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STRONGLY LARGE MODULE EXTENSIONS

BY

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Abstract. In this paper, we introduce strongly large submodules and investigate their properties. A submodule N of a right R-module M is said to be *strongly large* in case for any $m \in M$, $s \in R$ with $ms \neq 0$ there exists an $r \in R$ such that $mr \in N$ and $mrs \neq 0$. In this note, we also define and study strongly large closed submodules and strongly large complement submodules.

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1. Introduction

It is well-known that the concept of large submodule in an R-module M due to Johnson [4], named by Eckman and Schopf [3], plays an important role in the context of noncommutative algebras. A submodule N of a right R-module M is *a large submodule*, or M is *a large extension* of N, if $N \cap K \neq 0$ for each nonzero submodule K of M. A submodule N of M is called *a closed submodule* of M if it has no proper large extensions in M. If A and B are submodules of M, then A is *a complement* of B in M if A is a maximal submodule of M with the property that $A \cap B = 0$, and A is called *a complement submodule* of M. It is known that complement submodules and closed submodules of M coincide.

In section 2, we introduce the concept of strongly large extensions, and give some properties of such modules. In section 3, we study strongly singular submodules of a given module and then we define strongly singular modules. In section 4, we deal with strongly large closed submodules (SL-closed, for short) and strongly large complement (or simply, SL-complement) submodules. We study the conditions for which the property of strongly large extensions goes from modules to factor modules over SL-closed submodules. We show that every SL-complement submodule is SL-closed, and give an example showing that the converse is not true in general. In section 5, we compare the notions of strongly largeness and strongly essentiality concepts. The strongly essentiality is defined and investigated in [5]. We present by examples that there is no coincidence between strongly large submodules and strongly essential submodules.

Throughout all rings have an identity and all modules are unital. Let R be a ring, M a right R-module and N a submodule of M. Then $r_R(N) = \{r \in R \mid Nr = 0\}$. For $m \in M$, the right ideal $\{r \in R \mid mr \in N\}$ of R is denoted by $m^{-1}N$. When $N = 0$ the right ideal $m^{-1}N$ is denoted by $r_R(m)$. It is clear that N is a large submodule of M if and only if $m(m^{-1}N) \neq 0$ for each nonzero $m \in M$. For any subset X of M we let $(N : X) = \{r \in R \mid Xr \subseteq N\}$. For submodules A and B of M, $A \leq B$ will denote that A is a submodule of B. In what follows, by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_n$ and $\mathbb{Z}/n\mathbb{Z}$ we denote, respectively, integers, rational numbers, the ring of integers modulo n and the $\mathbb{Z}$-module of integers modulo n. For unexplained concepts and notations, we refer the reader to [1].

2. Strongly large submodules

In this section we introduce and investigate some properties of strongly large submodules. We start with the following definition to strengthen the concept of largeness of submodules of a given module.

Definition 2.1. A submodule N of an R-module M is called *strongly large in M* in case of, for each $m \in M$ and each right ideal I of R, if $mI \neq 0$ then $m(m^{-1}N)I \neq 0$.

Lemma 2.2. *Let N and L be submodules of an R-module M with $N \leq L$. If N is strongly large in L, then*

1. *$N \cap K = 0$ implies $L \cap K = 0$ for any submodule K of M.*
2. *$NI = 0$ implies $LI = 0$ for any right ideal I of R.*

Proof. (1) Let $N \cap K = 0$ for some $K \leq M$. Since N is large in L, then $L \cap K = 0$.

(2) Let $NI = 0$ for some right ideal I of R. Suppose $LI \neq 0$. Then there exist $m \in L$ and $x \in I$ such that $0 \neq mx \in LI$. So $mI \neq 0$. Since N is strongly large in L, then $m(m^{-1}N)I \neq 0$. Hence there exist $r \in m^{-1}N$ and

$s \in I$ such that $0 \neq mrs \in m(m^{-1}N)I$. Then $mrs \in NI$, which contradicts $NI = 0$. This shows that $LI = 0$. □

Every strongly large submodule of a module is always large, but the converse need not be true in general.

Example 2.3. Let F be a field and consider the ring $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ and the R-module $M = \begin{bmatrix} 0 & F \\ F & F \end{bmatrix}$. For the submodules $N_1 = \begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix}$ and $N_2 = \begin{bmatrix} 0 & 0 \\ F & F \end{bmatrix}$, it is obvious that N_1 is large in N_2. On the other hand $N_1 I = 0$ and $N_2 I \neq 0$ for the right ideal $I = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$ of R. So N_1 is not strongly large in N_2 by Lemma 2.2.

Proposition 2.4. *Let M be a module with a submodule N. Then the following are equivalent.*

1. *N is a strongly large submodule of M.*
2. *For each $m \in M$ and $s \in R$ with $ms \neq 0$, we have $m(m^{-1}N)s \neq 0$.*
3. *For each $m \in M$ and $s \in R$ with $ms \neq 0$, there exists $r \in R$ such that $mr \in N$ and $mrs \neq 0$.*
4. *For each $m \in M$, $m(m^{-1}N)$ is a strongly large submodule of mR.*
5. *For each $m \in M$, $mR \cap N$ is a strongly large submodule of mR.*

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4) Let $ms \in mR$ and I be a right ideal of R such that $(ms)I \neq 0$. Suppose $0 \neq (ms)t \in (ms)I$ for some $t \in I$. By assumption, there exists $r \in R$ such that $msr \in N$ and $msrt \neq 0$. Since $msr \in m(m^{-1}N)$, $r \in (ms)^{-1}m(m^{-1}N)$. So $0 \neq msrt \in (ms)[(ms)^{-1}m(m^{-1}N)]I$. Hence $m(m^{-1}N)$ is strongly large in mR.

(4) $\Rightarrow$ (1) Let $m \in M$ and I be a right ideal of R such that $mI \neq 0$. For $L = m(m^{-1}N)$, L is a submodule of N, we have $m^{-1}L \leq m^{-1}N$ and hence $m(m^{-1}L) \leq m(m^{-1}N)$. Since L is strongly large in mR, it follows that $0 \neq m(m^{-1}L)I \leq m(m^{-1}N)I$ and therefore N is strongly large in M.

(4) $\Leftrightarrow$ (5) is trivial. □

Let R be a commutative ring with unity, and S the set of nonzero divisors of R, R_S the total quotient ring of R. Let M be a unitary R-module and $T = \{t \in S : tm = 0$ for some $m \in M$ implies $m = 0\}$. Then T is a multiplicatively closed subset of S. For any submodule N of M, let $N' = \{x \in R_T : Nx \subseteq M\}$. If $NN' = M$, then N is called an *invertible submodule* of M.

We give some examples to show that strongly large submodules are in abundance.

Example 2.5. (1) Invertible submodules of modules over integral domains are strongly large. In particular, nonzero dense submodules of a faithful multiplication module are also strongly large [2, Proposition 3.4].

