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DUAL STRUCTURES ON THE PROLONGATIONS OF A LIE ALGEBROID

BY

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Abstract. In the present paper we study the properties of dual structures on the prolongations of a Lie algebroid. We introduce the dynamical covariant derivative on Lie algebroids and prove that the nonlinear connection induced by a regular Lagrangian is compatible with the metric and symplectic structures. The notions of mechanical structure and semi-Hamiltonian section are introduced on the prolongation of the Lie algebroid to its dual bundle and their properties are investigated. Finally, we prove the equivalence between the metric nonlinear connection and semi-Hamiltonian section, using the Legendre transformation induced by a regular Hamiltonian.

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1. Introduction

It is well known that the cotangent bundle of a differentiable manifold M plays a very important role in the symplectic geometry and its applications, since this carries a canonical symplectic structure induced by the Liouville form. The Hamiltonian formalism seems to be more straightforward that the Lagrangian formalism, because on the tangent bundle one cannot naturally define a symplectic structure. On the contrary, the tangent bundle has a naturally defined integrable tangent structure, which together with the notion of semispray (second order differential equation vector field) induce a nonlinear connection [4].

The notion of Lie algebroid (E, π, M) (MACKENZIE [12]) is a generalization of the concepts of Lie algebra and integrable distribution. The notion of prolongation of a Lie algebroid [8] over the vector bundle projections of a vector bundle and its dual generalize the concepts of tangent and cotangent bundle. WEINSTEIN [24] gives a generalized theory of Lagrangian systems on Lie algebroids and obtains the equations of motion, using the Poisson structure on the dual and Legendre transformation induced by a regular Lagrangian. The same equations were later obtained by MARTINEZ [13] using a modified version of symplectic formalism, in which the bundles tangent to E and E^* are replaced by the prolongations $\mathcal{T}E$ and $\mathcal{T}E^*$. In the last years diverse aspects of these subjects were studied by many authors. In [11] the aspects of Hamiltonian formalism on the prolongation of a Lie algebroid over the vector bundle projection of the dual bundle, and equivalence with the Lagrangian formalism are presented. In [1], [6], [7], [14], [18], [20], [23] the geometric properties of connections on Lie algebroids are investigated and in [9], [19], [22] the study of geometric structures on its dual is presented. For applications of Lie algebroids to optimal control see [6], [21].

The purpose of this paper is to study the dual structures induced by the regular Lagrangians and Hamiltonians on the prolongations of Lie algebroids. The paper is organized as follows. The second section contains the known results on a Lie algebroid and its prolongation over the vector bundle projection. In section three, we introduce the notion of dynamical covariant derivative (see [2], [3], [5], [10] for the case of tangent bundle) on $\mathcal{T}E$ induced by a semispray. We prove that the canonical nonlinear connection induced by a regular Lagrangian is the unique connection which is metric and compatible with the symplectic structure.

In section four we study the properties of connections on $\mathcal{T}E^*$ and introduce the notion of almost complex structure and characterize the integrability conditions in terms of torsion and curvature of the connection. We introduce some generalizations of the Hamilton sections, as mechanical structures and semi-Hamiltonian sections and characterize their properties.

In the last part, using the diffeomorpfism from $\mathcal{T}E^*$ to $\mathcal{T}E$ [11], induced by a regular Hamiltonian, we transfer many geometrical structures between these spaces (see [15], [16], [17] for the case of tangent and cotangent bundles). Thus, a semispray on $\mathcal{T}E$ is transformed into a semi-Hamiltonian section on $\mathcal{T}E^*$ if and only if the nonlinear connection determined by semispray is the canonical nonlinear connection induced by the regular Lagrangian,

via Legendre transformation.

2. Preliminaries on Lie algebroids

Let M be a differentiable, n-dimensional manifold and (TM, π_M, M) its tangent bundle. A Lie algebroid over the manifold M is the triple $(E, [\cdot,\cdot]_E, \sigma)$ where $\pi : E \to M$ is a vector bundle of rank m over M, whose $C^\infty(M)$-module of sections $\Gamma(E)$ is equipped with a Lie algebra structure $[\cdot,\cdot]_E$ and $\sigma : E \to TM$ is a vector bundle homomorphism (called the anchor) which induces a Lie algebra homomorphism (also denoted σ) from $\Gamma(E)$ to $\chi(M)$, satisfying the Leibniz rule $[s_1, fs_2]_E = f[s_1, s_2]_E + (\sigma(s_1)f)s_2$, for every $f \in C^\infty(M)$ and $s_1,\ s_2 \in \Gamma(E)$. Therefore, we get

$$\sigma[s_1, s_2]_E = [\sigma(s_1), \sigma(s_2)];\ [s_1, [s_2, s_3]_E]_E + [s_2, [s_3, s_1]_E]_E + [s_3, [s_1, s_2]_E]_E = 0.$$

If $\omega \in \bigwedge^k(E^*)$ then the *exterior derivative* $d^E\omega \in \bigwedge^{k+1}(E^*)$ is given by the formula

$$\begin{aligned} d^E\omega(s_1, ..., s_{k+1}) &= \sum_{i=1}^{k+1}(-1)^{i+1}\sigma(s_i)\omega(s_1, ..., \hat{s_i}, ..., s_{k+1}) \\ &\quad + \sum_{1\le i<j\le k+1}(-1)^{i+j}\omega([s_i, s_j]_E, s_1, ..., \hat{s_i}, ..., \hat{s_j}, ...s_{k+1}), \end{aligned}$$

where $s_i \in \Gamma(E)$, $i = \overline{1, k+1}$, and it follows that $(d^E)^2 = 0$. Also, for $\xi \in \Gamma(E)$ one can define the *Lie derivative* with respect to ξ by $\mathcal{L}_\xi = i_\xi \circ d^E + d^E \circ i_\xi$, where i_ξ is the contraction with ξ. If we take the local coordinates (x^i) on an open subset $U \subset M$, a local basis $\{s_\alpha\}$ of sections of the bundle $\pi^{-1}(U) \to U$ generates local coordinates (x^i, y^α) on E. The local functions $\sigma^i_\alpha(x)$, $L^\gamma_{\alpha\beta}(x)$ on M given by

$$\sigma(s_\alpha) = \sigma^i_\alpha \frac{\partial}{\partial x^i}, \quad [s_\alpha, s_\beta]_E = L^\gamma_{\alpha\beta}s_\gamma, \quad i = \overline{1, n}, \quad \alpha, \beta, \gamma = \overline{1, m},$$

are called the *structure functions* of the Lie algebroid, and satisfy the so called *structure equations* on Lie algebroid

$$\sigma^j_\alpha \frac{\partial \sigma^i_\beta}{\partial x^j} - \sigma^j_\beta \frac{\partial \sigma^i_\alpha}{\partial x^j} = \sigma^i_\gamma L^\gamma_{\alpha\beta}, \quad \sum_{(\alpha,\beta,\gamma)} \left(\sigma^i_\alpha \frac{\partial L^\delta_{\beta\gamma}}{\partial x^i} + L^\delta_{\alpha\eta} L^\eta_{\beta\gamma} \right) = 0.$$

2.1. The prolongation of a Lie algebroid over the vector bundle projection

Let (E,π,M) be a vector bundle. For the projection $\pi:E\to M$ we can construct the prolongation of E (see [8], [13], [11]). The associated vector bundle is $(\mathcal{T}E,\pi_2,E)$ where $\mathcal{T}E=\cup_{w\in E}\mathcal{T}_wE$ with

$$\mathcal{T}_wE=\{(u_x,v_w)\in E_x\times T_wE\mid \sigma(u_x)=T_w\pi(v_w),\quad \pi(w)=x\in M\},$$

and the projection $\pi_2(u_x,v_w)=\pi_E(v_w)=w$, where $\pi_E:TE\to E$ is the tangent projection. We have also the canonical projection $\pi_1:\mathcal{T}E\to E$ given by $\pi_1(u,v)=u$. The projection onto the second factor $\sigma^1:\mathcal{T}E\to TE$, $\sigma^1(u,v)=v$ will be the anchor of a new Lie algebroid over the manifold E. An element of $\mathcal{T}E$ is said to be vertical if it is in the kernel of the projection π_1. We will denote $(V\mathcal{T}E,\pi_{2|_{V\mathcal{T}E}},E)$ the vertical bundle of $(\mathcal{T}E,\pi_2,E)$. The local basis of $\Gamma(\mathcal{T}E)$ is given by $\{\mathcal{X}_\alpha,\mathcal{V}_\alpha\}$, where [13]

$$\mathcal{X}_\alpha(u)=\left(s_\alpha(\pi(u)),\sigma^i_\alpha\frac{\partial}{\partial x^i}\Big|_u\right),\quad \mathcal{V}_\alpha(u)=\left(0,\frac{\partial}{\partial y^\alpha}\Big|_u\right),$$

and $(\partial/\partial x^i,\partial/\partial y^\alpha)$ is the local basis on TE. The structure functions of $\mathcal{T}E$ are given by the following formulas

$$\sigma^1(\mathcal{X}_\alpha)=\sigma^i_\alpha\frac{\partial}{\partial x^i},\quad \sigma^1(\mathcal{V}_\alpha)=\frac{\partial}{\partial y^\alpha},$$

$$[\mathcal{X}_\alpha,\mathcal{X}_\beta]_{\mathcal{T}E}=L^\gamma_{\alpha\beta}\mathcal{X}_\gamma,\quad [\mathcal{X}_\alpha,\mathcal{V}_\beta]_{\mathcal{T}E}=0,\quad [\mathcal{V}_\alpha,\mathcal{V}_\beta]_{\mathcal{T}E}=0.$$

