,w} + \|f\|^t_{p,w} \right) \leq |||g|||^t_{p,w} |||f|||^t_{p,w}, \end{aligned} \tag{2.6}$$

for any $f, g \in L^t_{p,w}(G)$. Then $L^t_{p,w}(G)$ is a Banach algebra under the convolution. □

Remark 1. A general proof of the preceding proposition for locally compact groups can be found in [3]. Also, being essential Banach $L_{1,w}(G)$-module property of $L^t_{p,w}(G)$ spaces is discussed in Theorem 2.5 and the following remark in [3].

Proposition 6. *The set*

$$\Im = \overline{span \left\{ W_{f*g} \;\middle|\; f \in L^t_{p,w}(G), g \in C_c(G) \right\}} \tag{2.7}$$

is a complete subalgebra of $B_{p,w}$ and has a bounded approximate identity.

Proof. By the definition of $\Im$, it is easy to see that $\Im$ is a complete subalgebra of $B_{p,w}$ under the operator norm with the usual composition. Let K be a compact neighbourhood of the identity of G with $w(y) \leq A$, for all $y \in K$. Let F be the family of all neighborhoods of the identity contained in K. For $U, V \in F$, define $V \prec U$ if $U \subset V$. Then clearly $(F, \prec)$ is a directed set. For every $U \in F$, there exists a positive continuous function h_U on G such that $\int_G h_U(x)\, d\lambda(x) = 1$, the support of h_U is contained in U and $\|h_U\|_{1,w} \leq A$ for each $U \in F$. This net $\{h_U\}_{U \in F}$ is a bounded approximate identity for $L_{1,w}(G)$ ([17]). If $\{h_\gamma\}$ denotes the product net of $\{h_U\}_{U \in F}$ with itself i.e., $h_\gamma = h_U * h_V$, then $\{h_\gamma\}$ is again a bounded approximate identity for $L_{1,w}(G)$. It can be seen that the net $\{W_{h_\gamma}\}$ is in $\Im$ and $\lim_\gamma \|W_{h_\gamma}\| \leq A^2$. Also, for each $f \in C_c(G) \subset L^t_{p,w}(G)$, we have

$$\|W_f\| = \sup_{\|h\|_{p,w} \leq 1} \|W_f(h)\|_{p,w} = \sup_{\|h\|_{p,w} \leq 1} \|f * h\|_{p,w} \leq \|f\|_{1,w}. \tag{2.8}$$

Again, for all $f, g \in L^t_{p,w}(G)$ and $h \in L_{p,w}(G)$, we see that

$$(W_f - W_g)(h) = h * f - h * g = h * (f - g) = W_{f-g}(h). \tag{2.9}$$

Let $f \in L^t_{p,w}(G)$ and $g \in C_c(G)$. Since $\Im$ is a complete subalgebra of $B_{p,w}$, by using (2.8) and (2.9), we get

$$\begin{aligned} \overline{\lim_\gamma} \|W_{h_\gamma} \circ W_{f*g} - W_{f*g}\| &= \overline{\lim_\gamma} \|(W_{g*h_\gamma} - W_g) \circ W_f\| \\ &\leq \overline{\lim_\gamma} \|W_{g*h_\gamma - g}\| \, \|W_f\| \leq \overline{\lim_\gamma} \|g * h_\gamma - g\|_{1,w} \, \|W_f\| = 0. \end{aligned}$$

Consequently, we have $\overline{\lim_\gamma} \|W_{h_\gamma} \circ T - T\| = 0$, for all $T \in \Im$. □

Proposition 7. *$\Im$ is a complete subalgebra of $Hom_{L_{1,w}(G)}(L_{p,w}(G))$.*