(2) Let $\mathbb{Q}_1 = \left\{\frac{a}{b} \in \mathbb{Q} : b \notin 2\mathbb{Z}\right\}$. Consider the ring $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Q}_1 \\ 0 & \mathbb{Q}_1 \end{bmatrix}$, the R-module $M = R$ and the submodule $N = \begin{bmatrix} 4\mathbb{Z} & \mathbb{Q}_1 \\ 0 & \mathbb{Q}_1 \end{bmatrix}$ of M. For each $m \in M$ and $s \in R$ with $ms \neq 0$, there exists $r = \begin{bmatrix} 4 & 1 \\ 0 & 1 \end{bmatrix} \in R$ such that $mr \in N$ and $mrs \neq 0$. So N is strongly large in M.

Remark 2.6. Let R be a local ring with unique maximal ideal A. It is clear that A is a large right ideal of R if and only if $A \neq 0$. It is not the case when we deal with strongly large extensions. For example, the ring $R = \mathbb{Z}_4$ is local with a maximal ideal $A = \overline{2}\mathbb{Z}_4$, while A is not strongly large in R.

Lemma 2.7. *Let M be an R-module and N, K be submodules of M with $N \leq K$. Then N is strongly large in K if and only if $m(m^{-1}N)$ is strongly large in $m(m^{-1}K)$ for each $m \in M$.*

Proof. Necessity. Suppose that N is strongly large in K. Let $m \in M$, $x \in m(m^{-1}K)$ and $s \in R$ with $xs \neq 0$. Since $x = mr \in K$, where $r \in m^{-1}K$, there exists $\alpha \in x^{-1}N$ such that $0 \neq x\alpha s = mr\alpha s$. Then $r\alpha \in m^{-1}N$, and so $x\alpha = mr\alpha \in m(m^{-1}N)$. Hence $m(m^{-1}N)$ is strongly large in $m(m^{-1}K)$.

Sufficiency. It is obvious from Proposition 2.4 ($1 \Leftrightarrow 4$) since $k(k^{-1}N)$ is strongly large in $k(k^{-1}K) = kR$ for each $k \in K$. $\square$

Lemma 2.8. *Let M be an R-module and N, K be submodules of M with $N \leq K$. Then N is strongly large in M if and only if N is strongly large in K and K is strongly large in M.*

Proof. Necessity is obvious. Sufficiency. Let $m \in M$ and $s \in R$ with $ms \neq 0$. Since K is strongly large in M, we have $r \in R$ with $mr \in K$ and $mrs \neq 0$. Since N is strongly large in K, by Lemma 2.7, we have $m(m^{-1}N)$ is strongly large in $m(m^{-1}K)$, and hence there exists $t \in R$ such that $mrt \in N$ and $mrts \neq 0$. Therefore N is strongly large in M. $\square$

Lemma 2.9. *Let M be an R-module. If A_1 is strongly large in $B_1 \leq M$ and A_2 is strongly large in $B_2 \leq M$, then $A_1 \cap A_2$ is strongly large in $B_1 \cap B_2$.*

Proof. Let $b \in B_1 \cap B_2$ and $s \in R$ with $bs \neq 0$. There exists $r_1 \in R$ such that $br_1 \in A_1$ and $br_1 s \neq 0$. If $br_1 \in A_2$, then the proof is completed. If $br_1 \notin A_2$, observe that $br_1 \in B_2$ with $br_1 s \neq 0$, so there exists $r_2 \in R$ such that $br_1 r_2 \in A_2$ and $br_1 r_2 s \neq 0$. Hence $br_1 r_2 \in A_1 \cap A_2$ with $br_1 r_2 s \neq 0$. Therefore $A_1 \cap A_2$ is strongly large in $B_1 \cap B_2$. $\square$

The converse of Lemma 2.9 does not hold in general.

Example 2.10. Consider the ring $R = \left\{ \begin{bmatrix} a & x \\ 0 & a \end{bmatrix} : a, x \in \mathbb{Z}_4 \right\}$ and the R-module $M = R$. Let $A = \left\{ \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} : x \in \overline{2}\mathbb{Z}_4 \right\}$, $B_1 = \left\{ \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} : x \in \mathbb{Z}_4 \right\}$ and $B_2 = \left\{ \begin{bmatrix} a & x \\ 0 & a \end{bmatrix} : a, x \in \overline{2}\mathbb{Z}_4 \right\}$. Note that $B_1 \cap B_2 = A$. So A is strongly large in $B_1 \cap B_2$. On the other hand there is no $r \in R$ with $mr \in A$ and $mrs \neq 0$ for $m = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in B_1$ and $s = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \in R$. Hence A is not a strongly large submodule of B_1.

Corollary 2.11. *Let M be an R-module. If A_i is strongly large in $B_i \leq M$ $(i = 1, \ldots, n)$, then $\bigcap_{i=1}^n A_i$ is strongly large in $\bigcap_{i=1}^n B_i$.*

Proof. It follows from induction and by Lemma 2.9. $\square$

Arbitrary intersection of strongly large submodules need not be strongly large.

Example 2.12. Consider $\mathbb{Z}$ as a $\mathbb{Z}$-module. Let $A_n = n\mathbb{Z}$ and $B_n = \mathbb{Z}$ for all $n \in \mathbb{N}$. Although A_n is strongly large in B_n for each $n \in \mathbb{N}$, $\bigcap_{n=1}^\infty A_n = 0$ is not strongly large in $\bigcap_{n=1}^\infty B_n$.

Remark 2.13. Homomorphic images of strongly large submodules need not be strongly large. For example, consider the natural epimorphism $\theta : \mathbb{Z} \longrightarrow \mathbb{Z}/6\mathbb{Z}$ as a $\mathbb{Z}$-module, it is known that $2\mathbb{Z}$ is strongly large in $\mathbb{Z}$, but $\theta(2\mathbb{Z})$ is not strongly large in $\mathbb{Z}/6\mathbb{Z}$.

Proposition 2.14. *Let $f : M \longrightarrow N$ be an R-monomorphism and A, B be submodules of M with $A \leq B$. If A is strongly large in B, then $f(A)$ is strongly large in $f(B)$.*

Proof. Let $b \in B$ and $s \in R$ with $f(b)s \neq 0$. Then $bs \neq 0$. By hypothesis, there exists $r \in R$ such that $br \in A$ and $brs \neq 0$. So $f(b)r \in f(A)$. Since f is monomorphism, $f(b)rs \neq 0$. □

It is known that for any homomorphism $f : M \longrightarrow N$, if K is a large submodule of N, then $f^{-1}(K)$ is a large submodule of M. Unfortunately, this is not the case when we deal with strongly large extensions. By the help of the following proposition, if K is strongly large in N and f is a monomorphism, then we show that $f^{-1}(K)$ is also strongly large in M.