The differential of sections of $(\mathcal{T}E)^*$ is determined by $d^E\mathcal{X}^\alpha=-\frac{1}{2}L^\alpha_{\beta\gamma}\mathcal{X}^\beta\wedge\mathcal{X}^\gamma$, $d^E\mathcal{V}^\alpha=0$. Other canonical geometric objects (see [11]) are *Euler section* $C=y^\alpha\mathcal{V}_\alpha$ and the *vertical endomorphism* or *tangent structure* $J=\mathcal{X}^\alpha\otimes\mathcal{V}_\alpha$. A section $\mathcal{S}$ of $\mathcal{T}E$ is called *semispray* (or *second order differential equation -SODE*) if $J(\mathcal{S})=C$. In local coordinates a semispray has the expression $\mathcal{S}(x,y)=y^\alpha\mathcal{X}_\alpha+\mathcal{S}^\alpha(x,y)\mathcal{V}_\alpha$.

A nonlinear connection N on $\mathcal{T}E$ is an $m-$dimensional distribution (called *horizontal distribution*) $N:u\in E\to H\mathcal{T}_uE\subset\mathcal{T}E$ that is supplementary to the vertical distribution. This means that we have the following decomposition $\mathcal{T}_uE=H\mathcal{T}_uE\oplus V\mathcal{T}_uE$, for $u\in E$. A connection N on $\mathcal{T}E$ induces two projectors $\mathrm{h},\mathrm{v}:\mathcal{T}E\to\mathcal{T}E$ such that $\mathrm{h}(\rho)=\rho^{\mathrm{h}}$ and $\mathrm{v}(\rho)=\rho^{\mathrm{v}}$ for every $\rho\in\Gamma(\mathcal{T}E)$, where $\mathrm{h}=\frac{1}{2}(id+N)$, $\mathrm{v}=\frac{1}{2}(id-N)$. Locally, a connection can be expressed as $N(\mathcal{X}_\alpha)=\mathcal{X}_\alpha-2N^\beta_\alpha\mathcal{V}_\beta$, $N(\mathcal{V}_\beta)=-\mathcal{V}_\beta$, where $N^\beta_\alpha(x,y)$

are the local coefficients of N. The sections $\delta_\alpha = (\mathcal{X}_\alpha)^{\rm h} = \mathcal{X}_\alpha - N^\beta_\alpha \mathcal{V}_\beta$, generate a basis of $H\mathcal{T}E$. The frame $\{\delta_\alpha, \mathcal{V}_\alpha\}$ is a local basis of $\mathcal{T}E$ called the Berwald basis. The dual basis is $\{\mathcal{X}^\alpha, \delta\mathcal{V}^\alpha\}$ where $\delta\mathcal{V}^\alpha = \mathcal{V}^\alpha - N^\alpha_\beta \mathcal{X}^\beta$. The Lie brackets of the Berwald basis $\{\delta_\alpha, \mathcal{V}_\alpha\}$ are [20]

$$[\delta_\alpha, \delta_\beta]_{\mathcal{T}E} = L^\gamma_{\alpha\beta}\delta_\gamma + \mathcal{R}^\gamma_{\alpha\beta}\mathcal{V}_\gamma,\ [\delta_\alpha, \mathcal{V}_\beta]_{\mathcal{T}E} = \frac{\partial N^\gamma_\alpha}{\partial y^\beta}\mathcal{V}_\gamma,\ [\mathcal{V}_\alpha, \mathcal{V}_\beta]_{\mathcal{T}E} = 0,$$

$$\mathcal{R}^\gamma_{\alpha\beta} = \sigma^i_\beta \frac{\partial N^\gamma_\alpha}{\partial x^i} - \sigma^i_\alpha \frac{\partial N^\gamma_\beta}{\partial x^i} - \mathcal{N}^\varepsilon_\beta \frac{\partial N^\gamma_\alpha}{\partial y^\varepsilon} + \mathcal{N}^\varepsilon_\alpha \frac{\partial N^\gamma_\beta}{\partial y^\varepsilon} + L^\varepsilon_{\alpha\beta} N^\gamma_\varepsilon. \tag{2.1}$$

The curvature of a connection N on $\mathcal{T}E$ is given by $\Omega = -{\rm N}_{\rm h}$ where h is the horizontal projector and ${\rm N}_{\rm h}$ is the Nijenhuis tensor of h. In the local coordinates we have $\Omega = -\frac{1}{2}\mathcal{R}^\gamma_{\alpha\beta}\mathcal{X}^\alpha \wedge \mathcal{X}^\beta \otimes \mathcal{V}_\gamma$, where $\mathcal{R}^\gamma_{\alpha\beta}$ are given by (2.1) and represent the local coordinate functions of the curvature.

A semispray $\mathcal{S}$ with local coefficients $\mathcal{S}^\alpha$ determines an associated nonlinear connection $N = -\mathcal{L}_S J$ with local coefficients

$$N^\beta_\alpha = \frac{1}{2}\left(-\frac{\partial \mathcal{S}^\beta}{\partial y^\alpha} + y^\varepsilon L^\beta_{\alpha\varepsilon}\right). \tag{2.2}$$

Let us consider a regular Lagrangian L on E, i.e. the matrix $g_{\alpha\beta} = \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta}$ has constant rank m and is called the *metric tensor* of the space (E, L). The symplectic structure induced by the regular Lagrangian is [13]

$$\omega_L = \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta}\mathcal{V}^\beta \wedge \mathcal{X}^\alpha + \frac{1}{2}\left(\frac{\partial^2 L}{\partial x^i \partial y^\beta}\sigma^i_\alpha - \frac{\partial^2 L}{\partial x^i \partial y^\alpha}\sigma^i_\beta - \frac{\partial L}{\partial y^\gamma}L^\gamma_{\alpha\beta}\right)\mathcal{X}^\alpha \wedge \mathcal{X}^\beta.$$

and one can associate to L a remarkable *Lagrangian semispray*, given by

$$\mathcal{S}^\varepsilon = g^{\varepsilon\beta}\left(\sigma^i_\beta \frac{\partial L}{\partial x^i} - \sigma^i_\alpha \frac{\partial^2 L}{\partial x^i \partial y^\beta}y^\alpha - L^\theta_{\beta\alpha}y^\alpha \frac{\partial L}{\partial y^\theta}\right), \tag{2.3}$$

where $g_{\alpha\beta}g^{\beta\gamma} = \delta^\gamma_\alpha$.

The connection N with the coefficients given by (2.2), determined by the Lagrangian semispray from (2.3) will be called a *canonical nonlinear connection* induced by a regular Lagrangian L. Its coefficients are given by

$$\begin{aligned} N^\alpha_\beta \;=\; & \frac{1}{2}g^{\alpha\varepsilon}\Big[S(g_{\varepsilon\beta}) + \sigma^i_\beta \frac{\partial^2 L}{\partial x^i \partial y^\varepsilon} - \sigma^i_\varepsilon \frac{\partial^2 L}{\partial x^i \partial y^\beta} - L^\gamma_{\beta\varepsilon}\frac{\partial L}{\partial y^\gamma} \\ & + \left(g_{\gamma\varepsilon}L^\gamma_{\beta\theta} + g_{\gamma\beta}L^\gamma_{\varepsilon\theta}\right)y^\theta\Big]. \end{aligned} \tag{2.4}$$

3. Dynamical covariant derivative

Let us consider a semispray $\mathcal{S}$ and the associated nonlinear connection N with the coefficients (2.2).