Proof. We know that $\{W_{f*u} \mid f \in L^t_{p,w}(G), u \in C_c(G)\}$ is a fundamental set for $\Im$. Therefore it is enough to take a $W_f \in \Im$ where $f \in L^t_{p,w}(G)$. Since $L_{p,w}(G)$ is a Banach $L_{1,w}(G)$ -module we have $W_f(g * h) = g * h * f = g * W_f(h)$, for all $g \in L_{1,w}(G)$, $h \in L_{p,w}(G)$. Thus W_f belongs to $Hom_{L_{1,w}(G)}(L_{p,w}(G))$. Since $Hom_{L_{1,w}(G)}(L_{p,w}(G))$ is a Banach space under the usual operator norm, $\Im$ is a complete subalgebra of $Hom_{L_{1,w}(G)}(L_{p,w}(G))$. □

Proposition 8. *$\Im$ is an essential Banach $L_{1,w}(G)$-module by $(g \circ W_f)(u) = W_f(g * u)$, for all $u \in L_{p,w}(G)$ and $g \in L_{1,w}(G)$.*

Proof. Since $\left\{W_{f*g} \mid f \in L^t_{p,w}(G), g \in C_c(G)\right\}$ is a fundamental set for $\Im$, it is enough to take a $W_f \in \Im$ where $f \in L^t_{p,w}(G)$. Let us take any $g \in L_{1,w}(G)$ and define $g \circ W_f : L_{p,w}(G) \to L_{p,w}(G)$ by $(g \circ W_f)(u) = W_f(g * u)$ for each $u \in L_{p,w}(G)$. In this case

$$\|g \circ W_f\| = \sup_{\|u\|_{p,w} \leq 1} \|(g \circ W_f)(u)\|_{p,w} \leq \|W_f\| \|g\|_{1,w} = \|f\|^t_{p,w} \|g\|_{1,w}$$

is found. As a result, $\Im$ is a Banach $L_{1,w}(G)$ -module. Since any bounded approximate identity $\{e_\alpha\}$, $(e_\alpha \geq 0)$ of $L_{1,w}(G)$ is also an approximate identity for $L_{p,w}(G)$ ([17]), for any $W_{f*g} \in \Im$, we have

$$\begin{aligned} \|e_\alpha \circ W_{f*g} - W_{f*g}\| &= \sup_{\|u\|_{p,w} \leq 1} \|(e_\alpha \circ W_{f*g} - W_{f*g})(u)\|_{p,w} \\ &= \sup_{\|u\|_{p,w} \leq 1} \|u * e_\alpha * f * g - u * f * g\|_{p,w} \\ &= \sup_{\|u\|_{p,w} \leq 1} \|u * f\|_{p,w} \|e_\alpha * g - g\|_{1,w} \\ &\leq \|f\|^t_{p,w} \|e_\alpha * g - g\|_{1,w}, \end{aligned} \tag{2.10}$$

for all $u \in L_{p,w}(G)$. Therefore $\Im$ is an essential Banach $L_{1,w}(G)$ -module by the factorization theorem for Banach modules in [1, Corollary 15.3]. □

Remark 2. An alternative proof for general Banach spaces can be seen in the "only if" part of [4, Proposition 1.1]. Since $L^t_{p,w}(G) \subset \Im$ and $\Im$ is an essential Banach $L_{1,w}(G)$ -module, another discussion about not being essential Banach $L_{1,w}(G)$ -module property of $L^t_{p,w}(G)$ spaces can be derived.

Also for any $f \in L_{1,w}(G)$ and $W_{h_\gamma} \in \Im$, we have

$$\begin{aligned} \lim_\gamma \left\|f - f \circ W_{h_\gamma}\right\| &= \lim_\gamma \Big(\sup_{\|u\|_{p,w} \leq 1} \|f * u - h_\gamma * (f * u)\|_{p,w} \Big) \\ &\leq \lim_\gamma \Big(\sup_{\|u\|_{p,w} \leq 1} \|f - h_\gamma * f\|_{1,w} \|u\|_{p,w} \Big) \\ &\leq \lim_\gamma \|f - h_\gamma * f\|_{1,w} = 0. \end{aligned} \tag{2.11}$$

So $f \in \overline{L_{1,w}(G) \circ \Im}$, and $f \in \Im$ by Proposition 8. That is to say $L_{1,w}(G) \subset \Im$.