Proposition 2.15. *Let $f : M \longrightarrow N$ be an R-homomorphism with kernel K. If L is a strongly large submodule of N, then $f^{-1}(L)/K$ is strongly large in M/K.*

Proof. Let $\overline{m} \in M/K$ and $s \in R$ with $\overline{m}s \neq 0$. Then $ms \notin K$ and so $f(m)s \neq 0$. There exists $r \in R$ such that $f(m)r \in L$ and $f(m)rs \neq 0$ because of strongly largeness of L in N. Hence $\overline{m}r \in f^{-1}(L)/K$ and $\overline{m}rs \neq 0$. □

Corollary 2.16. *Let $f : M \longrightarrow N$ be an R-monomorphism. If L is a strongly large submodule of N, then $f^{-1}(L)$ is strongly large in M.*

Lemma 2.17. *Let M be an R-module and K, N be submodules of M with $K \leq N$ and N be strongly large in M. If $m^{-1}N/m^{-1}K$ is a faithful submodule of $R/m^{-1}K$ for each $m \in M \backslash N$, then N/K is strongly large in M/K.*

Proof. Let $\overline{m} \in M/K$, and $s \in R$ with $\overline{m}s \neq 0$. Since N is strongly large in M, there exists $r \in R$ such that $mr \in N$ and $mrs \notin K$ (otherwise if $mrs \in K$ for each $r \in m^{-1}N$, and thus $m^{-1}Ns \leq m^{-1}K$; which contradicts that $m^{-1}N/m^{-1}K$ is a faithful submodule of $R/m^{-1}K$). Therefore $\overline{m}r \in N/K$, and $\overline{m}rs \neq 0$. □

Proposition 2.18. *Let M be an R-module and K, N be submodules of M with $K \leq N$. Then the following are equivalent.*

1. *N/K is strongly large in M/K.*
2. *$m^{-1}N/m^{-1}K$ is strongly large in $R/m^{-1}K$ for each $m \in M$.*

Proof. (1) $\Rightarrow$ (2) Let $m \in M\backslash N$, $\overline{r} \in R/m^{-1}K$, and $s \in R$ with $\overline{r}s \neq 0$. Then $\overline{m}(rs) = \overline{mr}s$ is nonzero in M/K. By (1) there exists $r_1 \in R$ such that $\overline{mr}r_1 \in N/K$ and $\overline{mr}r_1s \neq 0$. Hence $\overline{r}r_1 \in m^{-1}N/m^{-1}K$ and $\overline{r}r_1s$ is nonzero in $R/m^{-1}K$.

(2) $\Rightarrow$ (1) Let $\overline{m} \in M/K$ and $s \in R$ with $\overline{m}s \neq 0$. Hence $ms \notin K$, and this follows that $0 \neq \overline{s} = \overline{1}s \in R/m^{-1}K$. By (2) there exists $r \in R$ such that $\overline{1}r \in m^{-1}N/m^{-1}K$ and $\overline{1}rs \neq 0$. Thus $\overline{m}r \in N/K$ and $\overline{m}rs$ is nonzero in M/K. This completes the proof. □

Corollary 2.19. *Let M be an R-module and N be a submodule of M. Then the following are equivalent.*

1. *N is a strongly large submodule of M.*
2. *$m^{-1}N/r_R(m)$ is a strongly large submodule of $R/r_R(m)$, for each $m \in M$.*

Proof. It follows by taking $K = 0$ in Proposition 2.18. □

Proposition 2.20. *Let M be an R-module, K and N be submodules of M with $K \leq N$, and N be strongly large in M. If $(N \cap mR)/(K \cap mR)$ is a faithful R-module for each $m \in M \setminus K$, then N/K is strongly large in M/K.*

Proof. Let $\overline{m} \in M/K$, and $s \in R$ with $\overline{m}s$ nonzero in M/K. By Lemma 2.9, we have $(N \cap mR) + K = N \cap (mR + K)$ is strongly large in $mR + K$. Due to $ms \neq 0$, there exists $r \in R$ such that $mr \in N \cap (mR + K)$ and $mrs \neq 0$. Now if each $r \in R$ with $mr \in N \cap (mR + K)$ and $mrs \neq 0$ implies that $mrs \in K$, then $[N \cap (mR + K)]s \leq K$. For such an s we have $([N \cap (mR + K)]/K)\, s = 0$ and so $s = 0$ since $(N \cap (mR + K))/K \cong (N \cap mR)/(K \cap mR)$ is faithful, which is a contradiction. Therefore there exists $r \in R$ such that $mr \in N \cap (mR + K)$ and $mrs \notin K$, i.e., there exists $r \in R$ with $\overline{m}r \in N/K$ and $\overline{m}rs \neq 0$. Hence N/K is strongly large in M/K. □

Let M be an R-module. M is said to satisfy condition (*) in case of, for each $0 \neq m \in M$ and $r_1, r_2 \in R$, if $r_i \notin r_R(m)$ for some $i = 1, 2$ and $r_1Rr_2 \leq r_R(m)$, then $r_j = 0$ for $j \neq i$. Recall that M is called *a prime module* if for any $m \in M$, $a \in R$, $mRa = 0$ implies $m = 0$ or $Ma = 0$.

Proposition 2.21. (1) *Every faithful prime module satisfies the condition $(*)$.*

(2) *Every free module over a domain satisfies the condition $(*)$.*

(3) *Every ring with no nonzero divisors of zero satisfies the condition $(*)$.*

Proof. (1) Let M be any faithful prime module, $m \in M$ and r_1, $r_2 \in R$. Assume first that $mr_1 \neq 0$ and $mr_1Rr_2 = 0$. Then $(mr_1)r_2 = 0$. Since M is prime, $mr_1 = 0$ or $Mr_2 = 0$. Since $mr_1 \neq 0$ and M is faithful, we have $r_2 = 0$. Assume now that $mr_2 \neq 0$ and $mr_1Rr_2 = 0$, then we have $mr_1 = 0$ or $Mr_2 = 0$. Since $mr_2 \neq 0$, we have $mr_1 = 0$. By the primeness of M, $m = 0$ or $Mr_1 = 0$. Since M is faithful, $r_1 = 0$.

(2) Let $M = \bigoplus_{i \in I} R$ be a free module over a domain R and $0 \neq m = (m_i)_{i \in I} \in M$ and r_1, $r_2 \in R$. Assume that $mr_1 \neq 0$ and $mr_1Rr_2 = 0$. Then $mr_1 \neq 0$ implies $m_ir_1 \neq 0$ for some $i \in I$. Then $m_ir_1Rr_2 = 0$ implies $r_2 = 0$ since R is a domain. Assume that $mr_2 \neq 0$ and $mr_1Rr_2 = 0$. Then we have $r_1 = 0$ in a similar way. This completes the proof.

(3) Let R be a ring with no nonzero divisors of zero and $0 \neq a \in R$. For $b_1, b_2 \in R$ suppose that $b_1 \notin r_R(a)$ and $b_1Rb_2 \leq r_R(a)$. Then $ab_1b_2 = 0$, and so $b_2 = 0$. Now let $b_2 \notin r_R(a)$ and $b_1Rb_2 \leq r_R(a)$. Since $b_2 \neq 0$ and R has no nonzero divisors of zero, we have $b_1 = 0$. $\square$

Lemma 2.22. *Let M be an R-module with the condition $(*)$. Then N is a large submodule of M if and only if N is strongly large in M.*

Proof. Let $m \in M \backslash N$ and $s \in R$ with $ms \neq 0$. There exists $r \in R$ such that $0 \neq mr \in N$ due to largeness of N in M. If $mrs \neq 0$, then the proof is completed. If $mrs = 0$, there exists $d \in R$ with $mrds \neq 0$ because of the condition (*). Therefore N is a strongly large submodule of M. $\square$

We now present an example to show that the sum of strongly large submodules need not be strongly large.