Definition 1. The dynamical covariant derivative that corresponds to $\mathcal{S}$ is defined by $\nabla : \Gamma(V\mathcal{T}E) \to \Gamma(V\mathcal{T}E)$ through

$$\nabla\left(\rho^{\alpha}\mathcal{V}_{\alpha}\right) = \left[\mathcal{S}(\rho^{\alpha}) - \frac{1}{2}\left(\frac{\partial \mathcal{S}^{\alpha}}{\partial y^{\beta}} + L^{\alpha}_{\beta\varepsilon}y^{\varepsilon}\right)\rho^{\beta}\right]\mathcal{V}_{\alpha}.$$

With respect to the local frame of $V\mathcal{T}E$ we have

$$\nabla\left(\mathcal{V}_{\alpha}\right) = -\frac{1}{2}\left(\frac{\partial \mathcal{S}^{\beta}}{\partial y^{\alpha}} + L^{\beta}_{\alpha\varepsilon}y^{\varepsilon}\right)\mathcal{V}_{\beta}.$$

In the case of standard Lie algebroid ($E = TM$, $\sigma = id$, $L^{\alpha}_{\beta\gamma} = 0$) we obtain the covariant derivative D from [2] or Γ-derivative in [10] (other point of view in given in [23]). Dynamical covariant derivative has the following properties

$$\begin{aligned}
&\nabla(\rho_1 + \rho_2) = \nabla\rho_1 + \nabla\rho_2, \quad \rho_1, \rho_2 \in \Gamma(V\mathcal{T}E),\\
&\nabla(f\rho) = S(f)\rho + f\nabla\rho, \quad \rho \in \Gamma(V\mathcal{T}E), \quad f \in C^{\infty}(M).
\end{aligned}$$

It is not difficult to extend the action of ∇ to the algebra of E -tensor by requiring for ∇ to preserve the tensor product. A E-tensor means a tensor on $\mathcal{T}E$, whose components, under a change of coordinates on $\mathcal{T}E$ behave like the components of a tensor on the base manifold E. For the metric g on $V\mathcal{T}E$ (i.e. a $(2,0)$-type symmetric tensor $g = g_{\alpha\beta}\mathcal{V}^{\alpha} \otimes \mathcal{V}^{\beta}$ of rank m on $V\mathcal{T}E$) we have $(\nabla g)(\rho_1, \rho_2) = \mathcal{S}(g(\rho_1, \rho_2)) - g(\nabla\rho_1, \rho_2) - g(\rho_1, \nabla\rho_2)$. In local coordinates we get

$$(\nabla g)(\mathcal{V}_{\alpha}, \mathcal{V}_{\beta}) = \mathcal{S}(g_{\alpha\beta}) + \frac{1}{2}\frac{\partial \mathcal{S}^{\gamma}}{\partial y^{\alpha}}g_{\gamma\beta} + \frac{1}{2}\frac{\partial \mathcal{S}^{\gamma}}{\partial y^{\beta}}g_{\gamma\alpha} + \frac{1}{2}y^{\varepsilon}\left(g_{\gamma\beta}L^{\gamma}_{\alpha\varepsilon} + g_{\gamma\alpha}L^{\gamma}_{\beta\varepsilon}\right),$$

which is equivalent to

$$(\nabla g)(\mathcal{V}_{\alpha}, \mathcal{V}_{\beta}) = \mathcal{S}(g_{\alpha\beta}) - g_{\gamma\beta}N^{\gamma}_{\alpha} - g_{\gamma\alpha}N^{\gamma}_{\beta} + y^{\varepsilon}\left(g_{\gamma\beta}L^{\gamma}_{\alpha\varepsilon} + g_{\gamma\alpha}L^{\gamma}_{\beta\varepsilon}\right).$$

Definition 2. ∇ is compatible with the metric g if $\nabla g = 0$ that is

$$\mathcal{S}(g(\rho_1, \rho_2)) = g(\nabla\rho_1, \rho_2) + g(\rho_1, \nabla\rho_2).$$

Let g be the metric tensor induced by a regular Lagrangian L. As in [23] we obtain:

Theorem 1. *The dynamical covariant derivative induced by a Lagrangian semispray is compatible with g.*

Theorem 2. *A nonlinear connection induced by a semispray satisfying*

$$\nabla g = 0, \tag{3.5}$$

$$\omega_L(\mathrm{h}\rho, \mathrm{h}\nu) = 0, \quad \forall \rho, \nu \in \Gamma(\mathcal{T}E \backslash \{0\}). \tag{3.6}$$

can only be the canonical nonlinear connection induced by a regular Lagrangian.

Proof. The equation $\omega_L(\mathrm{h}\rho, \mathrm{h}\nu) = 0$ yields [23]

$$N_{[\alpha,\beta]} = \frac{1}{2}(N_{\alpha\beta} - N_{\beta\alpha}) = \frac{1}{2}\left(\frac{\partial^2 L}{\partial x^i \partial y^\alpha}\sigma^i_\beta - \frac{\partial^2 L}{\partial x^i \partial y^\beta}\sigma^i_\alpha + \frac{\partial L}{\partial y^\varepsilon}L^\varepsilon_{\alpha\beta}\right),$$

where $N_{\alpha\beta} := g_{\alpha\gamma}N^\gamma_\beta$. It follows that the skew symmetric part of $N_{\alpha\beta}$ is uniquely determined by the condition (3.6). The symmetric part of $N_{\alpha\beta}$ is completely determined by the metric condition (3.5). Indeed, we get

$$2N_{(\alpha\beta)} = N_{\alpha\beta} + N_{\beta\alpha} = \mathcal{S}(g_{\alpha\beta}) + \left(g_{\gamma\beta}L^\gamma_{\alpha\varepsilon} + g_{\gamma\alpha}L^\gamma_{\beta\varepsilon}\right)y^\varepsilon.$$

The equations (3.5) and (3.6) uniquely determine the coefficients of the nonlinear connection $N^\gamma_\beta = g^{\gamma\alpha}N_{\alpha\beta} = g^{\gamma\alpha}(N_{(\alpha\beta)} + N_{[\alpha\beta]})$, and it results the canonical nonlinear connection (2.4). For the particular case of $(TM, [,], id)$ see for instance [2]. □

4. The prolongation of a Lie algebroid to its dual bundle

Let $\tau : E^* \to M$ be the dual bundle of $\pi : E \to M$ and $(E, [\cdot,\cdot]_E, \sigma)$ a Lie algebroid structure over M. One can construct a Lie algebroid structure over E^*, by taking the prolongation over $\tau : E^* \to M$ (see [8], [11], [9]). The associated vector bundle is $(\mathcal{T}E^*, \tau_1, E^*)$ where $\mathcal{T}E^* = \bigcup_{u^* \in E^*} \mathcal{T}_{u^*}E^*$,

$$\mathcal{T}_{u^*}E^* = \{(u_x, v_{u^*}) \in E_x \times T_{u^*}E^* | \sigma(u_x) = T_{u^*}\tau(v_{u^*}), \tau(u^*) = x \in M\},$$

and the projection $\tau_1 : \mathcal{T}E^* \to E^*$, $\tau_1(u_x, v_{u^*}) = u^*$. The anchor is the projection $\sigma^1 : \mathcal{T}E^* \to TE^*$, $\sigma^1(u, v) = v$. Notice that if $\mathcal{T}\tau : \mathcal{T}E^* \to$

E, $\mathcal{T}\tau(u,v)=u$ then $(V\mathcal{T}E^*,\tau_{1|V\mathcal{T}E^*},E^*)$ with $V\mathcal{T}E^*=Ker\mathcal{T}\tau$ is a subbundle of $(\mathcal{T}E^*,\tau_1,E^*)$, called the *vertical subbundle*. If (q^i,μ_α) are local coordinates on E^* at u^* and $\{s_\alpha\}$ is a local basis of sections of $\pi:E\to M$ then a local basis of $\Gamma(\mathcal{T}E^*)$ is $\{\mathcal{Q}_\alpha,\mathcal{P}^\alpha\}$ where [11]

$$\mathcal{Q}_\alpha(u^*)=\left(s_\alpha(\tau(u^*)),\sigma_\alpha^i\frac{\partial}{\partial q^i}|_{u^*}\right),\quad \mathcal{P}^\alpha(u^*)=\left(0,\frac{\partial}{\partial\mu_\alpha}|_{u^*}\right).$$