3. Identification for the multipliers spaces of $L_{1,w}(G)$-module with the multipliers space of certain normed algebra

Proposition 9. *Each multiplier of $L_{p,w}(G)$ is also a multiplier of $L^t_{p,w}(G)$.*

Proof. Let $f \in L_{p,w}(G), g \in L^t_{p,w}(G)$ and $T \in Hom_{L_{1,w}(G)}(L_{p,w}(G))$. Then, we get

$$\begin{aligned} \|T(g)\|^t_{p,w} &= \sup\left\{\|h * T(g)\|_{p,w} : \ h \in C_c(G), \|h\|_{p,w} \leq 1\right\} \\ &= \sup\left\{\|T(h * g)\|_{p,w} : \ h \in C_c(G), \|h\|_{p,w} \leq 1\right\} \\ &\leq \|T\| \, \|g\|^t_{p,w} < \infty. \end{aligned}$$

Since $C_c(G)$ is dense in $L_{p,w}(G)$, we can find a sequence $\{f_n\} \subset C_c(G)$ such that $\lim_n \|f_n - f\|_{p,w} = 0$ and $\lim_n \|f_n * g - f * g\|_{p,w} = 0$ by (2.2). Also, by the first part of the proof, we have

$$\lim_n \|f_n * T(g) - f * T(g)\|_{p,w} = 0$$

and $f * T(g) = \lim_n f_n * T(g) = \lim_n T(f_n * g) = T(f * g)$. □

Lemma 1. *For every T in $Hom_{L_{1,w}(G)}(L_{p,w}(G))$, $T \circ W \in \Im$, for all $W \in \Im$.*

Proof. Let $T \in Hom_{L_{1,w}(G)}(L_{p,w}(G))$, $f \in L^t_{p,w}(G)$ and $g \in C_c(G)$. Since the set $span\left\{W_{f*g} \mid f \in L^t_{p,w}(G), g \in C_c(G)\right\}$ is dense in $\Im$, for the proof, it is enough to show that $T \circ W_{f*g} \in \Im$. By Proposition 9, we get

$$(T \circ W_{f*g})(h) = T(h * f * g) = h * T(f * g) = W_{T(f*g)}(h) = W_{g*T(f)}(h),$$

for all $h \in L_{p,w}(G)$. Thus $T \circ W \in \Im$ for any $W \in \Im$. □

Theorem 2. *Let G be a locally compact abelian group and $1 \leq p < \infty$. Then $M(\Im, \Im)$, the space of multipliers on Banach algebra $\Im$, is algebraically isomorphic and topologically homeomorphic to $Hom_{L_{1,w}(G)}(L_{p,w}(G))$.*

Proof. Define a mapping $\Psi : Hom_{L_{1,w}(G)}(L_{p,w}(G)) \to M(\Im, \Im)$ by $\Psi(T) = \rho_T$ for each $T \in Hom_{L_{1,w}(G)}(L_{p,w}(G))$ where $\rho_T(S) = T \circ S$, for all $S \in \Im$. Note that Ψ is well-defined and moreover since $\rho_T(S \circ K) =$

$T \circ S \circ K = \rho_T(S) \circ K$, for all $S, K \in \Im$, we have $\Psi(T) = \rho_T \in M(\Im, \Im)$. It is obvious that the mapping Ψ is linear.

If we define a set P such that

$$P = \{W(f) : W \in \Im,\ f \in L_{p,w}(G)\}, \tag{3.1}$$

then by Proposition 9 and the density of $C_c(G)$ in $L_{p,w}(G)$, it is easy to show that P is dense in $L_{p,w}(G)$. Therefore, for any $T_1, T_2 \in Hom_{L_{1,w}(G)}(L_{p,w}(G))$ if $\Psi(T_1) = \rho_{T_1} = \Psi(T_2) = \rho_{T_2}$, then we have $T_1 = T_2$ on P. By the density of P in $L_{p,w}(G)$, the mapping Ψ is injective.

