ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
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HORIZONTAL LIFTS OF SOME GEOMETRIC OBJECTS TO THE BUNDLE OF PAIRS OF VOLUME FORMS

BY

ANNA GĄSIOR

Abstract. In this paper we present a bundle of pairs of volume forms V^2. We describe horizontal lift of a tensor of type $(1;1)$ and we show that horizontal lift of an almost complex structure on a manifold M is an almost complex structure on the bundle V^2. Next we give conditions under which the almost complex structure on V^2 is integrable. In the second part we find horizontal lift of vector fields, tensor fields of type $(0;2)$ and $(2;0)$, Riemannian metrics and we determine a family of a t-connections on the bundle of pairs of volume forms. At the end, we consider some properties of the horizontally lifted vector fields and certain infinitesimal transformations.

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1. Introduction

Throughout the paper we assume that indices $i, k, \ldots$ will run over the range $1, 2, 3, \ldots, n$ and $\alpha, \beta, \ldots$ will run over the range $0, 1, 2, \ldots, n$, $n+1$. The Einstein summation convention will be used with respect to these systems of indices.

In [7] THOMAS made a construction of a vector bundle of volume form. DHOOGHE gave some interesting properties of this bundle in [1]. Using Thomas construction in this paper we introduce a bundle of pairs of volume forms V^2.

Let M be an orientable n-dimensional manifold and (x^i) a set of coordinates on a neighbourhood $U \subset M$. Then volume forms ω and ν on U are determined as $\omega(x^i)dx^1 \wedge \ldots \wedge dx^n$ and $\nu(x^i)dx^1 \wedge \ldots \wedge dx^n$, where ω, ν are

positive C^∞-functions. The union of all pairs of volume forms (ω, ν) forms a bundle over M, which is generated by the set of all saturated neighborhoods equipped with the coordinates $(\omega, x^1, \ldots, x^n, \nu)$ and let $\pi : \mathcal{V}^2 \to M$ be a projection of the bundle. We call this bundle a bundle of pairs of volume forms and we denote this one by $\mathcal{V}^2$. We identify pairs of volume forms with sections of the bundle $\mathcal{V}^2$.

Let $x^{i'} = \phi(x^i)$ be an orientation-preserving transition functions on M. Then the lifted transition functions on the bundle of pairs of volume forms $\mathcal{V}^2$ are given by the following formula $w' = w\overline{\mathrm{J}}(\phi)(x^i)$, $x^{i'} = \phi(x^i)$, $v' = v\overline{\mathrm{J}}(\phi)(x^i)$, where $\overline{\mathrm{J}}(\phi)$ is the Jacobian of the map ϕ. Since we consider only positive volume forms, then we can put $x^0 = \ln\omega$, $x^{n+1} = \ln\nu$ and then we introduce a new system $(x^0, x^1, \ldots, x^n, x^{n+1})$ as local coordinates on $\mathcal{V}^2$. The transition functions in terms of these coordinates system are

$$\begin{aligned} x^{0'} &= x^0 + \ln\overline{\mathrm{J}}(\phi)(x^i), \\ x^{i'} &= \phi(x^i), \\ x^{(n+1)'} &= x^{n+1} + \ln\overline{\mathrm{J}}(\phi)(x^i). \end{aligned}$$

Then the Jacobi matrix of transition functions on $\mathcal{V}^2$ is given by

$$\frac{\partial x^{\alpha'}}{\partial x^\beta} = \begin{bmatrix} 1 & \dfrac{\partial\overline{\mathrm{J}}}{\partial x^j} & 0 \\ 0 & \dfrac{\partial x^{i'}}{\partial x^j} & 0 \\ 0 & \dfrac{\partial\overline{\mathrm{J}}}{\partial x^j} & 1 \end{bmatrix}.$$

2. The horizontal lift of a tensor field of type $(1,1)$ and the horizontal lift of an almost complex structure

At the beginning of this chapter, we are going to consider the horizontal lift of a tensor field of type $(1,1)$. Let (F^i_j) be components of a tensor F on the manifold M in the local coordinate system (U, x^i). For these tensor we have the following theorem.

Theorem 2.1. *Let $F = (F^i_j)$ be a tensor of type $(1,1)$ on the manifold M and let (Γ^k_{ij}) be components of a linear connection ∇ on M. Then*

$$\bar{F} = \begin{bmatrix} 0 & \Gamma^k_{jk} - \Gamma^k_{tk}F^t_j & 1 \\ 0 & F^i_j & 0 \\ -1 & -\Gamma^k_{jk} - \Gamma^k_{tk}F^t_j & 0 \end{bmatrix},$$

define a tensor of type $(1,1)$ *on the bundle of pairs of volume forms* $\mathcal{V}^2$. *The tensor* $\bar{F}$ *will be called a horizontal lift of* F.

Proof. We are going to check that the geometric object $F = (F_i^j)$ satisfies the transformation rule of the tensor of type $(1,1)$. We have

$$\begin{aligned}
&\left[\frac{\partial x^{\alpha'}}{\partial x^\gamma}\right]\left[\bar{F}_\delta^\gamma\right]\left[\frac{\partial x^\delta}{\partial x^{\beta'}}\right] \\
&= \begin{bmatrix} 0 & \Gamma_{ik}^k \dfrac{\partial x^i}{\partial x^{p'}} - \dfrac{\partial x^i}{\partial x^{p'}}\Gamma_{tk}^k F_i^t + \dfrac{\partial x^i}{\partial x^{p'}}\dfrac{\partial \bar{\mathrm{J}}}{\partial x^t}F_i^t - \dfrac{\partial \bar{\mathrm{J}}}{\partial x^i}\dfrac{\partial x^i}{\partial x^{p'}} & 1 \\ 0 & \dfrac{\partial x^i}{\partial x^{p'}}\dfrac{x^{i'}}{\partial x^t}F_i^t & 0 \\ -1 & \dfrac{\partial \bar{\mathrm{J}}}{\partial x^i}\dfrac{\partial x^i}{\partial x^{p'}} + \dfrac{\partial \bar{\mathrm{J}}}{\partial x^t}F_i^t\dfrac{\partial x^i}{\partial x^{p'}} - \dfrac{\partial x^i}{\partial x^{p'}}\Gamma_{ik}^k - \dfrac{\partial x^i}{\partial x^{p'}}\Gamma_{tk}^k F_i^t & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \Gamma_{p'k'}^{k'} - F_{p'}^{b'}\Gamma_{b'k'}^{k'} & 1 \\ 0 & F_{p'}^{i'} & 0 \\ -1 & -\Gamma_{p'k'}^{k'} - F_{p'}^{b'}\Gamma_{b'k'}^{k'} & 0 \end{bmatrix} = \left[\bar{F}_{\beta'}^{\alpha'}\right].
\end{aligned}$$

□

In the next part of this chapter we study some properties of the horizontally lifted tensor field $\bar{F}$. First, we show that horizontal lift of an almost complex structure is an almost complex structure on the bundle of pairs of volume forms.