The structure functions on $\mathcal{T}E^*$ are given by the following formulas

$$\sigma^1(\mathcal{Q}_\alpha)=\sigma_\alpha^i\frac{\partial}{\partial q^i},\quad \sigma^1(\mathcal{P}^\alpha)=\frac{\partial}{\partial\mu_\alpha},$$
$$[\mathcal{Q}_\alpha,\mathcal{Q}_\beta]_{\mathcal{T}E^*}=L^\gamma_{\alpha\beta}\mathcal{Q}_\gamma,\quad [\mathcal{Q}_\alpha,\mathcal{P}^\alpha]_{\mathcal{T}E^*}=0,\quad [\mathcal{P}^\alpha,\mathcal{P}^\beta]_{\mathcal{T}E^*}=0,$$

and therefore

$$d^E\mathcal{Q}^\gamma=-\frac{1}{2}L^\gamma_{\alpha\beta}\mathcal{Q}^\alpha\wedge\mathcal{Q}^\beta,\quad d^E\mathcal{P}_\alpha=0,\quad d^Eq^i=\sigma^i_\alpha\mathcal{Q}^\alpha,\quad d^E\mu_\alpha=\mathcal{P}_\alpha,$$

where $\{\mathcal{Q}^\alpha,\mathcal{P}_\alpha\}$ is the dual basis of $\{\mathcal{Q}_\alpha,\mathcal{P}^\alpha\}$. In local coordinates the *Liouville section* is given by $\theta_E=\mu_\alpha\mathcal{Q}^\alpha$. The *canonical symplectic structure* ω_E is defined by $\omega_E=-d^E\theta_E$. It follows that ω_E is a non degenerate 2-section, $d^E\omega_E=0$ and $\omega_E=\mathcal{Q}^\alpha\wedge\mathcal{P}_\alpha+\frac{1}{2}\mu_\alpha L^\alpha_{\beta\gamma}\mathcal{Q}^\beta\wedge\mathcal{Q}^\gamma$. The *Liouville-Hamilton section* $\mathcal{C}$ has local expression $\mathcal{C}=\mu_\alpha\mathcal{P}^\alpha$. We remark that $V\mathcal{T}E^*$ is Lagrangian for ω_E, i.e. $\omega_E(\rho_1,\rho_2)=0$, for every vertical sections $\rho_1,\rho_2\in\Gamma(V\mathcal{T}E^*)$. A nonlinear connection on $\mathcal{T}E^*$ is an almost product structure $\mathcal{N}$ on $\tau_1:\mathcal{T}E^*\to E^*$(i.e. a bundle morphism $\mathcal{N}:\mathcal{T}E^*\to\mathcal{T}E^*$, such that $\mathcal{N}^2=id$) smooth on $\mathcal{T}E^*\backslash\{0\}$ such that $V\mathcal{T}E^*=\ker(id+\mathcal{N})$. If $\mathcal{N}$ is a connection on $\mathcal{T}E^*$ then $H\mathcal{T}E^*=\ker(id-\mathcal{N})$ is the horizontal distribution associated to $\mathcal{N}$ and $\mathcal{T}E^*=V\mathcal{T}E^*\oplus H\mathcal{T}E^*$. A connection $\mathcal{N}$ on $\mathcal{T}E^*$ induces two projectors $\mathrm{h},\mathrm{v}:\mathcal{T}E^*\to\mathcal{T}E^*$ such that $\mathrm{h}(\rho)=\rho^{\mathrm{h}}$ and $\mathrm{v}(\rho)=\rho^{\mathrm{v}}$ for every $\rho\in\Gamma(\mathcal{T}E^*)$, given by $\mathrm{h}=\frac{1}{2}(id+\mathcal{N})$ and $\mathrm{v}=\frac{1}{2}(id-\mathcal{N})$. The local sections $\{\mathcal{P}^\alpha\}_{\alpha=\overline{1,m}}$ define a local frame of $V\mathcal{T}E^*$, and the sections $\delta^*_\alpha=(\mathcal{Q}_\alpha)^{\mathrm{h}}=\mathcal{Q}_\alpha+\mathcal{N}_{\alpha\beta}\mathcal{P}^\beta$, generate a local frame of $H\mathcal{T}E^*$. The frame $\{\delta^*_\alpha,\mathcal{P}^\alpha\}$ is a local basis of $\mathcal{T}E^*$ called *Berwald basis*. The dual basis is $\{\mathcal{Q}^\alpha,\delta\mathcal{P}_\alpha\}$ where $\delta\mathcal{P}_\alpha=\mathcal{P}_\alpha-\mathcal{N}_{\alpha\beta}\mathcal{Q}^\beta$. A connection $\mathcal{N}$ is called symmetric if $H\mathcal{T}E^*$ is Lagrangian for ω_E, that is $\omega_E(\mathrm{h}\rho_1,\mathrm{h}\rho_2)=0$, $\forall\rho_1,\rho_2\in\Gamma(\mathcal{T}E^*)$, and it follows that $\mathcal{N}$ is symmetric if and only if $\mathcal{N}_{\alpha\beta}-\mathcal{N}_{\beta\alpha}=\mu_\gamma L^\gamma_{\alpha\beta}$. The Lie brackets of the adapted basis $\{\delta^*_\alpha,\mathcal{P}^\alpha\}$ are [9]

$$[\delta^*_\alpha,\delta^*_\beta]_{\mathcal{T}E^*}=L^\gamma_{\alpha\beta}\delta^*_\gamma+\mathcal{R}_{\alpha\beta\gamma}\mathcal{P}^\gamma;[\delta^*_\alpha,\mathcal{P}^\beta]_{\mathcal{T}E^*}=-\frac{\partial\mathcal{N}_{\alpha\gamma}}{\partial\mu_\beta}\mathcal{P}^\gamma;[\mathcal{P}^\alpha,\mathcal{P}^\beta]_{\mathcal{T}E^*}=0,$$

$$\mathcal{R}_{\alpha\beta\gamma} = \delta^*_\alpha(\mathcal{N}_{\beta\gamma}) - \delta^*_\beta(\mathcal{N}_{\alpha\gamma}) - L^\varepsilon_{\alpha\beta}\mathcal{N}_{\varepsilon\gamma}. \tag{4.7}$$

The curvature of a connection $\mathcal{N}$ on $\mathcal{T}E^*$ is given by $\Omega = -\mathrm{N_h}$ where h is horizontal projector and $\mathrm{N_h}$ is the Nijenhuis tensor of h. In local coordinates $\Omega = -\frac{1}{2}\mathcal{R}_{\alpha\beta\gamma}\mathcal{Q}^\alpha \wedge \mathcal{Q}^\beta \otimes \mathcal{P}^\gamma$, where $\mathcal{R}_{\alpha\beta\gamma}$ is given by (4.7) and is called the *curvature tensor* of $\mathcal{N}$. The curvature is an obstruction to the integrability of $H\mathcal{T}E^*$, understanding that a vanishing curvature entails that horizontal sections are closed under the Lie algebroid bracket of $\mathcal{T}E^*$.

Definition 3. An almost tangent structure $\mathcal{J}$ on $\mathcal{T}E^*$ is a bundle morphism $\mathcal{J} : \mathcal{T}E^* \to \mathcal{T}E^*$ of $\tau_1 : \mathcal{T}E^* \to E^*$ of rank m, such that $\mathcal{J}^2 = 0$. An almost tangent structure $\mathcal{J}$ on $\mathcal{T}E^*$ is called *adapted* if $im\mathcal{J} = \ker\mathcal{J} = V\mathcal{T}E^*$.

Locally, an adapted almost tangent structure is given by $\mathcal{J} = t_{\alpha\beta}\mathcal{Q}^\alpha \otimes \mathcal{P}^\beta$, where the matrix $(t_{\alpha\beta}(x,\mu))$ is nondegenerate. It follows that $\mathcal{J}$ is an integrable structure if and only if [9]

$$\frac{\partial t^{\alpha\gamma}}{\partial \mu_\beta} = \frac{\partial t^{\beta\gamma}}{\partial \mu_\alpha}, \tag{4.8}$$

where $t^{\alpha\gamma}t_{\gamma\beta} = \delta^\alpha_\beta$. An adapted almost tangent structure $\mathcal{J}$ on $\mathcal{T}E^*$ is called symmetric if

$$\omega_E(\mathcal{J}\rho_1, \rho_2) = \omega_E(\mathcal{J}\rho_2, \rho_1), \quad \forall \rho_1, \rho_2 \in \Gamma(\mathcal{T}E^*).$$

Locally, this requires the symmetry of $t_{\alpha\beta}$. If g is a pseudo-Riemannian metric on the vertical subbundle $V\mathcal{T}E^*$ (i.e. a $(0,2)$-type symmetric E-tensor $g = g^{\alpha\beta}(q,\mu)\mathcal{P}_\alpha \otimes \mathcal{P}_\beta$ of rank m on $\mathcal{T}E^*$) then there exists a unique symmetric adapted almost tangent structure on $\mathcal{T}E^*$ such that

$$g(\mathcal{J}\rho, \mathcal{J}\upsilon) = -\omega_E(\mathcal{J}\rho, \upsilon), \quad \forall \rho_1, \rho_2 \in \Gamma(\mathcal{T}E^*), \tag{4.9}$$

and we say that $\mathcal{J}$ is induced by the metric g. Locally, the relation (4.9) implies $t^{\alpha\beta} = g^{\alpha\beta}$.