Example 2.23. There exist modules $A_i \leq B_i \leq M$ with A_i is strongly large in B_i for $i = 1, 2$ but $A_1 + A_2$ is not strongly large in $B_1 + B_2$. In order to see this, consider the $\mathbb{Z}$-module $M = \mathbb{Z} \oplus \mathbb{Z}_2$. Set $A_1 = A_2 = (2, \overline{0})\mathbb{Z}$, $B_1 = (1, \overline{0})\mathbb{Z}$ and $B_2 = (1, \overline{1})\mathbb{Z}$. Let $b_i \in B_i$ and $n_i \in \mathbb{Z}$ such that $b_in_i \neq 0$. For $2 \in \mathbb{Z}$ we have $b_i2 \in A_i$ and $b_i2n_i \neq 0$. So A_i is strongly large in B_i for $i = 1, 2$. On the other hand $(0, \overline{1})\mathbb{Z} \leq B_1 + B_2$, and $A_1 \cap (0, \overline{1})\mathbb{Z} = 0$. Thus A_1 is not a large submodule of $B_1 + B_2$. Therefore $A_1 + A_2$ is not strongly large in $B_1 + B_2$.

Proposition 2.24. *Let M be an R-module and N be a submodule of M. If M satisfies the condition $(*)$, then the following are equivalent.*

1. *N is strongly large in M.*
2. *$m^{-1}N$ is strongly large in R for each $m \in M$.*

Proof. (1)$\Rightarrow$ (2) Let $m \in M$ and $r, s \in R$ with $rs \neq 0$. If $mr = 0$, there is nothing to show due to $1 \in R$. Assume that $mr \neq 0$. If $mrs \neq 0$, there exists $r_1 \in R$ such that $(mr)r_1 \in N$ and $(mr)r_1 s \neq 0$, i.e., $rr_1 \in m^{-1}N$ and $rr_1 s \neq 0$. Now if $mrs = 0$, since M satisfies the condition (*) and $s \neq 0$, there exists $d \in R$ such that $mrds \neq 0$. Then there exists $r_1 \in R$ such that $(mrd)r_1 \in N$ and $(mrd)r_1 s \neq 0$, i.e., $r(dr_1) \in m^{-1}N$ and $r(dr_1)s \neq 0$. Therefore we have (2).

(2)$\Rightarrow$ (1) Let $m \in M$ and $s \in R$ with $ms \neq 0$. By (2) there exists $r \in R$ such that $mr \in N$ and $rs \neq 0$. By the condition (*) and $s \notin r_R(m)$, there exists $d \in R$ such that $m(rd)s \neq 0$. Since $m(rd) \in N$, we have (1). $\square$

Proposition 2.25. *Let M be an R-module with the condition $(*)$, $m \in M$ and I be a right ideal of R. Then*

1. *If I is strongly large in R_R, then mI is strongly large in mR.*
2. *If $r_R(m) \leq I$ and mI is strongly large in mR, then I is strongly large in R_R.*

Proof. (1) Let $mr \in mR$ and $s \in R$ with $mrs \neq 0$. There exists $r_1 \in R$ such that $rr_1 \in I$ and $rr_1 s \neq 0$. If $mrr_1 s \neq 0$, then the proof is completed. Now suppose that $mrr_1 s = 0$. Since $s \notin r_R(mr)$ and $r_1 \neq 0$, there exists $d \in R$ such that $mrr_1 ds \neq 0$, by the condition (*). Also $mrr_1 d \in mI$, thus mI is strongly large in mR.

(2) Let $r_R(m) \leq I$ and $r, s \in R$ with $rs \neq 0$. If $mr = 0$, then by assumption $r \in I$, and hence we have nothing to show. If $mr \neq 0$, then from the condition (*), there exists $d \in R$ with $mrds \neq 0$. Because of the strongly largeness of mI in mR, there exists $r_1 \in R$ such that $mrdr_1 \in mI$ and $mrdr_1 s \neq 0$. This gives that there exists $i \in I$ such that $m(rdr_1 - i) = 0$. Since $r_R(m) \leq I$, this follows that $rdr_1 \in I$. And so I is strongly large in R_R. $\square$

3. Strongly singular modules

Let M be an R-module. We define $Z_S(M) = \{m \in M : r_R(m)$ is strongly large in $R_R\}$. Since $0 \in Z_S(M)$, then $Z_S(M) \neq \emptyset$. Also $Z_S(M)$ is an additive subgroup of M, however it need not be a submodule in general. Let $Z(M)$ denote the singular submodule of M. Then $Z_S(M)$ is included in $Z(M)$. By Remark 3.4 we show that this inclusion is strict.

Lemma 3.1. *If M is an R-module with the condition $(*)$, then $Z_S(M)$ is a submodule of M.*

Proof. Let $m \in Z_S(M)$ and $t \in R$. Let $r, s \in R$ with $rs \neq 0$. We show that $mt \in Z_S(M)$. Since $r_R(m)$ is strongly large, there exists $r_1 \in R$ such that $rr_1 \in r_R(m)$ and $rr_1 s \neq 0$. Consider the following cases:

(i) If $mtrr_1 = 0$, then $rr_1 \in r_R(mt)$ and $rr_1 s \neq 0$.

(ii) Assume that $mtrr_1 \neq 0$. If $mtrr_1 Rs = 0$, by the condition (*), $s = 0$. This is a contradiction. So $mtrr_1 Rs \neq 0$. Thus there exists $c \in R$ such that $mtrr_1 cs \neq 0$. Also there exists $r_2 \in R$ such that $(trr_1)r_2 \in r_R(m)$ and $(trr_1)r_2(cs) \neq 0$. Hence $rr_1 r_2 c \in r_R(mt)$ and $rr_1 r_2 cs \neq 0$. Therefore $r_R(mt)$ is strongly large and so $mt \in Z_S(M)$. □

Lemma 3.2. *Let R be a commutative ring and M be an R-module. Then $Z_S(M)$ is a submodule of M.*

Proof. Let $m \in Z_S(M)$ and $t \in R$. Let $r, s \in R$ with $rs \neq 0$. Since $r_R(m)$ is strongly large in R_R, there exists $r_1 \in R$ such that $rr_1 \in r_R(m)$ and $rr_1 s \neq 0$. Then $mtrr_1 = mrr_1 t = 0$. Thus $rr_1 \in r_R(mt)$. This implies that $r_R(mt)$ is strongly large in R_R. Hence $mt \in Z_S(M)$. □

If $Z_S(M)$ is a submodule of M, then $Z_S(M)$ is called *a strongly singular submodule*. Also M is called *a strongly singular module* if $Z_S(M) = M$, and M is called *a non-strongly singular module* if $Z_S(M) = 0$.

Example 3.3. $\mathbb{Z}_4$ is strongly singular as a $\mathbb{Z}$-module. On the other hand $\mathbb{Z}_4$ is non-strongly singular as a $\mathbb{Z}_4$-module.