Theorem 2.2. *Let* $F = (F_i^j)$ *be an almost complex structure on the manifold* M. *Then horizontally lifted tensor field* $\bar{F}$ *is an almost complex structure on the bundle of pairs of volume forms* $\mathcal{V}^2$.

Proof. We have

$$\bar{F}^2 = \begin{bmatrix} -1 & \Gamma_{jk}^k F_i^j - \Gamma_{tk}^k F_j^t F_i^j - \Gamma_{ik}^k - \Gamma_{tk}^k F_i^t & 0 \\ 0 & F_j^i F_i^j & 0 \\ 0 & -\Gamma_{ik}^k + \Gamma_{tk}^k F_i^t - \Gamma_{jk}^k F_i^j - \Gamma_{tk}^k F_j^t F_i^j & -1 \end{bmatrix} = -\,\mathrm{I}_{\mathcal{V}^2}\,.$$

Thus $\bar{F}$ is an almost complex structure on $\mathcal{V}^2$. □

Now, we give the conditions under which the horizontally lifted almost complex structure is integrable. First we recall the definition of a Nijenhuis tensor.

Definition 2.1 ([3])**.** Let F be a tensor field of type $(1,1)$ and let X, Y be the vector fields on a manifold M. Then

$$N(X,Y) = F(FX,FY) - F(X,Y) - F(X,FY) - F(FX,Y)$$

is said to be a Nijenhuis tensor of the tensor F on the manifold M.

We need the following lemma to prove a next theorem.

Lemma 2.1. *Let (F_i^j) be components of a tensor field F of type $(1,1)$ on the manifold M. Let (Γ_{ij}^k) be components of the connection ∇ on M. Then nonzero components $(\bar{N}_{\alpha\beta}^{\gamma})$ of the Nijenhuis tensor $\bar{N}$ of the horizontally lifted tensor $\bar{F}$ are given by*

$$\bar{N}_{ij}^0 = \bar{N}_{ij}^{n+1} = \left[\Gamma_{dt|r}^t - \Gamma_{rt|d}^t\right]\left[-F_i^d\delta_j^r + F_i^d F_j^r + F_j^d\delta_i^r + \delta_j^d\delta_i^r\right],$$

$$\bar{N}_{ij}^k = N_{ij}^k,$$

where (N_{ij}^k) are components of the Nijenhuis tensor of F.

We recall ([2]) that the linear connection ∇ is locally volume preserving at $p \in M$ if there exists an neighbourhood U of p and a volume form ω on U such that $\nabla\omega = 0$. The integrability conditions of the existence of such a volume form in local coordinates (x^i) on a neighbourhood $p \in M$ are given by

$$\Gamma_{jk|i}^k - \Gamma_{ik|j}^k = 0,$$

where the (Γ_{ij}^k) are the coefficients of ∇ ([6]). The connection is called globally volume preserving if such a volume forme exists on the manifold M.

For a locally volume preserving connection we have

Theorem 2.3. *Let ∇ be a symmetric, locally volume preserving connection on the manifold M. Then the almost complex structure F on M is integrable if and only if the horizontally lifted almost complex structure $\bar{F}$ is integrable on the bundle of pairs of volume forms $\mathcal{V}^2$.*

Proof. Let the almost complex structure F be integrable one on the manifold M. Then the Nijenhuis tensor N of the tensor F vanishes. From the assumption that ∇ is the volume preserving connection and from lemma 2.1 we have that the Nijenhuis tensor $\bar{N}$ of the horizontally lifted

tensor $\bar{F}$ vanishes, too. So, we obtain that the horizontally lifted almost complex structure $\bar{F}$ is integrable on the bundle of pairs of volume forms $\mathcal{V}^2$.

From the other hand let $\bar{F}$ be the integrable almost complex structure on $\mathcal{V}^2$. Then $\bar{N}$ is the zero tensor and from lemma 2.1 we get

$$0 = \bar{N}^0_{ij} = N^k_{ij}$$

hence F is the integrable almost complex structure on the manifold M. □

3. Horizontal lift of some geometric objects to the bundle of pairs of volume forms $\mathcal{V}^2$

Mozgawa and Miernowski in ([4]) gave the horizontal lifts of vector fields, tensor field of type $(0,2)$ and $(2,0)$ and Riemannian metrics to the bundle of volume forms. Now, we are going to give formula for horizontal lifts of vector field, tensor fields of type $(0,2)$ and $(2,0)$ and Riemannian metrics to the bundle of pairs of volume forms $\mathcal{V}^2$. At the beginning we consider the problem of horizontal lift of vector fields to $\mathcal{V}^2$.

Theorem 3.1. *Let $v = v^i \frac{\partial}{\partial x^i}$ be a vector field on the manifold M and let $\nabla = (\Gamma^k_{ij})$ be a linear symmetric connection on the manifold M. Then*

$$\bar{v} = -v^i \Gamma^k_{ik} \frac{\partial}{\partial x^0} + v^i \frac{\partial}{\partial x^i} - v^i \Gamma^k_{ik} \frac{\partial}{\partial x^{n+1}},$$

is globally defined a vector field on $\mathcal{V}^2$ called a horizontal lift of the vector field on the bundle of pairs of volume forms $\mathcal{V}^2$.

Proof. We check the transformation rule of the vector fields for the components $(\bar{v}^\alpha)$ of the object $\bar{v}$. We have

$$\begin{aligned}
\bar{v}^{0'} &= -v^i \Gamma^k_{ik} + \frac{\partial \bar{\mathrm{J}}}{\partial x^d} v^d = -v^{d'} \Gamma^{t'}_{d't'}, \\
\bar{v}^{k'} &= \frac{\partial x^{k'}}{\partial x^d} v^d = v^{k'}, \\
\bar{v}^{(n+1)'} &= -v^i \Gamma^k_{ik} + \frac{\partial \bar{\mathrm{J}}}{\partial x^d} v^d = -v^{d'} \Gamma^{t'}_{d't'}.
\end{aligned}$$

Hence $\bar{v}$ is the vector field on the bundle $\mathcal{V}^2$. □

Now, we are going to describe a horizontal lift of Riemannian metric to the bundle of pairs of volume forms.

Theorem 3.2. *Let (Γ^k_{ij}) be the coefficients of the symmetric connection ∇ and let (g_{ij}) be coefficients of a tensor g of type $(0,2)$ on M. Then*

$$\bar{g} = \begin{bmatrix} 1 & \Gamma^k_{ik} & 0 \\ \Gamma^k_{jk} & g_{ij} + 2\Gamma^k_{ik}\Gamma^t_{jt} & \Gamma^k_{jk} \\ 0 & \Gamma^k_{ik} & 1 \end{bmatrix}$$

is globally defined $(0,2)$-tensor on $\mathcal{V}^2$ called the horizontal lift of the tensor of type $(0,2)$ to the bundle $\mathcal{V}^2$.