Also, for any $T \in Hom_{L_{1,w}(G)}(L_{p,w}(G))$ and $S \in \Im$, we have

$$\begin{aligned} \|T \circ S\| &= \sup_{\|g\|_{p,w} \le 1} \|(T \circ S)(g)\|_{p,w} = \sup_{\|g\|_{p,w} \le 1} \|T(S(g))\|_{p,w} \\ &\le \|T\| \sup_{\|g\|_{p,w} \le 1} \|S(g)\|_{p,w} = \|T\| \|S\| \end{aligned} \tag{3.2}$$

by Lemma 1 and

$$\|\rho_T\| = \sup_{S \in \Im} \frac{\|\rho_T(S)\|}{\|S\|} = \sup_{S \in \Im} \frac{\|T \circ S\|}{\|S\|} \le \|T\|$$

by (3.2). On the other hand, since $\{W_{h_\gamma}\}$ is a bounded approximate identity for the space $\Im$, by Proposition 6 we get

$$\|\rho_T\| = \sup_{S \in \Im} \frac{\|T \circ S\|}{\|S\|} \ge \sup_{\gamma} \frac{\|T \circ W_{h_\gamma}\|}{\|W_{h_\gamma}\|} \ge \sup_{\gamma} \frac{\|T \circ W_{h_\gamma}\|}{A^2} \ge \frac{\|T\|}{A^2}$$

and hence $\frac{\|T\|}{A^2} \le \|\rho_T\| \le \|T\|$. Finally we will show that the mapping $\Psi : Hom_{L_{1,w}(G)}(L_{p,w}(G)) \to M(\Im, \Im)$ is onto.

Let ρ be an element of $M(\Im, \Im)$ and $\{h_U\}$ approximate identity for $L_{1,w}(G)$. Since $\Im \subset Hom_{L_{1,w}(G)}(L_{p,w}(G))$ and $\rho h_U \in \Im$, for any $f \in L_{1,w}(G)$ and $g \in L_{p,w}(G)$, we have

$$\rho h_U (f * g) = (f \circ (\rho h_U))(g). \tag{3.3}$$

Also $M(\Im, \Im) \subset Hom_{L_{1,w}(G)}(\Im, \Im)$ implies that

$$\rho(f * h_U)(g) = (f \circ (\rho h_U))(g). \tag{3.4}$$

Therefore by (3.3) and (3.4) we get

$$\rho h_U (f * g) = (f \circ (\rho h_U))(g) = \rho(f * h_U)(g). \tag{3.5}$$

So for each $f \in L_{1,w}(G)$ and $g \in L_{p,w}(G)$

$$\begin{aligned}\lim_U \|\rho(f * h_U)(g) - \rho f(g)\|_{p,w} &= \lim_U \|(\rho(f * h_U) - \rho f)(g)\|_{p,w} \\ &= \lim_U \|\rho(f * h_U - f)(g)\|_{p,w} \\ &\leq \lim_U \|\rho(f * h_U - f)\| \|g\|_{p,w} \\ &\leq \|\rho\| \lim_U \|f * h_U - f\|_{1,w} \|g\|_{p,w} = 0,\end{aligned}$$

is obtained. Thus by (3.5), we get

$$\lim_U (\rho h_U)(f * g) = \lim_U (f \circ (\rho h_U))(g) = \lim_U \rho(f * h_U)(g) = \rho f(g).$$

Since the space $L_{p,w}(G)$ is an essential Banach $L_{1,w}(G)$-module, the limit of $(\rho h_U)(f * g) = (f \circ (\rho h_U))(g)$ exists and equal to $f * T(g) \in L_{p,w}(G)$ where T is an operator in $Hom_{L_{1,w}(G)}(L_{p,w}(G))$. Since $\lim_U (\rho h_U)(f * g) = \lim_U (f \circ (\rho h_U))(g) = \rho f(g)$ exists, we can write $f \circ T = \rho f$, for all $f \in L_{1,w}(G)$. Then $h_U \circ T \circ W = (\rho h_U) \circ W = \rho(h_U \circ W)$ can be written for all $W \in \Im$. By Proposition 8, for all $W \in \Im$,we get $T \circ W = \rho(W)$ or $\rho_T(W) = \rho(W)$. Therefore $\rho_T = \rho$. □