The torsion of a connection $\mathcal{N}$ is the vector valued two form $T := [\mathcal{J}, \mathrm{h}]$ where h is the horizontal projector and $[\mathcal{J}, \mathrm{h}]$ is the Frolicher-Nijenhuis bracket. The torsion T is a semibasic vector-valued form. Its local expression is [9]

$$T = \frac{1}{2}T_{\alpha\beta\gamma}\mathcal{Q}^\alpha \wedge \mathcal{Q}^\beta \otimes \mathcal{P}^\gamma,$$

$$T_{\alpha\beta\gamma} = t_{\alpha\epsilon}\frac{\partial \mathcal{N}_{\beta\gamma}}{\partial \mu_\varepsilon} - t_{\beta\epsilon}\frac{\partial \mathcal{N}_{\alpha\gamma}}{\partial \mu_\varepsilon} + \delta^*_\alpha(t_{\beta\gamma}) - \delta^*_\beta(t_{\alpha\gamma}) - L^\epsilon_{\alpha\beta}t_{\epsilon\gamma}.$$

Let us consider the linear mapping $\mathcal{F} : \mathcal{T}E^* \to \mathcal{T}E^*$ given by $\mathcal{F}(\mathrm{h}\rho) = \mathcal{J}\rho$, $\mathcal{F}(\mathcal{J}\rho) = -\mathrm{h}\rho$, where $\rho \in \Gamma(\mathcal{T}E^*)$ and h is the horizontal projector induced by nonlinear connection.

Proposition 1. *The mapping $\mathcal{F}$ has the properties:*
a) $\mathcal{F}$ is an almost complex structure $\mathcal{F} \circ \mathcal{F} = -id$.
b) Locally it is given by $\mathcal{F} = t_{\alpha\beta}\mathcal{P}^\beta \otimes \mathcal{Q}^\alpha - t^{\alpha\beta}\delta^*_\alpha \otimes \delta\mathcal{P}_\beta$.

Proof. It follows by definition that

$$(\mathcal{F} \circ \mathcal{F})(\mathrm{h}\rho) = \mathcal{F}(\mathcal{J}(\rho)) = -\mathrm{h}\rho, \quad (\mathcal{F} \circ \mathcal{F})(\mathcal{J}\rho) = \mathcal{F}(-\mathrm{h}\rho) = -\mathcal{J}\rho,$$

$$\mathcal{F}(\delta^*_\alpha) = t_{\alpha\beta}\mathcal{P}^\beta, \quad \mathcal{F}(\mathcal{P}^\alpha) = -t^{\alpha\beta}\delta^*_\beta,$$

which concludes the proof. □

Proposition 2. *The almost complex structure is integrable if and only if the torsion and curvature of the connection satisfy the relations $T_{\alpha\beta\gamma} = 0$, $\mathcal{R}_{\alpha\beta\gamma} = t_{\alpha\varepsilon}\frac{\partial t_{\beta\gamma}}{\partial \mu_\varepsilon} - t_{\beta\varepsilon}\frac{\partial t_{\alpha\gamma}}{\partial \mu_\varepsilon}$.*

Proof. Let $\mathrm{N}_\mathcal{F}$ be the Nijenjuis tensor of the almost complex structure. We get

$$\mathrm{N}_\mathcal{F}(\delta^*_\alpha, \delta^*_\beta) = \mathrm{N}^\gamma_{\alpha\beta}\delta^*_\gamma + \mathrm{N}_{\alpha\beta(\gamma)}\mathcal{P}^\gamma, \ \ \mathrm{N}_\mathcal{F}(\delta^*_\alpha, \mathcal{P}^\beta) = \mathrm{N}^{(\beta)\gamma}_\alpha\delta^*_\gamma + \mathrm{N}^{(\beta)}_{\alpha(\gamma)}\mathcal{P}^\gamma,$$
$$\mathrm{N}_\mathcal{F}(\mathcal{P}^\alpha, \mathcal{P}^\beta) = -t^{\varepsilon\alpha}t^{\beta\tau}\mathrm{N}_\mathcal{F}(\delta^*_\varepsilon, \delta^*_\tau),$$

where

$$\mathrm{N}^\gamma_{\alpha\beta} = T_{\alpha\beta\varepsilon}t^{\varepsilon\gamma}, \quad \mathrm{N}_{\alpha\beta(\gamma)} = t_{\alpha\varepsilon}\frac{\partial t_{\beta\gamma}}{\partial \mu_\varepsilon} - t_{\beta\varepsilon}\frac{\partial t_{\alpha\gamma}}{\partial \mu_\varepsilon} - \mathcal{R}_{\alpha\beta\gamma},$$
$$\mathrm{N}_{\alpha\beta(\gamma)} = \mathrm{N}^{(\varepsilon)\tau}_\alpha t_{\varepsilon\beta}t_{\gamma\tau}, \quad \mathrm{N}^\gamma_{\alpha\beta} = \mathrm{N}^{(\varepsilon)}_{\alpha(\tau)}t^{\tau\gamma}t_{\beta\varepsilon},$$

which ends the proof. □

Definition 4. Let $\mathcal{J}$ be an adapted almost tangent structure on $\mathcal{T}E^*$. A section ρ of $\mathcal{T}E^*$ is called $\mathcal{J}-$ *regular* if

$$\mathcal{J}[\rho, \mathcal{J}\nu]_{\mathcal{T}E^*} = -\mathcal{J}\nu, \quad \forall \nu \in \Gamma(\mathcal{T}E^*). \tag{4.10}$$

Locally, the section $\rho = \xi^\alpha \mathcal{Q}_\alpha + \rho_\beta \mathcal{P}^\beta$ is $\mathcal{J}-$regular if and only if [9] $t^{\alpha\beta} = \frac{\partial \xi^\beta}{\partial \mu_\alpha}$, where $t^{\alpha\beta} t_{\alpha\gamma} = \delta^\beta_\gamma$. We have to remark that if the equation (4.10) is satisfied for any section $\nu \in \Gamma(\mathcal{T}E^*)$ with $rank[t^{\alpha\beta}] = m$, then $\mathcal{J}$ is an integrable structure. Indeed, we have

$$\frac{\partial t^{\alpha\beta}}{\partial \mu_\gamma} = \frac{\partial^2 \xi^\beta}{\partial \mu_\gamma \partial \mu_\alpha} = \frac{\partial^2 \xi^\beta}{\partial \mu_\alpha \partial \mu_\gamma} = \frac{\partial t^{\gamma\beta}}{\partial \mu_\alpha},$$

and using (4.8) it follows that $\mathcal{J}$ is integrable.

Remark 1. Let $\mathcal{J}$ be an adapted almost tangent structure on $\mathcal{T}E^*$. If ρ is a $\mathcal{J}-$regular section of $\mathcal{T}E^*$ then

$$\mathcal{N} = -\mathcal{L}_\rho \mathcal{J}, \tag{4.11}$$

is a connection on $\mathcal{T}E^*$.

Its local coefficients are given by [9]

$$\mathcal{N}_{\alpha\beta} = \frac{1}{2}\left(t_{\alpha\gamma} \frac{\partial \rho_\beta}{\partial \mu_\gamma} - \sigma^i_\alpha t_{\gamma\beta} \frac{\partial \xi^\gamma}{\partial q^i} - \rho(t_{\alpha\beta}) + \xi^\gamma t_{\lambda\beta} L^\lambda_{\gamma\alpha} \right). \tag{4.12}$$

Definition 5. A section ψ on $\mathcal{T}E^*$ is called a *Hamilton section* if it is $\mathcal{J}-$regular and $\mathcal{L}_\psi \omega_E = 0$, where ω_E is the canonical symplectic section.