Proof. It is sufficient to check the transformation for the object $\bar{g}$. For the nonzero coefficients $(\bar{g}_{\alpha\beta})$ we have

$$\begin{aligned}
\bar{g}_{0'0'} &= \frac{\partial x^0}{\partial x^{0'}}\frac{\partial x^0}{\partial x^{0'}}\bar{g}_{00} = 1,\\
\bar{g}_{i'0'} &= \frac{\partial \mathrm{J}}{\partial x^{i'}} + \frac{\partial x^d}{\partial x^{i'}}\Gamma^k_{dk} = \frac{\partial \mathrm{J}}{\partial x^{i'}} + \frac{\partial x^d}{\partial x^{i'}}\left[\frac{\partial x^{r'}}{\partial x^d}\Gamma^{t'}_{r't'} + \frac{\partial \bar{\mathrm{J}}}{\partial x^d}\right] = \Gamma^{t'}_{i't'},\\
\bar{g}_{i'(n+1)'} &= \frac{\partial x^d}{\partial x^{i'}}\Gamma^k_{dk} + \frac{\partial x^{n+1}}{\partial x^{i'}} = \frac{\partial x^d}{\partial x^{i'}}\left[\frac{\partial x^{r'}}{\partial x^d}\Gamma^{t'}_{r't'} + \frac{\partial \bar{\mathrm{J}}}{\partial x^d}\right] + \frac{\partial \mathrm{J}}{\partial x^{i'}} = \Gamma^{t'}_{i't'},\\
\bar{g}_{i'j'} &= 2\frac{\partial \mathrm{J}}{\partial x^{i'}}\frac{\partial \mathrm{J}}{\partial x^{j'}} + 2\frac{\partial \mathrm{J}}{\partial x^{i'}}\frac{\partial x^d}{\partial x^{j'}}\Gamma^t_{dt} + 2\frac{\partial x^d}{\partial x^{i'}}\frac{\partial \mathrm{J}}{\partial x^{j'}}\Gamma^t_{dt}\\
&\quad + \frac{\partial x^d}{\partial x^{i'}}\frac{\partial x^k}{\partial x^{j'}}(g_{dk} + 2\Gamma^t_{dt}\Gamma^s_{ks})\\
&= 2\frac{\partial \mathrm{J}}{\partial x^{i'}}\frac{\partial \mathrm{J}}{\partial x^{j'}} + 2\frac{\partial \mathrm{J}}{\partial x^{i'}}\Gamma^{t'}_{j't'} - 2\frac{\partial \mathrm{J}}{\partial x^{i'}}\frac{\partial \mathrm{J}}{\partial x^{j'}}\\
&\quad + 2\frac{\partial \mathrm{J}}{\partial x^{j'}}\Gamma^{t'}_{i't'} - 2\frac{\partial \mathrm{J}}{\partial x^{j'}}\frac{\partial \mathrm{J}}{\partial x^{i'}} + g_{i'j'}\\
&\quad + 2\left[\Gamma^{t'}_{i't'} - \frac{\partial \mathrm{J}}{\partial x^{i'}}\right]\left[\Gamma^{s'}_{j's'} - \frac{\partial \bar{\mathrm{J}}}{\partial x^{j'}}\right] = g_{i'j'} + 2\Gamma^{t'}_{i't'}\Gamma^{s'}_{j's'},
\end{aligned}$$

so $\bar{g}$ is the tensor field of type $(0,2)$ on the bundle of volume forms $\mathcal{V}^2$. □

The horizontally lifted tensor field of type $(0,2)$ has the following property.

Corollary 3.1. *Let ∇ be a symmetric connection and let g be a positively defined tensor of type $(0,2)$ on the manifold M. Then the horizontal lift of the tensor g to the bundle of pairs of volume forms $\mathcal{V}^2$ is the positively defined tensor of type $(0,2)$ on $\mathcal{V}^2$.*

In the next part of this chapter we give formulas for the horizontal lift of a tensor fields of the type $(2,0)$ and a 1-form to the bundle of pairs of volume forms $\mathcal{V}^2$. We have the following theorem.

Theorem 3.3. *Let $h=(h^{ij})$ be a tensor of type $(2,0)$ on the manifold M. Then the geometric object*

$$\bar{h}=\begin{bmatrix} h^{ij}\Gamma^k_{ik}\Gamma^t_{jt} & -h^{ij}\Gamma^k_{jk} & h^{ij}\Gamma^k_{ik}\Gamma^t_{jt} \\ -h^{ij}\Gamma^k_{ik} & h^{ij} & -h^{ij}\Gamma^k_{ik} \\ h^{ij}\Gamma^k_{ik}\Gamma^t_{jt} & -h^{ij}\Gamma^t_{jt} & h^{ij}\Gamma^k_{ik}\Gamma^t_{jt}, \end{bmatrix}$$

is globally defined $(2,0)$-tensor on the bundle of pairs of volume forms $\mathcal{V}^2$. The tensor $\bar{h}$ is called the horizontal lift of tensor h to the bundle $\mathcal{V}^2$.

Proof. Let $(\bar{h}^{\alpha\beta})$ be the coefficients of the object $\bar{h}$. We show that the transformation rule of the tensor fields of the type $(2,0)$ follows for $\bar{h}$. For the coefficients $\bar{h}^{0\alpha}$ we have