4. The relationship between $M_{p,w}$ and $M_{p,w}^t$

At the beginning of the paper, we defined $M_{p,w}$ and $m_{p,w}$ spaces as:

$$M_{p,w}^t = \{T | T : L_{p,w}^t(G) \to L_{p,w}^t(G), T(f * g) = f * T(g), \forall f, g \in L_{p,w}^t(G)\}$$

and

$$M_{p,w} = \{T | T : L_{p,w}(G) \to L_{p,w}(G),\ TL_x(f) = L_x T(f)\ ,\ \forall f \in L_{p,w}(G)\}.$$

where $L_x f(y) = f(y - x)$. It is easy to see that the space $M_{p,w}^t$ is a normed algebra under usual operator norm and $M_{p,w}$ is a complete subalgebra of $B_{p,w}$.

Proposition 10. *Let T be in $M_{p,w}$.*

(i) T commutes with convolution by elements of $L_{1,w}(G)$.

(ii) if $f \in L_{p,w}^t(G)$, $T(f) \in L_{p,w}^t(G)$.

(iii) if $f \in L_{p,w}(G)$ *and* $g \in L^t_{p,w}(G)$, *then* $T(f * g) = f * T(g)$.

Proof. The proof of (i) follows from [3, Lemma 2.1] and the others can be proved by the same methods used in Proposition 9. □

Lemma 2. *For each* $T \in M^t_{p,w}$ *and* $V \in \Im$,

$$\sup\left\{\|T \circ V(h)\|_{p,w} : h \in L^t_{p,w}(G), \|h\|_{p,w} \leq 1\right\} \leq \|T\| \|V\|.$$

Proof. Let us define $D = \{W_f : f \in L^t_{p,w}(G), W_f \in \Im\}$ and take any $W \in \Im$. Then, by definition of $\Im$, we can find a $W_f \in span\{W_{f*g} \mid f \in L^t_{p,w}(G), g \in C_c(G)\}$ such that $\|W_f - W\| < \varepsilon$ for any $\varepsilon > 0$. This says that D is a dense subspace of $\Im$ and $\|W_f\| = \|f\|^t_{p,w}$, for all $W_f \in D$. Therefore, if $\pi' : D \to B_{p,w}$ is defined by $\pi'(W_f) = W_{T(f)}$, for all $W_f \in D$, then π' is bounded. So, we can find a unique extension of π', say π, such that $\pi : \Im \to B_{p,w}$. Let $h \in L^t_{p,w}(G)$ with $\|h\|_{p,w} \leq 1$ and $\{f_n\} \subset L^t_{p,w}(G)$ be a sequence such the $W_{f_n} \in D$ for each positive integer n and there exist $V \in \Im$ such that

$$\lim_n \|W_{f_n} - V\| = 0. \tag{4.1}$$

Also, since we have $W_g(L_x f) = g * L_x f = L_x(f * g) = L_x W_g(f)$ for each $f \in L_{p,w}(G)$ and $W_g \in \Im$, we can see that $\Im \subset M_{p,w}$ and $V \in M_{p,w}$. So, by Proposition 10, the equality

$$(V \circ W_h)(g) = V(W_h(g)) = V(g * h) = g * V(h) = W_{V(h)}(g)$$

can be written for all $g \in L_{p,w}(G)$, i.e. $V \circ W_h = W_{V(h)}$. Since $W_{W_{f_n}(h)} = W_{f_n} \circ W_h$ for each positive integer n, the equality

$$\begin{aligned} \|W_{f_n}(h) - V(h)\|^t_{p,w} &= \left\|W_{W_{f_n}(h)} - W_{V(h)}\right\| = \|W_{f_n} \circ W_h - V \circ W_h\| \\ &= \|(W_{f_n} - V) W_h\| \end{aligned}$$

is found by (2.2) and (2.9). Using (4.1), we get

$$\lim_n \|W_{f_n}(h) - V(h)\|^t_{p,w} \leq \lim_n \|W_{f_n} - V\| \|W_h\| = 0.$$