If $\psi = \xi^\alpha \mathcal{Q}_\alpha + \rho_\alpha \mathcal{P}^\alpha$ then the condition $\mathcal{L}_\psi \omega_E = 0$ is expressed locally by

$$\begin{aligned}
&i) \quad \frac{\partial \xi^\beta}{\partial \mu_\alpha} = \frac{\partial \xi^\alpha}{\partial \mu_\beta}, \\
&ii) \quad \sigma^i_\alpha \frac{\partial \xi^\beta}{\partial q^i} + \frac{\partial \rho_\alpha}{\partial \mu_\beta} = \xi^\gamma L^\beta_{\gamma\alpha} + \mu_\varepsilon L^\varepsilon_{\gamma\alpha} \frac{\partial \xi^\gamma}{\partial \mu_\beta}, \\
&iii) \quad \sigma^i_\beta \frac{\partial \rho_\alpha}{\partial q^i} - \sigma^i_\alpha \frac{\partial \rho_\beta}{\partial q^i} = \mu_\varepsilon \xi^\gamma \left(\sigma^i_\beta \frac{\partial L^\varepsilon_{\gamma\alpha}}{\partial q^i} - \sigma^i_\alpha \frac{\partial L^\varepsilon_{\gamma\beta}}{\partial q^i} + L^\varepsilon_{\nu\gamma} L^\nu_{\alpha\beta} \right) \\
&\qquad\qquad + \mu_\varepsilon \frac{\partial \xi^\gamma}{\partial q^i} \left(\sigma^i_\beta L^\varepsilon_{\gamma\alpha} - \sigma^i_\alpha L^\varepsilon_{\gamma\beta} \right) - \rho_\gamma L^\gamma_{\alpha\beta}.
\end{aligned} \tag{4.13}$$

Now, we deal with some generalizations of the Hamilton sections (see for instance [17] and its references for the particular case of the cotangent bundle).

Definition 6. The section $\psi = \xi^\alpha \mathcal{Q}_\alpha + \rho_\alpha \mathcal{P}^\alpha$ on $\mathcal{T}E^*$ defines a mechanical structure if ψ is $\mathcal{J}-$regular and $\omega_E(\mathcal{J}\rho_1, \rho_2) = \omega_E(\mathcal{J}\rho_2, \rho_1)$, for any $\rho_1, \rho_2 \in \Gamma(\mathcal{T}E^*)$.

The definition is equivalent with the symmetry of $t^{\alpha\beta} = \frac{\partial \xi^\beta}{\partial \mu_\alpha}$, which means that the property (4.13) i) is fulfilled.

Proposition 3. *The section ψ on $\mathcal{T}E^*$ defines a mechanical structure if and only if $\mathcal{L}_\psi \omega_E(\nu_1, \nu_2) = 0$, whenever $\nu_1, \nu_2 \in \Gamma(V\mathcal{T}E^*)$.*

Proof. Let us consider $\rho_1, \rho_2 \in \Gamma(\mathcal{T}E^*)$ and $\nu_1 = \mathcal{J}\rho_1$, $\nu_2 = \mathcal{J}\rho_2$. Then

$$\mathcal{L}_\psi \omega_E(\mathcal{J}\rho_1, \mathcal{J}\rho_2) = \psi(\omega_E(\mathcal{J}\rho_1, \mathcal{J}\rho_2) - \omega_E((\mathcal{L}_\psi \mathcal{J}\rho_1, \mathcal{J}\rho_2) - \omega_E((\mathcal{J}\rho_1, \mathcal{L}_\psi \mathcal{J}\rho_2) = \omega_E((\mathcal{N}\rho_1, \mathcal{J}\rho_2) + \omega_E((\mathcal{J}\rho_1, \mathcal{N}\rho_2).$$

Considering $\mathcal{N}\rho_1, \mathcal{N}\rho_2$ for ρ_1, ρ_2 in previous relation and using the properties $\mathcal{J}\mathcal{N} = \mathcal{J}$, $\mathcal{N}\mathcal{J} = -\mathcal{J}$ and $\mathcal{N}^2 = id$ we obtain

$$\mathcal{L}_\psi \omega_E(\mathcal{J}\rho_1, \mathcal{J}\rho_2) = \mathcal{L}_\psi \omega_E(\mathcal{J}\mathcal{N}\rho_1, \mathcal{J}\mathcal{N}\rho_2) = \omega_E(\rho_1, \mathcal{J}\rho_2) + \omega_E(\mathcal{J}\rho_1, \rho_2),$$

which ends the proof. □

Definition 7. The section ψ on $\mathcal{T}E^*$ is a semi-Hamilton section if ψ is $\mathcal{J}-$regular and $i_\upsilon(\mathcal{L}_\psi \omega_E) = 0$, whenever $\upsilon \in \Gamma(V\mathcal{T}E^*)$.

We have to remark that in the case of a semi-Hamilton section on $\mathcal{T}E^*$ only the conditions (4.13), i) and ii) are satisfied. Let us consider a section ψ defining a mechanical structure on $\mathcal{T}E^*$, then we can find the other connection. Indeed, we consider $g(\rho_1, \rho_2) = -\omega_E(\mathcal{J}\rho_1, \rho_2)$, for $\rho_1, \rho_2 \in \Gamma(\mathcal{T}E^*)$. It follows that $g(\rho_1, \rho_2) = g(\rho_2, \rho_1)$ and $g(\upsilon, \rho) = 0$ whenever $\upsilon \in \Gamma(V\mathcal{T}E^*)$. The local coordinate expression of g is given by $g = g_{\alpha\beta}\mathcal{Q}^\alpha \otimes \mathcal{Q}^\beta$.

Proposition 4. *If ψ defines a mechanical structure on $\mathcal{T}E^*$ then the section $\mathcal{N}'$ defined by*

$$\omega_E(\mathcal{N}'\rho_1, \rho_2) = (\mathcal{L}_\psi g)(\rho_1, \rho_2), \quad \rho_1, \rho_2 \in \Gamma(\mathcal{T}E^*), \tag{4.14}$$

determines a connection on $\mathcal{T}E^*$.

Proof. First, we prove that the following relation holds

$$(\mathcal{L}_\psi g)(\rho_1, \rho_2) = -(\mathcal{L}_\psi \omega_E)(\mathcal{J}\rho_1, \rho_2) + \omega_E(\mathcal{N}\rho_1, \rho_2).$$

Indeed, we get

$$(\mathcal{L}_\psi g)(\rho_1,\rho_2)=\psi(g(\rho_1,\rho_2))-g(\mathcal{L}_\psi\rho_1,\rho_2)-g(\rho_1,\mathcal{L}_\psi\rho_2)$$
$$=-\psi(\omega_E(\mathcal{J}\rho_1,\rho_2))+\omega_E(\mathcal{J}(\mathcal{L}_\psi\rho_1),\rho_2)+\omega_E(\mathcal{J}\rho_1,\mathcal{L}_\psi\rho_2).$$

But the relation

$$\psi(\omega_E(\mathcal{J}\rho_1,\rho_2))=(\mathcal{L}_\psi\omega_E)(\mathcal{J}\rho_1,\rho_2)+\omega_E(\mathcal{L}_\psi(\mathcal{J}\rho_1),\rho_2)+\omega_E(\mathcal{J}\rho_1,\mathcal{L}_\psi\rho_2),$$

yields $(\mathcal{L}_\psi g)(\rho_1,\rho_2) = -(\mathcal{L}_\psi\omega_E)(\mathcal{J}\rho_1,\rho_2)+\omega_E(-(\mathcal{L}_\psi\mathcal{J})\rho_1,\rho_2)$. Also, it follows

$$(\mathcal{L}_\psi\omega_E)(\mathcal{J}\rho_1,\rho_2)=\omega_E((\mathcal{N}-\mathcal{N}')\rho_1,\rho_2). \tag{4.15}$$