$$\begin{aligned}
\frac{\partial x^{0'}}{\partial x^\alpha}\frac{\partial x^{0'}}{\partial x^\beta}\bar{h}^{\alpha\beta} &= \bar{h}^{00}+\frac{\partial\bar{\mathrm{J}}}{\partial x^d}\bar{h}^{0d}+\frac{\partial\bar{\mathrm{J}}}{\partial x^d}\bar{h}^{d0}+\frac{\partial\bar{\mathrm{J}}}{\partial x^d}\frac{\partial\bar{\mathrm{J}}}{\partial x^r}\bar{h}^{dr}\\
&= h^{ij}\Gamma^t_{it}\Gamma^k_{jk}-2\frac{\partial\bar{\mathrm{J}}}{\partial x^d}h^{dj}\Gamma^t_{jt}+\frac{\partial\bar{\mathrm{J}}}{\partial x^d}\frac{\partial\bar{\mathrm{J}}}{\partial x^r}h^{dr}\\
&= h^{p'd'}\Gamma^{t'}_{p't'}\Gamma^{k'}_{d'k'}-\frac{\partial\,\mathrm{J}}{\partial x^{d'}}h^{p'd'}\Gamma^{t'}_{p't'}-\frac{\partial\,\mathrm{J}}{\partial x^{p'}}h^{p'd'}\Gamma^{t'}_{d't'}\\
&+\frac{\partial\,\mathrm{J}}{\partial x^{p'}}\frac{\partial\,\mathrm{J}}{\partial x^{d'}}h^{p'd'}+2\frac{\partial\,\mathrm{J}}{\partial x^{p'}}h^{p'd'}\Gamma^{t'}_{d't'}-\frac{\partial\,\mathrm{J}}{\partial x^{p'}}\frac{\partial\,\mathrm{J}}{\partial x^{d'}}h^{p'd'}\\
&= h^{p'd'}\Gamma^{t'}_{p't'}\Gamma^{k'}_{d'k'}=\bar{h}^{0'0'},\\
\frac{\partial x^{i'}}{\partial x^\alpha}\frac{\partial x^{0'}}{\partial x^\beta}\bar{h}^{\alpha\beta} &= \frac{\partial x^{i'}}{\partial x^d}\bar{h}^{d0}+\frac{\partial x^{i'}}{\partial x^d}\frac{\partial\bar{\mathrm{J}}}{\partial x^r}\bar{h}^{dr}=-\frac{\partial x^{i'}}{\partial x^d}h^{dj}\Gamma^t_{jt}+\frac{\partial x^{i'}}{\partial x^d}\frac{\partial\bar{\mathrm{J}}}{\partial x^r}h^{dr}\\
&= -h^{i'b'}\Gamma^{t'}_{b't'}+\frac{\partial\,\mathrm{J}}{\partial x^{b'}}h^{i'b'}-\frac{\partial\,\mathrm{J}}{\partial x^{b'}}h^{i'b'}=-h^{i'b'}\Gamma^{t'}_{b't'}=\bar{h}^{i'0'},\\
\frac{\partial x^{(n+1)'}}{\partial x^\alpha}\frac{\partial x^{0'}}{\partial x^\beta}\bar{h}^{\alpha\beta} &= \frac{\partial\bar{\mathrm{J}}}{\partial x^d}\frac{\partial x^{0'}}{\partial x^\beta}\bar{h}^{d\beta}+\frac{\partial x^{0'}}{\partial x^\beta}\bar{h}^{n+1\beta}=-2\frac{\partial\bar{\mathrm{J}}}{\partial x^d}h^{dj}\Gamma^t_{jt}\\
&+\frac{\partial\bar{\mathrm{J}}}{\partial x^d}\frac{\partial\bar{\mathrm{J}}}{\partial x^r}h^{dr}+h^{ij}\Gamma^t_{it}\Gamma^k_{jk}=2\frac{\partial\,\mathrm{J}}{\partial x^{a'}}h^{a'b'}\Gamma^{t'}_{b't'}\\
&-2\frac{\partial\,\mathrm{J}}{\partial x^{a'}}h^{a'b'}\frac{\partial\,\mathrm{J}}{\partial x^{b'}}+h^{a'b'}\Gamma^{t'}_{a't'}\Gamma^{k'}_{b'k'}-2\frac{\partial\,\mathrm{J}}{\partial x^{b'}}h^{a'b'}\Gamma^{t'}_{a't'}\\
&+\frac{\partial\,\mathrm{J}}{\partial x^{a'}}\frac{\partial\,\mathrm{J}}{\partial x^{b'}}h^{a'b'}+\frac{\partial\,\mathrm{J}}{\partial x^{a'}}\frac{\partial\,\mathrm{J}}{\partial x^{b'}}h^{a'b'}\\
&= h^{a'b'}\Gamma^{t'}_{a't'}\Gamma^{k'}_{b'k'}=\bar{h}^{(n+1)'0'}.
\end{aligned}$$

For the components $\bar{h}^{i'j'}$ and $\bar{h}^{(n+1)'j'}$ we get

$$\begin{aligned}\frac{\partial x^{i'}}{\partial x^{\alpha}}\frac{\partial x^{j'}}{\partial x^{\beta}}\bar{h}^{\alpha\beta} &= \frac{\partial x^{i'}}{\partial x^{d}}\frac{\partial x^{j'}}{\partial x^{r}}\bar{h}^{dr} = h^{i'j'} = \bar{h}^{i'j'},\\ \frac{\partial x^{(n+1)'}}{\partial x^{\alpha}}\frac{\partial x^{j'}}{\partial x^{\beta}}\bar{h}^{\alpha\beta} &= \frac{\partial \bar{\mathrm{J}}}{\partial x^{d}}\frac{\partial x^{j'}}{\partial x^{r}}\bar{h}^{dr} + \frac{\partial x^{j'}}{\partial x^{d}}\bar{h}^{n+1,d} = \frac{\partial \bar{\mathrm{J}}}{\partial x^{d}}\frac{\partial x^{j'}}{\partial x^{r}}h^{dr}\\ &\quad - \frac{\partial x^{j'}}{\partial x^{d}}h^{rd}\Gamma^{t}_{rt} = -h^{a'j'}\Gamma^{t'}_{a't'} = \bar{h}^{(n+1)'j'}.\end{aligned}$$

At the end we have

$$\begin{aligned}&\frac{\partial x^{(n+1)'}}{\partial x^{\alpha}}\frac{\partial x^{(n+1)'}}{\partial x^{\beta}}\bar{h}^{\alpha\beta} = \frac{\partial \bar{\mathrm{J}}}{\partial x^{d}}\frac{\partial \bar{\mathrm{J}}}{\partial x^{r}}\bar{h}^{dr} + \frac{\partial \bar{\mathrm{J}}}{\partial x^{d}}\bar{h}^{d,n+1} + \frac{\partial \bar{\mathrm{J}}}{\partial x^{d}}\bar{h}^{n+1,d} + \bar{h}^{n+1,n+1}\\ &= \frac{\partial \mathrm{J}}{\partial x^{a'}}\frac{\partial \mathrm{J}}{\partial x^{b'}}h^{a'b'} + \frac{\partial \mathrm{J}}{\partial x^{a'}}h^{a'b'}\left(\Gamma^{t'}_{b't'} - \frac{\partial \mathrm{J}}{\partial x^{b'}}\right)\\ &+ \frac{\partial \mathrm{J}}{\partial x^{b'}}h^{a'b'}\left(\Gamma^{t'}_{a't'} - \frac{\partial \mathrm{J}}{\partial x^{a'}}\right) + h^{a'b'}\left(\Gamma^{t'}_{a't'} - \frac{\partial \mathrm{J}}{\partial x^{a'}}\right)\left(\Gamma^{k'}_{b'k'} - \frac{\partial \mathrm{J}}{\partial x^{b'}}\right)\\ &= h^{a'b'}\Gamma^{t'}_{a't'}\Gamma^{k'}_{b'k'} = \bar{h}^{(n+1)'(n+1)'}.\end{aligned}$$

Hence, $\bar{h}$ is the tensor field of type $(2,0)$ on the bundle $\mathcal{V}^2$. □

Now, we will examine a problem of a horizontal lift of a 1-form to the bundle of pairs of volume forms $\mathcal{V}^2$. We have

Theorem 3.4. *Let (Γ^{k}_{ij}) be the coefficients of a symmetric connection ∇ and let (ω_{ij}) be coefficients of a 1-form ω on the manifold M. Then $\bar{\omega} = (\omega_i + 2\Gamma^{k}_{ik})dx^i + dx^0 + dx^{n+1}$ is a 1-form on $\mathcal{V}^2$ which is called the horizontal lift of the 1-form ω to the bundle of pairs of volume forms $\mathcal{V}^2$*

Proof. This is sufficient to check the transformation rule for the 1-form. □

Using theorems 3.3 and 3.4 one can prove that the horizontally lifted tensor field of type $(2,0)$ has the following property.