Remark 3.4. Let M be a module and $Z(M)$ denote the singular submodule of M. It is obvious that $Z_S(M) \subseteq Z(M)$. However $Z(M)$ need not be equal to $Z_S(M)$ in general. For example, let F be any field and consider the ring $R = \begin{bmatrix} F[x] & 0 \\ \frac{F[x,y]}{(x,y^2)} & \frac{F[y]}{(y^2)} \end{bmatrix}$ and the R-module $M = \begin{bmatrix} 0 & 0 \\ \frac{F[x,y]}{(x,y^2)} & \frac{F[y]}{(y^2)} \end{bmatrix}$. Let $m = \begin{bmatrix} 0 & 0 \\ 0 & y + (y^2) \end{bmatrix} \in M$. Then $r_R(m) = \begin{bmatrix} F[x] & 0 \\ \frac{F[x,y]}{(x,y^2)} & \frac{(y)}{(y^2)} \end{bmatrix}$. For the right ideal $I = \begin{bmatrix} 0 & 0 \\ \frac{(y)}{(x,y^2)} & 0 \end{bmatrix}$ of R, $RI \neq 0$ despite $r_R(m)I = 0$. So $r_R(m)$ is not strongly large in R, and then $m \notin Z_S(M)$. Clearly $r_R(m)$ is large in R_R. So $m \in Z(M)$.

Proposition 3.5. *If the ring R satisfies the condition $(*)$, then $Z_S(M) = Z(M)$ for all R-modules M.*

Proof. Clear from Lemma 2.22. □

Let M and N be modules and $f : M \longrightarrow N$ an R-homomorphism. It is obvious that $f(Z_S(M)) \subseteq Z_S(N)$. In particular $Z_S(M)$ is fully invariant. Let Mod_R denote the category of all right R-modules. A functor $Z_S : \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_R$ can be defined as follows:

(i) for each module M in Mod_R, $Z_S(M)$ is strongly singular submodule of M in Mod_R.

(ii) for each R-homomorphism $f : M \longrightarrow N$, $Z_S(f) : Z_S(M) \longrightarrow Z_S(N)$ is an R-homomorphism with $Z_S(f)(m) = f(m)$ for any $m \in Z_S(M)$.

Proposition 3.6. *If M is non-strongly singular R-module, then $Hom_R(N, M) = 0$ for all strongly singular modules N. The converse holds if $Z_S(M)$ is a submodule of M.*

Proof. Let $f : N \longrightarrow M$ be an R-homomorphism, where N is strongly singular R-module. Hence $f(N) = f(Z_S(N)) \subseteq Z_S(M) = 0$, and so $f = 0$. Conversely, let $Z_S(M)$ be a submodule of M, and consider the inclusion map $i : Z_S(M) \longrightarrow M$. By assumption we have $Z_S(M) = 0$. □

It is known that if K is a large submodule of a module M, then M/K is singular. But this statement is not true when we deal with strongly large extensions.

Example 3.7. Let F be a field. Consider the subring

$$R = \left\{ (x_1, x_2, \ldots, x_n, a, a, a, \ldots) : n \in \mathbb{Z}^+, x_i, a \in \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}, i = 1, \ldots, n \right\}$$

of the ring $\prod_{i=1}^{\infty} \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}_i$ with componentwise addition and multiplication and the submodule

$$I = \left\{ (x_1, x_2, \ldots, x_n, 0, 0, 0, \ldots) : n \in \mathbb{Z}^+, x_i, 0 \in \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}, i = 1, \ldots, n \right\}$$

of R_R. It can be seen easily that I is strongly large in R. Let

$$\overline{x} = \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \ldots \right) + I \in R/I$$

and let j be the first nonzero component of $\overline{x}$. Then

$$r_R(\overline{x}) = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}_1 \oplus \cdots \oplus \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}_{j-1} \oplus \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}_j \oplus \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}_{j+1} \oplus \cdots$$

Since $r_R(\overline{x})$ is not large in R, then $r_R(\overline{x})$ is not strongly large in R. So $\overline{x} \notin Z_S(R/I)$ and R/I is not a strongly singular R-module.

Proposition 3.8. *Let M be an R-module with the condition $(*)$. If K is a strongly large submodule of M, then M/K is a strongly singular R-module.*

Proof. Let $\overline{m} \in M/K$ and $r, s \in R$ with $rs \neq 0$. If $mr \notin K$, consider the following cases:

(i) Assume that $mrs \neq 0$. Then there exists $r_1 \in R$ such that $(mr)r_1 \in K$ and $(mr)r_1 s \neq 0$ due to strongly largeness of K.

(ii) Assume that $mrs = 0$. Since $r \notin r_R(m)$ and $s \neq 0$, by the condition (*) there exists $r_2 \in R$ such that $mrr_2 s \neq 0$. There exists $r_3 \in R$ such that $(mrr_2)r_3 \in K$ and $(mrr_2)r_3 s \neq 0$ due to strongly largeness of K. So $rr_2r_3 \in r_R(\overline{m})$ and $rr_2r_3s \neq 0$. Thus $r_R(\overline{m})$ is a strongly large right ideal of R. Hence $Z_S(M/K) = M/K$. $\square$

The converse of Proposition 3.8 can easily fail. Let n be a positive integer and $M = \mathbb{Z}/n\mathbb{Z}$. Then M is strongly singular as a $\mathbb{Z}$-module since every nonzero submodule of $\mathbb{Z}$ is strongly large. Let N denote the zero submodule of M. M/N is strongly singular but N is not strongly large in M.

4. SL-closed submodules and SL-complement submodules

In this section we define and study SL-closed and SL-complement submodules.

Definition 4.1. Let N be a submodule of a module M. Then N is called an *SL-closed submodule* if N has no proper strongly large extensions in M.

For example 0 and M are always SL-closed submodules of M. There are modules with SL-closed submodules which are not closed.

Example 4.2. Let F be any field. Consider the ring $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ and the right R-module $M = R$. Then $N = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$ is SL-closed and large in M.

Lemma 4.3. *If N is a closed submodule of a module M, then N is SL-closed. The converse holds if M has the condition $(*)$.*

Proof. First statement is clear. Assume that M has the condition (*). Let N be any SL-closed submodule in M and K a submodule in which N is large in K. By Lemma 2.22, N is strongly large in K and so $N{=}K$. $\square$

Proposition 4.4. *Direct summands of a module are SL-closed submodules.*

Proof. Let $M = N \oplus K$ be an R-module. Assume that N is strongly large in a submodule $D \leq M$. Hence $D \cap K = 0$ by Lemma 2.2, and so $D = N$. Therefore N has no proper strongly large extensions in M. $\square$

Remark 4.5. There are modules with SL-closed submodules which are not direct summands. For example, consider the $\mathbb{Z}$-module $M = \mathbb{Z}_2 \oplus \mathbb{Z}_8$ and the submodule $N = (\overline{1}, \overline{2})\mathbb{Z}$ of M. Since N is a closed submodule of M, then N is an SL-closed submodule of M. On the other hand N is not a direct summand of M.

The intersection of SL-closed submodules of a module M need not be SL-closed.

Example 4.6. Consider the $\mathbb{Z}$-module $M = \mathbb{Z} \oplus \mathbb{Z}_2$. Let $A = (1, \overline{0})\mathbb{Z}$ and $B = (1, \overline{1})\mathbb{Z}$. Observing that $M = A \oplus (0, \overline{1})\mathbb{Z} = B \oplus (0, \overline{1})\mathbb{Z}$. Since A and B are direct summands of M, A and B are SL-closed in M. Note that $A \cap B = (2, \overline{0})\mathbb{Z}$. We know that $A \cap B$ is strongly large in both A and B from Example 2.23. Hence $A \cap B$ is not SL-closed in A, B or M.