Consequently, for $T \in M^t_{p,w}$,

$$\lim_n \|T(W_{f_n}(h)) - T(V(h))\|^t_{p,w} = 0. \tag{4.2}$$

Since $W_{T(f_n)}(g) = g * T(f_n) = T(g * f_n) = T(W_{f_n}(g)) = (T \circ W_{f_n})(g)$ for each $n \in \mathbb{N}$, and for all $g \in L^t_{p,w}(G)$, we can write $W_{T(f_n)} = T \circ W_{f_n}$ and $\pi(W_{f_n}) = \pi'(W_{f_n}) = W_{T(f_n)} = T \circ W_{f_n}$. As a result, by (4.1)

$$\lim_n \|T \circ W_{f_n} - \pi(V)\| = \lim_n \|\pi(W_{f_n}) - \pi(V)\| = 0$$

is found. Therefore, since $\lim_n \|(T \circ W_{f_n})(h) - \pi(V)(h)\|_{p,w} = 0$, we get

$$\overline{\lim_n} \|g * ((T \circ W_{f_n})(h)) - g * (\pi(V)(h))\|_{p,w} = 0, \tag{4.3}$$

for all $g \in C_c(G)$. Using (4.2) and (4.3), we get $g * (\pi(V)(h)) = g * (T \circ V)(h)$, for all $g \in C_c(G)$ and thus $\pi(V)(h) = (T \circ V)(h)$ by [6, 20.15]. Consequently $\pi(V)$ and $(T \circ V)$ coincide on $L^t_{p,w}(G)$. The normed algebra property of $L^t_{p,w}(G)$ and (4.1) show that

$$\begin{aligned}\|(T \circ V)(h)\|_{p,w} = \|\pi(V)(h)\|_{p,w} &= \lim_n \|\pi(W_{f_n})(h)\|_{p,w} = \lim_n \|W_{T(f_n)}(h)\|_{p,w} \\ &= \lim_n \|h * T(f_n)\|_{p,w} \leq \|h\|_{p,w} \lim_n \|T(f_n)\|^t_{p,w}\end{aligned}$$

exists. Hence by $\|h\|_{p,w} \leq 1$, (2.2) and (4.1), we get$\|(T \circ V)(h)\|_{p,w} \leq \|T\| \lim_n \|f_n\|^t_{p,w} = \|T\| \lim_n \|W_{f_n}\| \leq \|T\| \|V\|$. □

Proposition 11. *For all $T \in M^t_{p,w}$, $V \in \Im$ and $f \in L^t_{p,w}(G)$,*

$$\|(T \circ V)(f)\|_{p,w} \leq \|T\| \|V(f)\|_{p,w}.$$

Proof. Let ε be a positive number. Since $\Im$ is a Banach algebra with a bounded approximate identity, Cohen's factorization theorem [7, 32.26] implies that there exist P and S in $\Im$ such that $\|P\| = 1, \|V - S\| < \varepsilon$ and $V = P \circ S$. Thus

$$\begin{aligned}\|S(f)\|_{p,w} &= \|(S - V + V)(f)\|_{p,w} \leq \|(S - V)(f)\|_{p,w} + \|V(f)\|_{p,w} \\ &\leq \varepsilon \|f\|_{p,w} + \|V(f)\|_{p,w}\end{aligned}$$

and

$$\begin{aligned}\|(T \circ V)(f)\|_{p,w} &= \|(T \circ (P \circ S))(f)\|_{p,w} = \|(T \circ P)(S(f))\|_{p,w} \\ &\leq \|T\| \|P\| \left(\varepsilon \|f\|_{p,w} + \|V(f)\|_{p,w}\right) \\ &= \|T\| \left(\varepsilon \|f\|_{p,w} + \|V(f)\|_{p,w}\right)\end{aligned}$$

can be found by Lemma 2. Therefore, it follows that $\|T(V(f))\|_{p,w} \leq \|T\| \|V(f)\|_{p,w}$. □