But,

$$\omega_E(\mathcal{N}'\mathcal{J}\rho_1,\rho_2)=(\mathcal{L}_\psi g)(\mathcal{J}\rho_1,\rho_2)=\psi(g(\mathcal{J}\rho_1,\rho_2))-g((\mathcal{L}_\psi\mathcal{J})\rho_1,\rho_2)$$
$$-g(\mathcal{J}\rho_1,\mathcal{L}_\psi\rho_2)=g(\mathcal{N}\rho_1,\rho_2)=-\omega_E(\mathcal{J}\mathcal{N}\rho_1,\rho_2)=-\omega_E(\mathcal{J}\rho_1,\rho_2),$$

whence $\mathcal{N}'\mathcal{J}=-\mathcal{J}$. Next

$$\omega_E(\mathcal{J}\mathcal{N}'\rho_1,\rho_2)=-\omega_E(\mathcal{N}'\rho_1,\mathcal{J}\rho_2)=-(\mathcal{L}_\psi g)(\rho_1,\mathcal{J}\rho_2)$$
$$=(\mathcal{L}_\psi\omega_E)(\mathcal{J}\rho_1,\mathcal{J}\rho_2)-\omega_E(\mathcal{N}\rho_1,\mathcal{J}\rho_2)=\omega_E(\mathcal{J}\mathcal{N}\rho_1,\rho_2)=\omega_E(\mathcal{J}\rho_1,\rho_2),$$

whence $\mathcal{J}\mathcal{N}'=\mathcal{J}$, which ends the proof. □

In local coordinates the connection $\mathcal{N}'$ has the coefficients given by

$$\mathcal{N}'^{i}_{\alpha\beta}=\frac{1}{2}\left(-\psi t_{\alpha\beta}-\sigma^i_\beta\frac{\partial\xi^\varepsilon}{\partial q^i}t_{\varepsilon\alpha}-\sigma^i_\alpha\frac{\partial\xi^\varepsilon}{\partial q^i}t_{\varepsilon\beta}+\xi^\gamma L^\varepsilon_{\gamma\beta}t_{\varepsilon\alpha}+\xi^\gamma L^\varepsilon_{\gamma\alpha}t_{\varepsilon\beta}+\mu_\varepsilon L^\varepsilon_{\beta\alpha}\right).$$

Using the relation (4.15) we obtain

Remark 2. In the case that ψ is both semi-Hamiltonian and mechanical, and moreover Hamiltonian, the connections $\mathcal{N}$ and $\mathcal{N}'$ coincide.

5. Duality between Lagrangian and Hamiltonian formalism

Let us consider a regular Hamiltonian $\mathcal{H}: E^*\to\mathbb{R}$ i.e. the matrix $g^{\alpha\beta}(q,\mu)=\frac{\partial^2\mathcal{H}}{\partial\mu_\alpha\partial\mu_\beta}$, is nondegenerate. Any regular Hamiltonian $\mathcal{H}$ on E^* induces a pseudo-Riemannian metric on $V\mathcal{T}E^*$ (the metric tensor is $g^{\alpha\beta}(q,\mu)$) therefore, it induces a unique symmetric adapted almost tangent structure (denoted $\mathcal{J}_\mathcal{H}$) such that (4.9) is verified. Moreover, this is a tangent structure i.e., $\mathcal{J}_\mathcal{H}$ is integrable. A $\mathcal{J}-$regular section induced by the regular Hamiltonian $\mathcal{H}$ is given by $\rho=\frac{\partial\mathcal{H}}{\partial\mu_\alpha}\mathcal{Q}_\alpha+\rho_\alpha\mathcal{P}^\alpha$. Since ω_E is a symplectic section and $d\mathcal{H}\in\Gamma(\mathcal{T}E^*)$, we get [11]

Remark 3. There exists a unique section $\rho_{\mathcal{H}} \in \Gamma(\mathcal{T}E^*)$ such as

$$i_{\rho_{\mathcal{H}}}\omega_E = d\mathcal{H},$$

and $\rho_{\mathcal{H}}$ is a *Hamilton section* in the sense of (4.13).

With respect to the local basis $\{\mathcal{Q}_\alpha, \mathcal{P}^\alpha\}$, the local expression of $\rho_{\mathcal{H}}$ is

$$\rho_{\mathcal{H}} = \frac{\partial \mathcal{H}}{\partial \mu_\alpha}\mathcal{Q}_\alpha - (\sigma^i_\alpha \frac{\partial \mathcal{H}}{\partial q^i} + \mu_\gamma L^\gamma_{\alpha\beta}\frac{\partial \mathcal{H}}{\partial \mu_\beta})\mathcal{P}^\alpha.$$

Corollary 1. *The symmetric nonlinear connection $\mathcal{N} = -\mathcal{L}_{\rho_{\mathcal{H}}}\mathcal{J}_{\mathcal{H}}$ has the coefficients given by*

$$\begin{aligned}\mathcal{N}_{\alpha\beta} = \frac{1}{2}(\sigma^i_\gamma\{g_{\alpha\beta}, \mathcal{H}\} - \frac{\partial^2 \mathcal{H}}{\partial q^i \partial \mu_\varepsilon}(\sigma^i_\beta g_{\alpha\varepsilon} + \sigma^i_\alpha g_{\beta\varepsilon}) \\ + \mu_\gamma L^\gamma_{\varepsilon\kappa}\frac{\partial \mathcal{H}}{\partial \mu_\varepsilon}\frac{\partial g_{\alpha\beta}}{\partial \mu_\kappa} + \mu_\gamma L^\gamma_{\alpha\beta} + \frac{\partial \mathcal{H}}{\partial \mu_\delta}(g_{\alpha\varepsilon}L^\varepsilon_{\delta\beta} + g_{\beta\varepsilon}L^\varepsilon_{\delta\alpha})),\end{aligned}$$

where

$$\{g_{\alpha\beta}, \mathcal{H}\} = \frac{\partial g_{\alpha\beta}}{\partial \mu_\gamma}\frac{\partial \mathcal{H}}{\partial q^i} - \frac{\partial g_{\alpha\beta}}{\partial q^i}\frac{\partial \mathcal{H}}{\partial \mu_\gamma},$$

is the Poisson bracket.

We have a local diffeomorphism Φ to E^* to E given locally by

$$x^i = q^i, \quad y^\alpha = \xi^\alpha(q, \mu) = \frac{\partial \mathcal{H}}{\partial \mu_\alpha}, \tag{5.16}$$

and its inverse Φ^{-1} has the following local coordinates expression

$$q^i = x^i, \quad \mu_\alpha = \zeta_\alpha(x, y) = \frac{\partial \mathcal{L}}{\partial y^\alpha}, \tag{5.17}$$

where $\mathcal{L}(x, y) = \zeta_\alpha y^\alpha - \mathcal{H}(x, \mu)$ and the components $\zeta_\alpha(x, y)$ define an 1-section on E. From the condition for Φ^{-1} to be the inverse of Φ we get the following formulas

$$\begin{cases}\mathcal{V}_\beta(\zeta_\alpha) \circ \Phi = g_{\alpha\beta}, \quad \mathcal{X}_\beta(\zeta_\alpha) \circ \Phi = -g_{\alpha\gamma}\mathcal{Q}_\beta(\xi^\gamma), \\ \Phi_*\mathcal{P}^\alpha = (g^{\alpha\beta} \circ \Phi^{-1})\mathcal{V}_\beta, \quad \Phi_*(\mathcal{Q}_\alpha) = \mathcal{X}_\alpha + (\mathcal{Q}_\alpha(\xi^\beta) \circ \Phi^{-1})\mathcal{V}_\beta, \\ \Phi_*^{-1}(\mathcal{V}_\alpha) = g_{\alpha\beta}\mathcal{P}^\beta, \quad \Phi_*^{-1}(\mathcal{X}_\alpha) = \mathcal{Q}_\alpha - g_{\gamma\varepsilon}\mathcal{Q}_\alpha(\xi^\varepsilon)\mathcal{P}^\gamma,\end{cases} \tag{5.18}$$

where Φ_* is the tangent map of Φ [11] and $g^{\alpha\beta} = \partial\xi^\alpha/\partial\mu_\beta$, $g^{\alpha\beta}g_{\beta\gamma} = \delta^\alpha_\gamma$.