Corollary 3.2. *Let ∇ be a symmetric connection and let h be a tensor h of type $(2,0)$ on the manifold M. Then the horizontal lift $\bar{h}$ of a tensor h is unique and such that*

1. $\bar{h}(\tau,\tau) = 0$,
2. $\bar{h}(\tau,\bar{\omega}) = \bar{h}(\bar{\omega},\tau) = 0$,

3. $\bar{h}(\bar{\omega}, \bar{\phi}) = (h(\omega, \phi))^V$,

where $\bar{\omega}$, $\bar{\phi}$ are the horizontal lift of the 1-form ω, ϕ to the bundle $\mathcal{V}^2$, $\tau = \bar{0}$ and $(h(\omega, \phi))^V = (h(\omega, \phi)) \circ \pi$, respectively.

From the theorem 3.2 and the corollary 3.1 we obtain that the horizontally lifted tensor field $\bar{g}$ of type $(0, 2)$ is nonsingular and positively defined. By multiplication one can directly check that $(\bar{g})^{-1} \circ \bar{g} = \bar{g} \circ (\bar{g})^{-1} = I_{\mathcal{V}^2}$. Hence we have the following theorem.

Theorem 3.5. *Let g be a Riemannian metric on M. Then $\bar{g}$ is a Riemannian metric on $\mathcal{V}^2$ and*

$$(\bar{g})^{-1} = \overline{(g^{-1})} + \frac{\partial}{\partial x^0} \otimes \frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^{n+1}} \otimes \frac{\partial}{\partial x^{n+1}}.$$

From the above theorem we know that horizontal lift of a Riemannian metric g is a Riemannian metric on the bundle $\mathcal{V}^2$ we can calculate coefficients of the Levi-Civita connection. The proof of next theorem follows from the direct calculations of coefficients of the Levi-Civita connection.

Theorem 3.6. *Let $\nabla = (\Gamma^k_{ij})$ be the symmetric linear connection and let $g = (g_{ij})$ be the Riemannian metric on the manifold M. Then the nonzero coefficients of the Levi-Civita connection $\bar{\tilde{\nabla}}$ for $\bar{g}$ are given by*

$$\begin{aligned}
\bar{\tilde{\nabla}}_0 \frac{\partial}{\partial x^i} &= \bar{\tilde{\nabla}}_i \frac{\partial}{\partial x^0} = \frac{1}{2} g^{dm} \left(\Gamma^k_{dk|i} - \Gamma^k_{ik|d} \right) \frac{\partial}{\partial x^m} \\
&\quad - \frac{1}{2} g^{dt} \Gamma^r_{tr} \left(\Gamma^k_{dk|i} - \Gamma^k_{ik|d} \right) \left(\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^{n+1}} \right), \\
\bar{\tilde{\nabla}}_i \frac{\partial}{\partial x^j} &= \bar{\tilde{\nabla}}_j \frac{\partial}{\partial x^i} = \left[\tilde{\Gamma}^m_{ij} + g^{dm} (\Gamma^t_{dt|a} - \Gamma^t_{at|d})(\Gamma^k_{jk} \delta^a_i + \Gamma^t_{it} \delta^a_j) \right] \frac{\partial}{\partial x^m} \\
&\quad + \left[\frac{1}{2} (\Gamma^k_{jk|i} + \Gamma^k_{ik|j})(1 - 2 g^{dt} \Gamma^s_{ts} \Gamma^k_{dk}) - \Gamma^s_{ts} \tilde{\Gamma}^t_{ij} \right. \\
&\quad \left. - g^{dt} \Gamma^s_{ts} (\Gamma^t_{dt|a} - \Gamma^t_{at|d})(\Gamma^k_{jk} \delta^a_i + \Gamma^k_{ik} \delta^a_j) \right] \left(\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^{n+1}} \right), \\
\bar{\tilde{\nabla}}_i \frac{\partial}{\partial x^{n+1}} &= \bar{\tilde{\nabla}}_{n+1} \frac{\partial}{\partial x^i} = \frac{1}{2} g^{dm} \left(\Gamma^k_{dk|i} - \Gamma^k_{ik|d} \right) \frac{\partial}{\partial x^m} \\
&\quad - \frac{1}{2} g^{dt} \Gamma^t_{tr} \left(\Gamma^k_{dk|i} - \Gamma^k_{ik|d} \right) \left(\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^{n+1}} \right),
\end{aligned}$$

where $(g^{ij}) = (g_{ij})^{-1}$ and $\tilde{\nabla} = (\tilde{\Gamma}^k_{ij})$ is the Levi-Civita connection for the metric g on M.

For the symmetric, volume preserving and linear connection we have

Corollary 3.3. *Let $\nabla = (\Gamma^k_{ij})$ be the symmetric, volume preserving, linear connection and $g = (g_{ij})$ be the Riemannian metric on M. Then the nonzero coefficients of the Levi-Civita connection for the horizontally lifted Riemannian metric $\bar{g}$ are given by*

$$\bar{\tilde{\nabla}}_i\partial_j = \bar{\tilde{\nabla}}_j\partial_i = \left[\Gamma^k_{jk|i}\left(1 - 2g^{dt}\Gamma^s_{ts}\Gamma^k_{dk}\right) - \Gamma^s_{ts}\tilde{\Gamma}^t_{ij}\right]\left(\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^{n+1}}\right) + \tilde{\Gamma}^m_{ij}\frac{\partial}{\partial x^m},$$

where $(g^{ij}) = (g_{ij})^{-1}$ and $\tilde{\nabla} = (\tilde{\Gamma}^k_{ij})$ is the Levi-Civita connection for the metric g on M.

At the end of this chapter we determine a family of a t-connections on the bundle of pairs of volume forms $\mathcal{V}^2$, where $t \in [0, 1]$.

Theorem 3.7. *Let $\nabla = (\Gamma^k_{ij})$ and $\nabla_1 = (\Phi^k_{ij})$ be the symmetric connections on the manifold M. Then the nonzero coefficients of a connection $\bar{\nabla}_1$ are given by the formulas*

$$\begin{aligned}\bar{\nabla}_{1i}\frac{\partial}{\partial x^j} &= \bar{\nabla}_{1j}\frac{\partial}{\partial x^i}\\ &= \left[t(\Gamma^t_{it|j} - \Phi^d_{ij}\Gamma^t_{dt}) + (1-t)(\Gamma^t_{ij|t} - \Gamma^r_{id}\Gamma^d_{jr})\right]\left(\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^{n+1}}\right)\\ &\quad + \left[t\Phi^k_{ij} + (1-t)\Gamma^k_{ij}\right]\frac{\partial}{\partial x^k},\end{aligned}$$

where $t \in [0, 1]$. The connection $\bar{\nabla}_1$ is called the $t-$connection on the bundle of pairs of volume forms $\mathcal{V}^2$.