Lemma 4.7. *Let N be a submodule of a module M. Then the union of any chain of SL-extensions of N is an SL-extensions of N in M.*

Proof. Let $\{L_\alpha\}_{\alpha \in \Lambda}$ be a family of submodules of M with N is strongly large in L_α for each $\alpha \in \Lambda$. Let $x \in \bigcup_{\alpha \in \Lambda} L_\alpha$ and I be a right ideal of R such that $xI \neq 0$, hence $x \in L_\alpha$ for some $\alpha \in \Lambda$. Since N is strongly large in L_α, we have that $x(x^{-1}N)I \neq 0$. Therefore N is strongly large in $\bigcup_{\alpha \in \Lambda} L_\alpha$. $\square$

Theorem 4.8. *Every submodule N of a module M is contained in an SL-closed submodule K of M in which N is strongly large.*

Proof. Let N be a submodule of M, and let $\Re = \{L \leq M : N$ is strongly large in $L\}$. Order $\Re$ by inclusion, then by Lemma 4.7, $\Re$ is inductive, and hence by Zorn's Lemma $\Re$ has a maximal element K. By Lemma 2.8, K is SL-closed in M. □

Proposition 4.9. *Let M be an R-module and N be an SL-closed submodule of M. If K is strongly large in M, then $N \cap K$ is SL-closed in K.*

Proof. Let L be a strongly large extension of $N \cap K$ in K. We show that N is strongly large in $N + L$. Let $n \in N$, $l \in L$ and $s \in R$ with $(n + l)s \neq 0$. Since K is strongly large in M, there exists $r \in R$ such that $(n+l)r \in K$ and $(n+l)rs \neq 0$. So $nr \in N \cap K$, and hence $(n+l)r \in L$ with $(n + l)rs \neq 0$. Since $N \cap K$ is strongly large in L, there exists $r_1 \in R$ such that $(n + l)rr_1 \in N \cap K \leq N$ and $(n + l)rr_1s \neq 0$. Therefore N is strongly large in $N + L$. Since N is an SL-closed submodule of M, it follows that $N = N + L$. Hence $N \cap K = L$, and so $N \cap K$ is an SL-closed submodule of K. □

Lemma 4.10. *Let M be an R-module and N, K be submodules of M with $K \leq N$. If N is strongly large and K is SL-closed in M, then $m^{-1}K \neq m^{-1}N$ for each $m \in M\backslash K$.*

Proof. Let $m \in M\backslash K$ and $m^{-1}K = m^{-1}N$. We show that K is strongly large in $K + mR$, which contradicts that K is SL-closed. To this end, let $k + mr \in K + mR$ and $s \in R$ with $(k + mr)s \neq 0$. Since N is strongly large in M, there exists $b \in R$ such that $(k+mr)b \in N$ and $(k+mr)bs \neq 0$. This follows that $rb \in m^{-1}N = m^{-1}K$, and hence $(k + mr)b \in K$. Then K is strongly large in $K + mR$, which contradicts that K is an SL-closed submodule of M. Thus $m^{-1}K \neq m^{-1}N$. □

Proposition 4.11. *Let M be an R-module and N, K be submodules of M with $K \leq N$. If N is strongly large and K is SL-closed in M, then N/K is large in M/K.*

Proof. Let $0 \neq \overline{m} \in M/K$. It is easy to see that $\overline{m}^{-1}\overline{N} = m^{-1}N$. Now if $\overline{m}(\overline{m}^{-1}\overline{N}) = 0$, then $m(m^{-1}N) \leq K$. It follows that $m^{-1}N = m^{-1}K$, and hence, by Lemma 4.10, $m \in K$ which contradicts that $\overline{m} \neq \overline{0}$. This shows that N/K is large in M/K. □

Proposition 4.12. *Let M be an R-module and N, K be submodules of M with $K \leq N$. If N is strongly large and K is SL-closed in M, then the following are equivalent.*

1. *N/K is strongly large in M/K.*
2. *$(m^{-1}K : m^{-1}N) \leq (m^{-1}K : R)$ for each $m \in M \setminus K$.*

Proof. (1)$\Rightarrow$(2) Let $m \notin K$. By Lemma 4.10, we have $m^{-1}K \neq m^{-1}N$, i.e., $(m^{-1}K : m^{-1}N) \neq R$.

On the other hand $(m^{-1}N)/(m^{-1}K)$ is strongly large in $R/m^{-1}K$ from Proposition 2.18. Clearly $[(m^{-1}N)/(m^{-1}K)](m^{-1}K : m^{-1}N) = 0$. By Lemma 2.2, we have that $[R/m^{-1}K](m^{-1}K : m^{-1}N) = 0$, i.e., $R(m^{-1}K : m^{-1}N) \leq m^{-1}K$. Therefore $(m^{-1}K : m^{-1}N) \leq (m^{-1}K : R)$ for each $m \in M \setminus K$.

(2) $\Rightarrow$ (1) Let $\overline{m} \in M/K$ and $s \in R$ with $\overline{m}s \neq 0$. It is clear that $ms \notin K$. Then $ms \neq 0$, there exists $r \in R$ such that $mr \in N$ and $mrs \neq 0$ because of strongly largeness of N in M. Now if each $r \in R$ with $0 \neq mrs \in N$ implies that $mrs \in K$, then $s \in (m^{-1}K : m^{-1}N)$. Since $m \notin K$, we have by (2) that $Rs \leq m^{-1}K$, and hence $s \in m^{-1}K$. This shows that $\overline{m}s = 0$, which is a contradiction. Therefore there exists $r \in R$ such that $mr \in N$ and $mrs \notin K$, i.e., $\overline{m}rs \neq 0$. This completes the proof.□

Corollary 4.13. *Let M be an R-module, where R is a commutative ring and N, K be submodules of M with $K \leq N$. If N is strongly large and K is SL-closed in M, then the following are equivalent.*

1. *N/K is strongly large in M/K.*
2. *$m^{-1}N/m^{-1}K$ is a faithful $R/m^{-1}K$-module for each $m \in M \setminus K$.*

Proof. It is clear that $(m^{-1}K : R) = m^{-1}K$, when R is commutative. It follows that (2) of Proposition 4.12 is equivalent to say $(m^{-1}K : m^{-1}N) \leq m^{-1}K$ for each $m \notin K$, this in turn is equivalent to say that $m^{-1}N/m^{-1}K$ is a faithful $R/m^{-1}K$-module for each $m \notin K$. □

Definition 4.14. Let N, N' be submodules of a module M with $N \cap N' = 0$. Then N' is called an *SL-complement of N in M* if N' is an SL-closed submodule of M and $N \oplus N'$ is strongly large in M. A submodule K is called an *SL-complement in M* if there exists a submodule L such that K is an SL-complement of L in M.