Lemma 3. *$E = \{V(f) : f \in L^t_{p,w}(G), V \in \Im\}$ is a dense subspace of $L_{p,w}(G)$.*

Proof. Let $\varepsilon > 0$ and $g \in L_{p,w}(G)$. Since $C_c(G)$ is a dense subspace of $L_{p,w}(G)$, there exists $f \in C_c(G)$ such that $\|g - f\|_{p,w} < \frac{\varepsilon}{2}$. By Proposition 6, while $\{W_{h_\gamma}\}$ is the bounded approximate identity for $\Im$, we get $W_{h_\gamma}(f) \in L_{p,w}(G)$. Since the net $\{h_\gamma\}$ is an approximate identity for $L_{p,w}(G)$,

$$\lim_\gamma \|W_{h_\gamma}(f) - f\|_{p,w} = \lim_\gamma \|h_\gamma * f - f\|_{p,w} = 0$$

is found. Thus, for some index γ, $\|W_{h_\gamma}(f) - f\|_{p,w} < \frac{\varepsilon}{2}$. So, $\{W_{h_\gamma}(f)\} \subset E$ and $\|W_{h_\gamma}(f) - g\|_{p,w} < \varepsilon$. □

Lemma 4. *Let V be in $B_{p,w}$ and D be a dense subset of $L_{p,w}(G)$ such that $V(h * f) = h * V(f)$, for all $h \in C_c(G)$ and $f \in D$. Then V is in $M_{p,w}$.*

Proof. Let us take any $x \in G$. By using the same method in [6, 20.15], we can find a net $\{f_\alpha\} \subset C_c(G)$ such that $\lim_\alpha \|L_x h - f_\alpha * h\|_{p,w} = 0$, for all $h \in L_{p,w}(G)$. Since V is bounded, we get $\lim_\alpha \|V(L_x h) - V(f_\alpha * h)\|_{p,w} = 0$ and $\lim_\alpha \|V(L_x h) - f_\alpha * V(h)\|_{p,w} = 0$ by the hypothesis. Therefore for any $h \in D$, we get

$$\begin{aligned}\|V(L_x h) - L_x V(h)\|_{p,w} &= \lim_\alpha \|V(L_x h) - V(f_\alpha * h) + V(f_\alpha * h) - L_x V(h)\|_{p,w} \\ &\le \lim_\alpha \|V(L_x h) - f_\alpha * V(h)\|_{p,w} + \\ &+ \lim_\alpha \|f_\alpha * V(h) - L_x V(h)\|_{p,w} = 0,\end{aligned}$$

by using the property of V. Since $\overline{D} = L_{p,w}(G)$, we get $V \in M_{p,w}$. □

Theorem 3. *Let $\omega : M_{p,w} \to M^t_{p,w}$ be a mapping defined by $\omega(T) = \omega_T$ such that $\omega(T)(f) = \omega_T(f) = T(f)$, for all $T \in M_{p,w}$ and $f \in L^t_{p,w}$. Then ω is an isometric isomorphism of $M_{p,w}$ onto $M^t_{p,w}$. Furthermore, if T is an operator in $M^t_{p,w}$, then there exists some $S \in M_{p,w}$ such that $\omega_S(V(f)) = T(V(f))$ for all $V \in \Im$ and $f \in L^t_{p,w}$.*

Proof. The map ω is well-defined by Proposition 10. Since $L^t_{p,w}$ is a dense subset of $L_{p,w}$, it is easy to see that ω is linear and injective. Let T be an arbitrary element of $M^t_{p,w}$. It follows from Proposition 11 and Lemma 3 that there exists a unique operator $S \in B_{p,w}$ such that $S(V(f)) =$