Proof. We are going to check that the coefficients $(\bar{\Phi}^\sigma_{\delta\varrho})$ of $\bar{\nabla}_1$ satisfy the transformation rule of the connection. The transformation rule for the zero coefficients of the connection $\bar{\nabla}_1$ follow from simple calculation. We check this rule for the nonzero coefficients of the object $(\bar{\Phi}^\sigma_{\delta\varrho})$. Now, we consider the coefficients $\bar{\Phi}^{0'}_{i'j'}$

$$\begin{aligned}\bar{\Phi}^{0'}_{i'j'} &= \frac{\partial x^d}{\partial x^{i'}}\frac{\partial x^a}{\partial x^{j'}}\left[t\left(\Gamma^k_{da|a} - \Phi^r_{da}\Gamma^k_{rk}\right) + (1-t)\left(\Gamma^k_{da|k} - \Gamma^r_{dk}\Gamma^k_{ar}\right)\right]\\ &\quad + \frac{\partial x^d}{\partial x^{i'}}\frac{\partial x^a}{\partial x^{j'}}\frac{\partial \bar{\mathrm{J}}}{\partial x^b}\left[t\Phi^b_{da} + (1-t)\Gamma^b_{da}\right] + \frac{\partial^2 \mathrm{J}}{\partial x^{i'}\partial x^{j'}} + \frac{\partial \bar{\mathrm{J}}}{\partial x^d}\frac{\partial^2 x^d}{\partial x^{i'}\partial x^{j'}}\end{aligned}$$

$$
\begin{aligned}
&+\frac{\partial^2 \mathrm{J}}{\partial x^{i'}\partial x^{j'}}+\frac{\partial \bar{\mathrm{J}}}{\partial x^d}\frac{\partial^2 x^d}{\partial x^{i'}\partial x^{j'}}\\
&=t\left[\Gamma^{k'}_{i'k'|j'}-\Phi^{m'}_{i'j'}\Gamma^{k'}_{m'k'}\right]+(1-t)\left[\Gamma^{s'}_{i'j'|s'}-\Gamma^{m'}_{i'p'}\Gamma^{p'}_{j'm'}\right]\\
&+t\frac{\partial^2\bar{\mathrm{J}}}{\partial x^d\partial x^a}\frac{\partial x^d}{\partial x^{i'}}\frac{\partial x^a}{\partial x^{j'}}(1-t)\frac{\partial^3 x^{c'}}{\partial x^d\partial x^a\partial x^k}\frac{\partial x^k}{\partial x^{c'}}\frac{\partial x^a}{\partial x^{j'}}\frac{\partial x^d}{\partial x^{i'}}\\
&-(1-t)\frac{\partial^2 x^{c'}}{\partial x^d\partial x^k}\frac{\partial x^r}{\partial x^{c'}}\frac{\partial x^d}{\partial x^{i'}}\frac{\partial x^k}{\partial x^{s'}}\frac{\partial^2 x^{s'}}{\partial x^a\partial x^r}\frac{\partial x^a}{\partial x^{j'}}+\frac{\partial^2 \mathrm{J}}{\partial x^{i'}\partial x^{j'}}+\frac{\partial\bar{\mathrm{J}}}{\partial x^d}\frac{\partial^2 x^d}{\partial x^{i'}\partial x^{j'}}\\
&=t\left[\Gamma^{k'}_{i'k'|j'}-\Phi^{m'}_{i'j'}\Gamma^{k'}_{m'k'}\right]+(1-t)\left[\Gamma^{s'}_{i'j'|s'}-\Gamma^{m'}_{i'p'}\Gamma^{p'}_{j'm'}\right]+T_{i'j'},
\end{aligned}
$$

where

$$
\begin{aligned}
T_{i'j'}=&\,t\frac{\partial^2\bar{\mathrm{J}}}{\partial x^d\partial x^a}\frac{\partial x^d}{\partial x^{i'}}\frac{\partial x^a}{\partial x^{j'}}(1-t)\frac{\partial^3 x^{c'}}{\partial x^d\partial x^a\partial x^k}\frac{\partial x^k}{\partial x^{c'}}\frac{\partial x^a}{\partial x^{j'}}\frac{\partial x^d}{\partial x^{i'}}\\
&-(1-t)\frac{\partial^2 x^{c'}}{\partial x^d\partial x^k}\frac{\partial x^r}{\partial x^{c'}}\frac{\partial x^d}{\partial x^{i'}}\frac{\partial x^k}{\partial x^{s'}}\frac{\partial^2 x^{s'}}{\partial x^a\partial x^r}\frac{\partial x^a}{\partial x^{j'}}+\frac{\partial^2 \mathrm{J}}{\partial x^{i'}\partial x^{j'}}+\frac{\partial\bar{\mathrm{J}}}{\partial x^d}\frac{\partial^2 x^d}{\partial x^{i'}\partial x^{j'}}.
\end{aligned}
$$

In the next part of the prove we use the following equality.

$$
\begin{aligned}
&\frac{\partial^2 x^{c'}}{\partial x^d\partial x^k}\frac{\partial x^r}{\partial x^{c'}}=-\frac{\partial x^{c'}}{\partial x^k}\frac{\partial^2 x^r}{\partial x^{c'}\partial x^{v'}}\frac{\partial x^{v'}}{\partial x^d},\\
&\frac{\partial^2 x^{c'}}{\partial x^d\partial x^k}\frac{\partial x^r}{\partial x^{c'}}\frac{\partial x^d}{\partial x^{i'}}\frac{\partial x^k}{\partial x^{s'}}\frac{\partial^2 x^{s'}}{\partial x^a\partial x^r}\frac{\partial x^a}{\partial x^{j'}}\\
&\qquad=-\frac{\partial x^{c'}}{\partial x^k}\frac{\partial^2 x^r}{\partial x^{c'}\partial x^{v'}}\frac{\partial x^{v'}}{\partial x^d}\frac{\partial x^d}{\partial x^{i'}}\frac{\partial x^k}{\partial x^{s'}}\frac{\partial^2 x^{s'}}{\partial x^a\partial x^r}\frac{\partial x^a}{\partial x^{j'}}\\
&\qquad=-\frac{\partial^2 x^r}{\partial x^{c'}\partial x^{i'}}\frac{\partial^2 x^{c'}}{\partial x^a\partial x^r}\frac{\partial x^{a\,\prime}}{\partial x^{j'}}\\
&\frac{\partial^2 x^{r'}}{\partial x^a\partial x^b}\frac{\partial x^a}{\partial x^{i'}}=-\frac{\partial x^{r'}}{\partial x^a}\frac{\partial^2 x^a}{\partial x^{i'}\partial x^{s'}}\frac{\partial x^{s'}}{\partial x^b},\\
&\frac{\partial^2 x^{r'}}{\partial x^a\partial x^b}\frac{\partial x^a}{\partial x^{i'}}\frac{\partial \mathrm{J}}{\partial x^{r'}}\frac{\partial x^b}{\partial x^{j'}}=-\frac{\partial x^{r'}}{\partial x^a}\frac{\partial^2 x^a}{\partial x^{i'}\partial x^{s'}}\frac{\partial x^{s'}}{\partial x^b}\frac{\partial \mathrm{J}}{\partial x^{r'}}\frac{\partial x^b}{\partial x^{j'}}=\frac{\partial\bar{\mathrm{J}}}{\partial x^a}\frac{\partial^2 x^a}{\partial x^{i'}\partial x^{j'}}.
\end{aligned}
$$