Example 4.15. Consider the ring $R = \begin{bmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Z} \end{bmatrix}$, the R-module $M =$

R and the submodule $L = \begin{bmatrix} 0 & 0 \\ 0 & 2\mathbb{Z} \end{bmatrix}$ of M. For the submodule $N = \begin{bmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & 0 \end{bmatrix}$ of M, $N \oplus L$ is strongly large in M and N is SL-closed in M. Hence N is an SL-complement of L in M.

It is clear that every SL-complement is SL-closed, while there are modules in which SL-closed submodules need not be SL-complements.

Remark 4.16. It is known that complements in any module M always exist, and a submodule N is closed in M if and only if it is complement of some submodule of M. Unfortunately, in the case of strongly large there are modules M and submodules N that have no SL-complements in M. For example if $M = \mathbb{Z}_{p^2}$ as a $\mathbb{Z}$-module (for a prime integer p) and $N = \overline{p}\mathbb{Z}$, then 0 is the only submodule N' of M such that $N \cap N' = 0$, while $N \oplus N'$ is not strongly large in M. This also shows that $N = \overline{p}\mathbb{Z}$ is an SL-closed submodule of $\mathbb{Z}_{p^2}$, which is not an SL-complement in $\mathbb{Z}_{p^2}$.

Proposition 4.17. *Let N and L be submodules of a module M with L strongly large in N. If K is an SL-complement of L, then K is an SL-complement of N in M.*

Proof. Clear from definition of SL-complements. □

Theorem 4.18. *Let N, N' be submodules of a module M with $N \cap N' = 0$. Then N' is an SL-complement of N in M if and only if N' is maximal with respect to being $N' \oplus N$ strongly large in M.*

Proof. Let N' be an SL-complement of N in M and $N' \subseteq K \leq M$ with $K \oplus N$ strongly large in M. Let $k \in K$ and $s \in R$ with $ks \neq 0$. There exists $r \in R$ such that $kr \in N' \oplus N$ and $krs \neq 0$. Then $kr \in N'$. Hence N' is strongly large in K. Since N' is SL-closed in M, we have $N' = K$. This shows the maximality of N' with respect to being $N' \oplus N$ strongly large in M.

Conversely, let N' be maximal with respect to being $N' \oplus N$ strongly large in M and K be a strongly large extension of N' in M. Then $N \cap K = 0$, and hence by Lemma 2.8, $K \oplus N$ is strongly large in M. By the maximality of N', $N' = K$. Therefore N' is SL-closed in M and N' is an SL-complement of N in M. □

Proposition 4.19. *Let M be a module and L, N be submodules of M. If N is an SL-complement of L in M, then N is a complement of L in M.*

Proof. Let $K \leq M$ with $N \leq K$ and $L \cap K = 0$. Since M is a strongly large extension of $N \oplus L$, $K \oplus L$ is a strongly large submodule of M by Lemma 2.8. So we have that $N = K$ from Theorem 4.18. Hence N is maximal with respect to being $L \cap N = 0$. $\square$

Remark 4.20. Complements of a submodule need not be SL-complements for the same submodule. For example, the zero submodule is a complement of $\overline{p}\mathbb{Z}$, which is not an SL-complement of $\overline{p}\mathbb{Z}$ in $\mathbb{Z}_{p^2}$ (for a prime integer p). This example shows also that if $K \leq N \leq M$ and B is an SL-complement of K in M, then B need not contain an SL-complement of N in M. This fact work properly in case of complements.

Lemma 4.21. *Let M be an R-module with the condition $(*)$. Then complements and SL-complements of M coincide.*

Proof. By Lemma 2.22 and Lemma 4.3. $\square$

Proposition 4.22. *Let K be an SL-complement of N in a module M and $K = X \oplus Y$, then X is an SL-complement of $Y \oplus N$ in M. In particular, direct summands of SL-complements are SL-complements.*

Proof. Let $H \leq M$ with $X \leq H$ and $H \oplus Y \oplus N$ is strongly large in M. Then $K = X \oplus Y \leq H \oplus Y$. Since K is maximal with respect to being $K \oplus N$ strongly large in M, then $K = X \oplus Y = H \oplus Y$. So $X = H$. Thus by Theorem 4.18, X is an SL-complement of $Y \oplus N$ in M. $\square$

Proposition 4.23. *Let M be a module and A, B be submodules of M with $A \leq B$. If A is an SL-complement in B and B is an SL-complement in M, then B contains every strongly large extension of A in M.*

Proof. Let A be an SL-complement of a submodule A' in B, and B be an SL-complement of a submodule B' in M. By Proposition 4.18, A is maximal in B with respect to being $A' \oplus A$ strongly large in B and B is maximal in M with respect to being $B' \oplus B$ strongly large in M. Let C be a strongly large extension of A in M. Then $C + (A' \oplus B')$ is direct, and hence $[(C \oplus B') \cap B] + A'$ is direct. Since $A \leq (C \oplus B') \cap B \leq B$, and $[(C \oplus B') \cap B] \oplus A'$ is strongly large in B, we have $(C \oplus B') \cap B = A$. We claim that $(C + B) \cap B' = 0$. Let $b' = c + b \in B'$, where $c \in C$ and $b \in B$. Then $b \in (C \oplus B') \cap B = A \leq C$. It follows that $b' \in C \cap B' = 0$. Then we have our claim. It is clear that M is a strongly large extension of $(C+B) \oplus B'$. By the maximality of B, we have $C + B = B$ and so $C \leq B$. $\square$

Corollary 4.24. *Let M be a module and A, B be submodules of M with $A \leq B$. If A is an SL-complement in B and B is an SL-complement in M, then A is an SL-closed submodule of M.*

Proof. Clear from Proposition 4.23. $\square$

Proposition 4.25. *Let M be a module and A, B be submodules of M with $A \leq B$. If C is an SL-complement of A in M, then $C \cap B$ is an SL-complement of A in B.*

Proof. Since $C \oplus A$ is strongly large in M, by Lemma 2.9, $(C \oplus A) \cap B = (C \cap B) \oplus A$ is strongly large in B. It remains to show that $C \cap B$ is an SL-closed submodule of B. To end this, let D be an strongly large extension of $C \cap B$ in B. Then $(C + D) \cap A = (C + D) \cap B \cap A = [(C \cap B) + D] \cap A = D \cap A = 0$. It is clear that $(C + D) \oplus A$ is strongly large in M, and hence $C = C + D$ by the maximality of C. Thus $D = C \cap B$, and therefore $C \cap B$ is an SL-closed submodule of B. $\square$

Corollary 4.26. *Let A be an SL-complement in M and B an SL-complement in M for some submodule of A. Then $A \cap B$ is an SL-closed submodule of M.*

Proof. Let $C \leq A \leq M$ with B an SL-complement of C in M. Then by Proposition 4.25, $A \cap B$ is an SL-complement of C in A. Since A is an SL-complement in M, we have by Corollary 4.24 that $A \cap B$ is an SL-closed submodule of M. $\square$

Proposition 4.27. *Let $M = M_1 \oplus M_2$ be a module, $A \leq M_1$ and C an SL-complement of $A \oplus M_2$ in M. If $C \leq M_1$, then C is an SL-complement of A in M_1.*