$T(V(f))$, for all $V \in \Im$ and $f \in L^t_{p,w}$. For $h \in C_c(G)$, $V \in \Im$ and $f \in L^t_{p,w}$, Proposition 10 implies that

$$\begin{aligned} S(h * V(f)) &= S(V(h * f)) = T(V(h * f)) = T(h * V(f)) \\ &= h * T(V(f)) = h * S(V(f)). \end{aligned} \tag{4.4}$$

By Lemma 3, Lemma 4 and (4.4), we get $S \in M_{p,w}$. Consequently, $\omega_S(V(g)) = S(V(g)) = T(V(g))$, for all $V \in \Im$ and $g \in L^t_{p,w}$.

On the other hand, suppose that $\omega_S(h) \neq T(h)$ for some $h \in L^t_{p,w}(G)$. Then there exists $g \in C_c(G)$ such that $g * (\omega_S(h) - T(h)) \neq 0$ and $\|g * (\omega_S(h) - T(h))\|_{p,w} \neq 0$. Since $h \in L^t_{p,w}(G) \subset L_{p,w}(G)$, there exists $\{h_n\} \subset C_c(G)$ such that $\lim_n \|h_n - h\|_{p,w} = 0$. By Lemma 3, we get

$$\begin{aligned} \|g * (\omega_S(h) - T(h))\|_{p,w} &= \|g * (\omega_S - T)(h)\|_{p,w} = \|(\omega_S - T)(g * h)\|_{p,w} \\ &= \|h * (\omega_S - T)(g)\|_{p,w} \\ &= \lim_n \|h_n * (\omega_S - T)(g)\|_{p,w} \\ &= \lim_n \|(\omega_S - T)(h_n * g)\|_{p,w} \\ &= \lim_n \|(\omega_S - T)(W_{h_n}(g))\|_{p,w}. \end{aligned} \tag{4.5}$$

Since $W_{h_n}(g) \in E$ and $\overline{E} = L_{p,w}$, it follows that $\|g * (\omega_S(h) - T(h))\|_{p,w} = 0$. Thus, with this contradiction we get $\omega(S) = \omega_S = T$ or ω is surjective.

Now, let us show that ω is an isometry. Take any T from $M_{p,w}$, let $f \in L^t_{p,w}$ and ε be a positive number. Let us choose $g \in L^t_{p,w}$ for which $\|g\|_{p,w} \leq 1$ and $\|\omega_T(f)\|^t_{p,w} < \|g * \omega_T(f)\|_{p,w} + \varepsilon$. Then by proposition 11, we get

$$g * \omega_T(f) = g * T(f) = T(g * f) = T(W_f(g)) = (T \circ W_f)(g).$$

So, this implies with (2.2) that

$$\begin{aligned} \|\omega_T(f)\|^t_{p,w} &< \|g * \omega_T(f)\|_{p,w} + \varepsilon \\ &= \|(T \circ W_f)(g)\|_{p,w} + \varepsilon \leq \|T\| \|W_f\| \|g\|_{p,w} + \varepsilon \\ &\leq \|T\| \|f\|^t_{p,w} + \varepsilon. \end{aligned} \tag{4.6}$$

Therefore $\|\omega_T\| = \|\omega(T)\| \leq \|T\|$. Conversely, with Proposition 11 and Lemma 3, for all $T \in M_{p,w}$,

$$\begin{aligned} \|T\| &= \sup\{\|T(V(h))\|_{p,w} : V(h) \in E, \|V(h)\|_{p,w} \leq 1\} \\ &= \sup\{\|\omega_T(V(h))\|_{p,w} : V(h) \in E, \|V(h)\|_{p,w} \leq 1\} \leq \|\omega_T\|, \end{aligned}$$

is found. As a result, $\|\omega_T\| = \|T\|$ and $M_{p,w} \cong M^t_{p,w}$. □

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