We obtain from above identity

$$
\bar{\Phi}^{0'}_{i'j'}=t\left[\Gamma^{k'}_{i'k'|j'}-\Phi^{m'}_{i'j'}\Gamma^{k'}_{m'k'}\right]+(1-t)\left[\Gamma^{s'}_{i'j'|s'}-\Gamma^{m'}_{i'p'}\Gamma^{p'}_{j'm'}\right].
$$

Now, we observe that

$$
\bar{\Phi}^{n+1}_{ij}=\bar{\Phi}^{0}_{ij}
$$

and

$$\frac{\partial^2 x^{n+1}}{\partial x^{i'}\partial x^{j'}} = \frac{\partial^2 x^0}{\partial x^{i'}\partial x^{j'}},$$

so we have

$$\bar{\Phi}^{(n+1)'}_{i'j'} = t\left(\Gamma^{t'}_{i't'|j'} - \Phi^{c'}_{i'j'}\Gamma^{t'}_{c't'}\right) + (1-t)\left(\Gamma^{p'}_{i'j'|p'} - \Gamma^{c'}_{i'b'}\Gamma^{b'}_{j'c'c}\right).$$

At the end we check the transformation rule for the coefficients $\bar{\Phi}^{k'}_{i'j'}$, and we get

$$\begin{aligned}
\bar{\Phi}^{k'}_{i'j'} &= \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^{k'}}{\partial x^k}\bar{\Phi}^k_{ij} + \frac{\partial x^{k'}}{\partial x^d}\frac{\partial^2 x^d}{\partial x^{i'}x^{j'}} \\
&= \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^{k'}}{\partial x^k}\left(t\Phi^k_{ij} + (1-t)\Gamma^k_{ij}\right) + [t+(1-t)]\frac{\partial x^{k'}}{\partial x^d}\frac{\partial^2 x^d}{\partial x^{i'}x^{j'}} \\
&= t\left[\frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^{k'}}{\partial x^k}\Phi^k_{ij} + \frac{\partial x^{k'}}{\partial x^d}\frac{\partial^2 x^d}{\partial x^{i'}x^{j'}}\right] \\
&+ (1-t)\left[\frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^{k'}}{\partial x^k}\Gamma^k_{ij} \;+\frac{\partial x^{k'}}{\partial x^d}\frac{\partial^2 x^d}{\partial x^{i'}x^{j'}}\right] = t\Phi^{k'}_{i'j'} + (1-t)\Gamma^{k'}_{i'j'}.
\end{aligned}$$

We proved that $\bar{\nabla}_1$ satisfy the transformation rule of the connection, so $\bar{\nabla}_1$ is the connection on the bundle of pairs of volume forms $\mathcal{V}^2$. □

4. Some properties of a horizontally lifted vector field on the bundle of pairs of volume forms $\mathcal{V}^2$

In this chapter we give some properties of the horizontally lifted vector fields $\bar{X}$ on the bundle of pairs of volume forms $\mathcal{V}^2$. Let us consider under which assumptions the horizontally lifted vector field $\bar{X}$ is a Killing field on the bundle of pairs of volume forms $\mathcal{V}^2$. We have

Theorem 4.1. *Let ∇ be the symmetric, locally volume preserving connection on the Riemannian manifold (M, g). Then the horizontally lifted vector field $\bar{X}$ is the Killing field on the Riemannian manifold $(\mathcal{V}^2, \bar{g})$ if and only if the vector field X is the Killing field on the Riemannian manifold (M, g).*

Proof. Let X be the Killing field on the Riemannian manifold (M, g). Then we have

$$\mathcal{L}_X g_{ij} = 0,$$

where $\mathcal{L}$ is a Lie derivative ([3]). We have to show that

$$\mathcal{L}_{\bar{X}}\bar{g}_{\alpha\beta} = 0.$$

The coefficients of the Lie derivative of the horizontally lifted Riemannian metric $\bar{g}$ are the following

$$\begin{aligned}
\mathcal{L}_{\bar{X}}\bar{g}_{00} &= \bar{X}^{\alpha}\bar{g}_{00|\alpha} + \bar{g}_{\alpha 0}\bar{X}^{\alpha}_{|0} + \bar{g}_{0\alpha}\bar{X}^{\alpha}_{|0} = 0,\\
\mathcal{L}_{\bar{X}}\bar{g}_{b0} &= X^{d}\left(\Gamma^{t}_{bt|d} - \Gamma^{t}_{dt|b}\right) = 0,\\
\mathcal{L}_{\bar{X}}\bar{g}_{n+1,0} &= X^{d}\left(\Gamma^{t}_{bt|d} - \Gamma^{t}_{dt|b}\right) = 0,\\
\mathcal{L}_{\bar{X}}\bar{g}_{n+1,n+1} &= \bar{X}^{\alpha}\bar{g}_{n+1,n+1|\alpha} + \bar{g}_{\alpha,n+1}\bar{X}^{\alpha}_{|n+1} + \bar{g}_{n+1\alpha}\bar{X}^{\alpha}_{|n+1} = 0,\\
\mathcal{L}_{\bar{X}}\bar{g}_{bc} &= X^{d}g_{bc|d} + g_{dc}X^{d}_{|b} + g_{bd}X^{d}_{|c}\\
&\quad + 2X^{d}\Gamma^{t}_{ct}\left(\Gamma^{k}_{bk|d} - \Gamma^{k}_{dk|b}\right) + 2X^{d}\Gamma^{k}_{bk}\left(\Gamma^{t}_{ct|d} - \Gamma^{t}_{dt|c}\right) = \mathcal{L}_{X}g_{bc}.
\end{aligned}$$

Using the assumptions we get that all coefficients of $\mathcal{L}_{\bar{X}}\bar{g}$ are equal to zero, so $\bar{X}$ is a Killing filed on $\mathcal{V}^2$.

Let $\bar{X}$ be the Killing filed on the bundle of pairs of volume forms $\mathcal{V}^2$. Then we have $\mathcal{L}_{\bar{X}}\bar{g} = 0$ and from the firs part of the proof we get $0 = \mathcal{L}_{\bar{X}}\bar{g}_{bc} = \mathcal{L}_{X}g_{bc}$, so X is the Killing filed on the manifold (M, g). □

Yamauchi in ([8]) studied certain types of an infinitesimal transformations on a tangent bundle. Now, we show that the horizontally lifted vector field $\bar{X}$ is an infinitesimal fibre-preserving transformation on the bundle of pairs of volume forms $\mathcal{V}^2$.