Proof. Since $C \oplus (A \oplus M_2)$ is strongly large in M, by Lemma 2.9, $[C \oplus (A \oplus M_2)] \cap M_1$ is strongly large in M_1. By the modularity condition we have $[C \oplus (A \oplus M_2)] \cap M_1 = [M_1 \cap (A \oplus M_2)] \oplus C = [A \oplus (M_1 \cap M_2)] \oplus C = A \oplus C$. So M_1 is a strongly large extension of $A \oplus C$. Since C is SL-closed in M, also C is SL-closed in M_1. Therefore C is an SL-complement of A in M_1. $\square$

Proposition 4.28. *Let $M = M_1 \oplus M_2$ be a module, $A, C \leq M_1$ and $C \oplus M_2$ an SL-complement of A in M. Then C is an SL-complement of A in M_1.*

Proof. In the proof of Proposition 4.27, we have $A \oplus C$ is strongly large in M_1. Let $N \leq M_1$ with $C \leq N$ and $A \oplus N$ is strongly large in M_1. By Lemma 2.8, we have that $A \oplus N \oplus M_2$ is strongly large in M. So $N \oplus M_2 = C \oplus M_2$ by the maximality of $C \oplus M_2$. Thus $N = C$. Therefore C is an SL-complement of A in M_1 from the Theorem 4.18. □

5. Strongly largeness versus strongly essentiality

It is known by Lemma 2.22 that the largeness forces the strongly largeness in any module which satisfy the condition (*). So we can say that every large submodule of a faithful prime module is strongly large, and every large submodule of a free module over a domain is strongly large. In [5], a strong version of largeness is introduced. A submodule N of an R-module M is called *strongly essential* in M if for any nonzero subset X of M, there exists $r \in R$ such that $0 \neq Xr \subseteq N$. In this section, we investigate modules in which whether large submodules, strongly large submodules and strongly essential submodules coincide or not.

Proposition 5.1. *Let M be a torsion-free module. Then every essential submodule of M is strongly large.*

Proposition 5.2. *Let R be a ring, where R has no nonzero divisors of zero. Then essentiality implies strongly largeness in R.*

Proof. Assume that I is an essential right ideal of R. Let $r, s \in R$ with $rs \neq 0$. Then there exists $x \in R$ such that $0 \neq rx \in I$ by the essentiality of I. Since R has no nonzero divisors of zero, we have $rxs \neq 0$. □

Proposition 5.3. *Let M be an R-module. Then each essential submodule of M which has an SL-complement in M is strongly large.*

Proof. Clear from the definitions. □

Proposition 5.4. *The following are equivalent for a ring R.*

1. *If N is an essential submodule of an R-module M, then N is strongly large.*
2. *M is strongly large in the injective envelope $E(M)$ of M, for any R-module M.*

Proof. (2) $\Rightarrow$ (1) Let N be essential in M. Then $E(N) = E(M)$ because of essentiality of N. Hence N is strongly large in $E(M)$. Also N is strongly large in M from Lemma 2.8.

(1) $\Rightarrow$ (2) Clear. □

Proposition 5.5. *Let R be a prime ring and M be a singular R-module. Then for any $m \in M$ the right annihilator $r_R(m)$ of m in R is a strongly large submodule of R_R.*

Proof. Let $x, s \in R$ with $xs \neq 0$. Since M is singular, for any $m \in M$, $r_R(m)$ is an essential right ideal of R. Hence $xR \cap r_R(m) \neq 0$. So there exists $t \in R$ such that $0 \neq xt \in r_R(m)$. Since R is a prime ring, we have that $xtRs \neq 0$. Hence there exists $t_1 \in R$ such that $xtt_1s \neq 0$, and $xtt_1 \in r_R(m)$. Therefore $r_R(m)$ is a strongly large submodule of R_R. □

Proposition 5.6. *If R is a prime ring, then each nonzero ideal of R is a strongly large submodule of R_R. In particular essential submodules and strongly large submodules of R_R coincide.*

Proof. Let I be a nonzero ideal of a prime ring R. It is clear that I is an essential right ideal of R. Let $r, s \in R$ with $rs \neq 0$. Then $rR \cap I$ is a nonzero right ideal. Since R is prime, $(rR \cap I)(sR) \neq 0$. So there exists $x \in rR \cap I$ such that $xs \neq 0$. Since $x \in rR$, there exists $a \in R$ such that $x = ra$. Therefore $ra \in I$ and $ras \neq 0$. □

Proposition 5.7. *Let M be a prime R-module. Then essentiality implies strongly largeness in M.*

Proof. Let N be an essential submodule of M, and $m \in M$, $s \in R$ with $ms \neq 0$. So there exists $r \in R$ with $0 \neq mr \in N$. Since M is prime, $mrRs \neq 0$. Thus we have $t \in R$ such that $mrt \in N$ and $mrts \neq 0$. □

Now we give some examples to show that the notion of strongly essentiality and strongly largeness are distinct from each other.

Example 5.8. Let $\mathbb{Q}_1 = \left\{\frac{a}{b} \in \mathbb{Q} : 2 \text{ does not divide } b\right\}$ and R denote the ring $\begin{bmatrix} \mathbb{Z} & 0 \\ 0 & \mathbb{Z} \end{bmatrix}$ and consider the R-module $M = \begin{bmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{bmatrix}$. Take $N = \begin{bmatrix} 4\mathbb{Z} & \mathbb{Q}_1 \\ 0 & \mathbb{Q}_1 \end{bmatrix}$ submodule of M. For the nonzero subset

$$X = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & \frac{n}{2^{n+1}} \end{bmatrix} : n = 0, 1, 2, 3, \dots \right\}$$

of M, it is easy to observe that R has no element r such that $0 \neq Xr \subseteq N$. Hence N is not a strongly essential submodule of M. We claim N is strongly large in M. Consider the following cases:

Case (i). Let $m = \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \in M \backslash N$, $s = \begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix} \in R$ where x and y are nonzero integers. For $r = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \in R$, we have $mr \in N$ and $mrs \neq 0$.

Case (ii). Let $m = \begin{bmatrix} 0 & \frac{1}{2^x} \\ 0 & 0 \end{bmatrix} \in M \backslash N$ and $s = \begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix} \in R$ such that x is positive integer and y is nonzero integer. For $r = \begin{bmatrix} 0 & 0 \\ 0 & 2^x \end{bmatrix} \in R$ we have $mr \in N$ and $mrs \neq 0$.

Case (iii). Let $m = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2^x} \end{bmatrix} \in M \backslash N$ and $s = \begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix} \in R$ such that x is positive integer and y is nonzero integer. Then $mr \in N$ and $mrs \neq 0$ for $r = \begin{bmatrix} 0 & 0 \\ 0 & 2^x \end{bmatrix} \in R$. Therefore N is a strongly large submodule of M.

Example 5.9. Let F be a field, R the upper triangular matrix ring $\begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ and M denote the R-module R. Consider the submodule $N = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$ of M and the right ideal $I = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$ of R. Then $NI = 0$ but $MI \neq 0$. So N is not strongly large in M. On the other hand e_{ij} denote the 2×2 matrix unit in R with (i, j) entry 1, all other entries 0. Let $r = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \in R$, then $e_{ij}r \in N$. Hence $0 \neq Xr \subseteq N$ for any nonzero subset X of M. Thus N is a strongly essential submodule of M.

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