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Partial trivial structures in real vector bundles

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Abstract In this paper we study a class of real vector bundles from the partial trivial viewpoint. We define a partial trivial vector bundle as a couple (ξ, ρ_p) where $\xi = (E, \pi, M)$ is a vector bundle and ρ_p is a global partial frame of ξ. We give an incomplete answer to the question: when and how is it possible to obtain a partial trivial structure in a real vector bundle? The construction of some partial trivial structures is reduced to a problem of "adjunction and lifting" for Gauss map in a certain commutative diagram homotopy. Pontryagin classes of a partial trivial vector bundle are considered. The structure of induced partial trivial tangent bundle of a foliated manifold is presented.

Keywords partial trivial structures · vector bundle · global partial frame · Gauss map · Pontryagin classes

Mathematics Subject Classification (2010) 53C10 · 53C12 · 55R10

Introduction

We have written these pages with the intention to find (in the future) some properties of differentiable manifolds endowed with global vector fields.

In this paper we analyze the real vector bundles endowed with partial trivial structures and we describe some examples. The notion of "partial trivial structure" is inspired from the notions of a $\mathcal{F}-$flat structure ([3] , [4]), G_T-structure ([1] , [2]) and parallelizable manifolds. The key property of vector bundle ξ from this paper is the existence of a global partial frame $\rho_p = (s_1, ..., s_p)$ of ξ, $1 \leq p < n$, where $n = rank\xi$. We give some characterizations and the geometric structure of a partial trivial vector bundle. The existence of a global partial frame $\rho_p = (s_1, ..., s_p)$ in the vector bundle $\xi = (E, \pi, M)$ is equivalent to the partial triviality of ξ expressed by the isomorphism $E \simeq (M \times \mathbb{R}^p) \oplus E/E(\rho_p)$. This isomorphism is independent of the choice of ρ_p, i. e. two different global partial frames ρ_p, $\widetilde{\rho}_p$ of ξ give rise to some isomorphism. The quotient vector bundle $E/E(\rho_p)$ obstructs the triviality of ξ. In this situation the vector

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bundle $E/E(\rho_p)$ can be interpreted as a "geometrical obstruction" to triviality of ξ. We construct some partial trivial vector bundles with given "geometrical obstruction" to triviality. Pontryagin algebra and the total class of a partial trivial vector bundle are considered. Finally, a special partial trivial tangent bundle of a foliated manifold is examined.

We use some results of fibre bundle theory and characteristic classes presented in [6], [7], [10], [12], [13], [14]. Basic notions and results on theory of vector fields on manifolds can be found in [5] and [15]. In [6] and [16], different properties of Grassmann manifolds are studied.

1 Definition and some properties

Sometimes a vector bundle will be denoted by its total space. The geometrical objects considered in this paper are of C^∞ (smooth) class.

Let $\xi = (E, \pi, M)$ be a (real) vector bundle of rank (fiber dimension) n, over a differentiable manifold M of dimension m, where the total space is E and π is its projection. In the sequel, $\Gamma(E)$ denotes the set of sections of ξ. Let p be a fixed natural number, where $1 \le p < n$. We assume that ξ is endowed with a global partial frame (g.p.f.) $\rho_p = (s_1, ..., s_p)$, i. e. $s_1, ..., s_p \in \Gamma(E)$ and the vectors $s_1(x), ..., s_p(x)$ are independent, $\forall x \in M$. The g.p.f. ρ_p yields a trivial vector subbundle $E(\rho_p)$ of ξ. Let $E_x = \pi^{-1}(x)$ be the fiber of E over $x \in M$ and $E_x(\rho_p)$ be the subspace of E_x spanned by $s_1(x), ..., s_p(x)$, $\forall x \in M$. Consider the set $E(\rho_p) = \bigcup\limits_{x \in M} E_x(\rho_p)$ and let π' be the restriction of π to $E(\rho_p)$. We suppose for indices the values: $i, j, k, ... = 1, 2, ..., p$; $\widehat{i}, \widehat{j}, \widehat{k}, ... = p+1, p+2, ..., n$; $a, b, c, ... = 1, 2, ..., n$. The standard summation convention as usual is used. We suppose that the morphisms of vector bundles are of constant rank.

Lemma 1.1 *The set $\xi(\rho_p) = (E(\rho_p), \pi', M)$ is a trivial vector subbundle of rank p of ξ. The subbundle $\xi(\rho_p)$ is determined up to an isomorphism of vector bundles.*