Theorem 3. *Let ρ be a $\mathcal{J}-$regular section on $\mathcal{T}E^*$ induced by the regular Hamiltonian $\mathcal{H}$, and $\Phi : E^* \to E$ the diffeomorpfism given by* (5.16)*, then the section $\Phi_*\rho$ is a semispray on $\mathcal{T}E$ whose induced connection N is the image by Φ of the connection $\mathcal{N}$ defined by ρ.*

Proof. Let us consider $\rho = \xi^\alpha \mathcal{Q}_\alpha + \rho_\alpha \mathcal{P}^\alpha$ and we obtain the semispray

$$\vartheta = \Phi_*\rho = \xi^\alpha(\mathcal{X}_\alpha + (\mathcal{Q}_\alpha(\xi^\beta) \circ \Phi^{-1})\mathcal{V}_\beta) + \rho_\alpha(g^{\alpha\beta} \circ \Phi^{-1})\mathcal{V}_\beta = y^\alpha \mathcal{X}_\alpha + \vartheta^\alpha \mathcal{V}_\alpha,$$

where ϑ^α is given by

$$\vartheta^\alpha \circ \Phi = \xi^\beta \mathcal{Q}_\beta(\xi^\alpha) + \rho_\beta \mathcal{P}^\beta(\xi^\alpha) = \rho(\xi^\alpha).$$

Considering $\widetilde{\Phi}$ the map induced by Φ at the level of tensors and using (5.18) we obtain $\widetilde{\Phi}\mathcal{J} = J$, which yields $N = -\mathcal{L}_\vartheta J = -\mathcal{L}_{\Phi_*\rho}\widetilde{\Phi}\mathcal{J} = -\widetilde{\Phi}(\mathcal{L}_\rho \mathcal{J}) = \widetilde{\Phi}\mathcal{N}$, thus N is the image of $\mathcal{N}$ by $\widetilde{\Phi}$. □

This theorem shows that the splitting $V\mathcal{T}E^* \oplus H\mathcal{T}E^*$ defined by $\mathcal{N}$ is mapped by Φ_* into the spliting $V\mathcal{T}E \oplus H\mathcal{T}E$ defined by N, and it follows:

Corollary 2. *The following relations hold*

$$\begin{aligned}
&\Phi_*\delta^*_\alpha = \delta_\alpha, \quad \Phi_*^{-1}\delta_\alpha = \delta^*_\alpha,\\
&\mathcal{N}_{\alpha\beta}(q,\mu) = -\left(N^\gamma_\alpha(x,y) + \mathcal{Q}_\alpha(\xi^\gamma)\right) g_{\beta\gamma},\\
&\mathcal{R}_{\alpha\beta\gamma}g^{\alpha\varepsilon} = \mathcal{R}^\varepsilon_{\beta\gamma} \circ \Phi, \quad \mathcal{R}^\varepsilon_{\beta\gamma}\frac{\partial \zeta_\alpha}{\partial y^\varepsilon} = \mathcal{R}_{\alpha\beta\gamma} \circ \Phi^{-1},\\
&N^\beta_\alpha \circ \Phi = -\delta^*_\alpha(\xi^\beta), \quad \mathcal{N}_{\alpha\beta} \circ \Phi^{-1} = -\delta_\alpha(\zeta_\beta).
\end{aligned}$$

Proof. We have on one hand $\Phi_*^{-1}(\delta_\alpha) = \delta^*_\alpha = \mathcal{Q}_\alpha + \mathcal{N}_{\alpha\gamma}\mathcal{P}^\gamma$, and on the other hand $\Phi_*^{-1}(\delta_\alpha) = \Phi_*^{-1}(\mathcal{X}_\alpha - N^\beta_\alpha \mathcal{V}_\beta) = \mathcal{Q}_\alpha - g_{\gamma\varepsilon}\mathcal{Q}_\alpha(\xi^\alpha)\mathcal{P}^\gamma - N^\beta_\alpha g_{\beta\gamma}\mathcal{P}^\gamma$. Therefore, we get $\mathcal{N}_{\alpha\gamma} = -\left(N^\varepsilon_\alpha + \mathcal{Q}_\alpha(\xi^\varepsilon)\right) g_{\gamma\varepsilon}$. Next, since Φ^{-1} is a diffeomorphism, we obtain

$$\begin{aligned}
&[\Phi_*^{-1}(\delta_\alpha), \Phi_*^{-1}(\delta_\beta)]_{\mathcal{T}E^*} = L^\gamma_{\alpha\beta}\Phi_*^{-1}(\delta_\gamma) + \mathcal{R}^\gamma_{\alpha\beta}\Phi_*^{-1}(\mathcal{V}_\gamma),\\
&[\Phi_*^{-1}(\delta_\alpha), \Phi_*^{-1}(\delta_\beta)]_{\mathcal{T}E^*} = [\delta^*_\alpha, \delta^*_\beta]_{\mathcal{T}E^*} = L^\gamma_{\alpha\beta}\delta^*_\gamma + \mathcal{R}_{\alpha\beta\gamma}\mathcal{P}^\gamma,
\end{aligned}$$

and using (5.18) it follows the relation between the curvature tensors of connections N on $\mathcal{T}E$ and $\mathcal{N}$ on $\mathcal{T}E^*$. By direct computation the other relations are obtained. □

Let us consider $\mathcal{S}$ a semispray on $\mathcal{T}E$ and $\Phi^{-1} : E \to E^*$ the diffeomorphism given by (5.17). Then we set:

Theorem 4. *The section $\rho = \Phi_*^{-1}\mathcal{S}$ is a semi-Hamiltonian section on $\mathcal{T}E^*$ if and only if the nonlinear connection on $\mathcal{T}E$ induced by semispray is the canonical nonlinear connection induced by the regular Lagrangian.*

Proof. Considering $\mathcal{S} = y^\alpha \mathcal{X}_\alpha + \mathcal{S}^\alpha \mathcal{V}_\alpha$ we have from (5.18)

$$\Phi_*^{-1}\mathcal{S} = \xi^\alpha \mathcal{Q}_\alpha + \left(-\xi^\alpha g_{\gamma\varepsilon}\mathcal{Q}_\alpha(\xi^\varepsilon) + \mathcal{S}^\alpha g_{\alpha\gamma}\right)\mathcal{P}^\gamma.$$

The conditions (4.13) *i*), *ii*) and (5.18) lead to

$$\begin{aligned} g^{\theta\beta}\left(\mathcal{X}_\theta(\zeta_\alpha) - \mathcal{X}_\alpha(\zeta_\theta) + \mathcal{V}_\theta(\mathcal{S}^\varepsilon)g_{\varepsilon\alpha} + \xi^\varepsilon \mathcal{X}_\varepsilon(g_{\alpha\theta})\right. \\ \left. + \mathcal{S}^\varepsilon \mathcal{V}_\varepsilon(g_{\alpha\theta}) + \zeta_\varepsilon L^\varepsilon_{\alpha\theta}\right) + y^\varepsilon L^\beta_{\alpha\varepsilon} = 0, \end{aligned}$$

and using (2.2) it follows

$$\mathcal{X}_\alpha(\zeta_\theta) - \mathcal{X}_\theta(\zeta_\alpha) = \mathcal{S}(g_{\alpha\theta}) - 2N^\varepsilon_\theta g_{\varepsilon\alpha} + y^\beta L^\varepsilon_{\theta\beta} g_{\varepsilon\alpha} + y^\beta L^\varepsilon_{\alpha\beta} g_{\varepsilon\theta} + \zeta_\varepsilon L^\varepsilon_{\alpha\theta},$$

which concludes that (change α with θ and totalizing)

$$\mathcal{S}(g_{\alpha\theta}) - N^\varepsilon_\theta g_{\varepsilon\alpha} - N^\varepsilon_\alpha g_{\varepsilon\theta} + y^\beta\left(L^\varepsilon_{\theta\beta}g_{\varepsilon\alpha} + L^\varepsilon_{\alpha\beta}g_{\varepsilon\theta}\right) = 0 \overset{(3.5)}{\Leftrightarrow} \nabla g\left(\mathcal{V}_\alpha, \mathcal{V}_\theta\right) = 0,$$

and $\mathcal{X}_\alpha(\zeta_\theta) - \mathcal{X}_\theta(\zeta_\alpha) = N^\varepsilon_\theta g_{\varepsilon\alpha} - N^\varepsilon_\alpha g_{\varepsilon\theta} + \zeta_\varepsilon L^\varepsilon_{\alpha\theta}$, which is equivalent with

$$N^\varepsilon_\theta g_{\varepsilon\alpha} - N^\varepsilon_\alpha g_{\varepsilon\theta} + \frac{\partial^2 L}{\partial x^i \partial y^\alpha}\sigma^i_\beta - \frac{\partial^2 L}{\partial x^i \partial y^\beta}\sigma^i_\alpha - \frac{\partial L}{\partial y^\varepsilon}L^\varepsilon_{\alpha\beta} = 0 \overset{(3.6)}{\Leftrightarrow} \omega_L(\mathrm{h}\rho, \mathrm{h}\nu) = 0.$$

From the Theorem 2 we get that the section $\rho = \Phi_*^{-1}\mathcal{S}$ is semi-Hamiltonian if and only if the connection induced by semispray is the canonical nonlinear connection (2.4). When $\rho = \Phi_*^{-1}\mathcal{S}$ is a Hamiltonian section we have the same result, but moreover, the condition (4.13) iii) seems to lead to the third Helmholtz condition on a Lie algebroid [23] and the work is in progress. □

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