Theorem 4.2. *Let X be the vector filed on the manifold M and let $\bar{X}$ be the horizontal lift of X to the bundle of pairs of volume forms $\mathcal{V}^2$. Then $\bar{X}$ is the infinitesimal fibre-preserving transformations on $\mathcal{V}^2$.*

Proof. Since the fibres of the bundle $\mathcal{V}^2$ form a trivial foliation we have ([5]) that the horizontally lifted vector field is the fibre-preserving infinitesimal transformation on $\mathcal{V}^2$ if and only if the coordinates $\bar{X}^1, \ldots, \bar{X}^n$ of the vector field $\bar{X} = \bar{X}^0\frac{\partial}{\partial x^0} + \bar{X}^i\frac{\partial}{\partial x^i} + \bar{X}^{n+1}\frac{\partial}{\partial x^{n+1}}$ depend only of the coordinates $(x^1, x^2, \ldots, x^n)$. From the above and theorem 3.1 we get that the horizontally lifted vector field $\bar{X}$ is the fibre-preserving infinitesimal transformation on the bundle $\mathcal{V}$. □

In the next part of this chapter, we prove that the horizontally lifted vector field $\bar{X}$ is an infinitesimal affine transformation of the t-connection

$\bar{\nabla}_1$ if $t = 1$ and X is an infinitesimal affine transformation of the connection ∇_1 on the manifold M. We have the following theorem.

Theorem 4.3. *Let ∇ be a locally volume preserving, symmetric connection, ∇_1 be a linear connection and let X be a vector field on the manifold M. If $\bar{\nabla}_1$ is a t-connection on $\mathcal{V}^2$, then the horizontally lifted vector field $\bar{X}$ is the infinitesimal affine transformation of the t-connection if $t = 1$ and X is the infinitesimal affine transformation of the connection ∇_1 on M.*

Proof. Let $t = 1$. Then the nonzero coefficients of the t-connection $\bar{\nabla}_1$ are given by the following formulas

$$\bar{\nabla}_{1i}\frac{\partial}{\partial x^j} = \bar{\nabla}_{1j}\frac{\partial}{\partial x^i} = \left(\Gamma^t_{it|j} - \Phi^d_{ij}\Gamma^t_{dt}\right)\left(\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^{n+1}}\right) + \Phi^k_{ij}\frac{\partial}{\partial x^k},$$

where $\nabla = (\Gamma^k_{ij})$ and $\nabla_1 = (\Phi^k_{ij})$. Then from ([8]) we get

$$\mathcal{L}_X\Phi^k_{ij} = 0.$$

Let X be the infinitesimal affine transformation of the connection $\nabla_1 = (\Phi^k_{ij})$. For the t-connection $\bar{\nabla}_1 = \bar{\Phi}^\gamma_{\alpha\beta}$ and the horizontally lifted vector field $\bar{X}$ we have that the nonzero coefficients of the Lie derivative of the t-connection $\bar{\nabla}_1 = (\bar{\Phi}^\gamma_{\alpha\beta})$ are given by

$$\mathcal{L}_{\bar{X}}\bar{\Phi}^0_{ki} = \mathcal{L}_{\bar{X}}\bar{\Phi}^{n+1}_{ki} = -\Gamma^u_{ru}\mathcal{L}_X\Phi^r_{ki},$$
$$\mathcal{L}_{\bar{X}}\bar{\Phi}^j_{ki} = \mathcal{L}_X\Phi^j_{ki}.$$

From the above calculations and the assumptions we get that $\mathcal{L}_{\bar{X}}\bar{\Phi}^\gamma_{\alpha\beta} = 0$. Hence the horizontally lifted vector field $\bar{X}$ is the infinitesimal affine transformation on $\mathcal{V}^2$. □

At the end of this chapter we show that the horizontally lifted vector field $\bar{X}$ is never an infinitesimal projective transformation and it is never an infinitesimal conformal transformation.

Theorem 4.4. *Let ∇ be the symmetric connection and let ∇_1 be the linear connection on the manifold M. Then the horizontally lifted vector field $\bar{X}$ is never the infinitesimal projective transformation on $\mathcal{V}^2$.*

Proof. A vector field X is an infinitesimal projective transformation on a Riemannian manifold (M, g) if and only if exist a non-zero 1-form ϕ such that ([8])

$$\mathcal{L}_X\Gamma^k_{ij} = \phi_i\delta^k_j + \phi_j\delta^k_i,$$

where $\nabla = (\Gamma^k_{ij})$ is a connection on M.

For the horizontally lifted vector fields such a non-zero 1-form does not exist, so $\bar{X}$ is never the infinitesimal projective transformation on the manifold $(\mathcal{V}^2, \bar{g})$. □

Lemma 4.1. *Let ∇ be the symmetric, locally volume preserving connection on the Riemannian manifold (M, g). If a nonzero function ϱ exists on M and $\mathcal{L}_X g = 2g\varrho$ then $\mathcal{L}_{\bar{X}}\bar{g}_{ab} = 2\varrho^V g_{ab}$, $\mathcal{L}_{\bar{X}}\bar{g}_{\alpha 0} = \mathcal{L}_{\bar{X}}\bar{g}_{\alpha,n+1} = \mathcal{L}_{\bar{X}}\bar{g}_{n+1,n+1} = 0$, where $\varrho^V = \varrho \circ \pi$, where $\pi : \mathcal{V}^2 \to M$ is the projection of the bundle.*

Using the above lemma we get the following

Theorem 4.5. *Let X be the vector field on the Riemannian manifold (M, g) and let ∇ be the symmetric, locally volume preserving connection on the manifold M. Then the horizontally lifted vector field $\bar{X}$ on the bundle $\mathcal{V}^2$ is never the infinitesimal conformal transformation on the bundle of pairs of volume forms $\mathcal{V}^2$.*

Proof. Let $\bar{X}$ be the infinitesimal conformal transformation on the bundle of pairs of volume forms $\mathcal{V}^2$. Then there are exist a nonzero function ϱ such that $\mathcal{L}_{\bar{X}} = 2\varrho g$. In particular we have $\mathcal{L}_{\bar{X}}\bar{g}_{00} = 2\varrho g_{00}$. But from the Lemma 4.1 we obtain $\mathcal{L}_{\bar{X}}\bar{g}_{00} = 0$. We have that $\bar{g}_{00} = 1$, so $\varrho = 0$ and this is contradiction of the assumption that $\bar{X}$ is the infinitesimal conformal transformation on $\mathcal{V}^2$. □

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Institute of Mathematics,
UMCS,
pl. Marii Curie Skłodowskiej 1,
20-031 Lublin,
POLAND
anna.gasior@poczta.umcs.lublin.pl