Proof. A local frame of ξ over an open set $U \subset M$, adapted to the structure ρ_p is $r_u = \left(\underset{U}{s_i}, \underset{U}{\sigma_{\widehat{i}}} \right)$ where $\underset{U}{s_i} = s_i/_U$ (the restriction of s_i to U), $\underset{U}{\sigma_{\widehat{i}}}$ are independent sections of $E_U = \pi^{-1}(U)$. Let $(e_a) = \left(e_i, e_{\widehat{i}}\right)$, $(e_i) \subset \mathbb{R}^p$, $\left(e_{\widehat{i}}\right) \subset \mathbb{R}^q$ be the standard basis of $\mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q$, $p + q = n$. Consider (U, φ_U) the bundle chart of ξ defined by the frame r_U, i. e. $\varphi_U : \pi^{-1}U \to U \times \mathbb{R}^n$, $\varphi_U(z) = \varphi\left(z^i \underset{U}{s_i} + z^{\widehat{i}} \underset{U}{\sigma_{\widehat{i}}} \right) = (x, z^a e_a)$, where $z = z^i \underset{U}{s_i} + z^{\widehat{i}} \underset{U}{\sigma_{\widehat{i}}} \in \pi^{-1}U$, $x = \pi(z) \in U$. Using bundle chart (U, φ_U) we obtain a bundle chart (U, φ'_U) of $\xi(\rho_p)$, where $\varphi'_U : \pi'^{-1}U \to U \times \mathbb{R}^p$, $\varphi'_U(z) = \varphi_U\left(z^i s_i\right) = \left(x, z^i e_i\right)$. Thus, using the family of frames $\{r_U\}_{U \in \mathcal{U}}$ ($\mathcal{U}$ an open covering of M), we obtain on $E(\rho_p)$ a subbundle structure of ξ. Denote $\theta^p = (M \times \mathbb{R}^p, p_M, M)$ the product bundle over manifold M and the projection p_M. Finally, the map $f : M \times \mathbb{R}^p \to E(\rho_p)$, $f(x, \lambda^i e_i) = \lambda^i s_i(x)$, $\lambda^i \in \mathbb{R}$, $(s_i, ..., s_p) = \rho_p$, is a bundle isomorphism, q. e. d. $\square$

Remark 1.1 The triviality of $\xi(\rho_p)$ results from the fact that ρ_p is a global frame of $\xi(\rho_p)$, but we have preferred a direct proof.

Remark 1.2 a) The set of matrices

$$G_1 = \left\{ \begin{pmatrix} A & 0 \\ 0 & I_q \end{pmatrix}, \text{ where } A \in GL(p, \mathbb{R}),\ I_q \text{ denote the unit matrix of } GL(q, \mathbb{R}) \right\}$$

forms a subgroup of $GL(n, \mathbb{R})$. We identify the group G_1 and $GL(p, \mathbb{R})$ using the isomorphism $A \leftrightarrow \begin{pmatrix} A & 0 \\ 0 & I_q \end{pmatrix}$.

b) The structure group of $\xi(\rho_p)$ is $\{I_p\} \subset GL(p, \mathbb{R})$.

The following proposition describes the structure of a vector bundle endowed with a g.p.f.

Lemma 1.2 *a) Let ρ_p be a g. p. f. of $\xi = (E, \pi, M)$. If the base M of ξ is a paracompact manifold, then there is a fibre bundle isomorphism $\rho : E \simeq (M \times \mathbb{R}^p) \oplus E/E(\rho_p)$;*

b) This decomposition is, up to isomorphism, independent of ρ_p.

Proof. a) Let $i = \{i_x : E_x(\rho_p) \to E_x,\ x \in M\}$ be the inclusion map of $E(\rho_p)$ in E, and $j = \{j_x : E_x \to E_x/E_x(\rho_p)\}$ be the canonical projection map $E \to E/E(\rho_p)$. If O denotes the vector bundle $(M, 1_M, M)$, we have the following short exact sequence of vector bundles over M ([11], [13]):

$$O \to E(\rho_p) \xrightarrow{i} E \xrightarrow{j} E/E(\rho_p) \to O.$$

Since M is a paracompact manifold, this exact sequence gives the following isomorphism of vector bundles: $g : E \simeq E(\rho_p) \oplus E/E(\rho_p)$. This is a canonical direct sum bundle decomposition. Using the isomorphism $f : M \times \mathbb{R}^p \to E(\rho_p)$ and g, we obtain the isomorphism

$$E \simeq (M \times \mathbb{R}^p) \oplus E/E(\rho_p).$$

To show b), let $\widetilde{\rho}_p = (\widetilde{s}_1, ..., \widetilde{s}_p)$ be a g.p.f. of ξ, $\widetilde{\rho}_p \neq \rho_p$, and $E(\widetilde{\rho}_p)$ be the subbundle generated by $\widetilde{\rho}_p$. Via Lemma 1.1 we have $E(\widetilde{\rho}_p) \simeq M \times \mathbb{R}^p \simeq E(\rho_p)$ and $E \simeq (M \times \mathbb{R}^p) \oplus E/E(\widetilde{\rho}_p)$, q.e.d.

We note that the vector bundle $(\xi; \rho_p)$ has a partial trivial structure defined by the trivial bundle θ^p. Lemma 1.2 indicates in what way the partial trivial vector bundle $(\xi; \rho_p)$ differs from a trivial vector bundle. $\square$

Definition 1.1 A vector bundle endowed with a partial trivial structure is said to be partial trivial vector bundle (p.t.v.b.).

Remark 1.3 We view the quotient vector bundle $E/E(\rho_p)$ as a "geometrical obstruction" to triviality of ξ.

Let $G(\rho_p)$ be the subgroup of $GL(n, \mathbb{R})$ consisting of all matrices $\begin{pmatrix} I_p & A \\ 0 & B_q \end{pmatrix}$ (or the transpose $\begin{pmatrix} I_p & 0 \\ A & B_q \end{pmatrix}$), where $I_p \in GL(p; \mathbb{R})$ is the unit matrix, $B_q \in GL(q; \mathbb{R})$, and A

is a $p \times q-$matrix. We consider $G(\rho_p)$ a properly imbedded subgroup of $GL(n, \mathbb{R})$. Then, the vector bundle $\xi = (E, \pi, M)$ admits a $G(\rho_p)-$reduction iff the frame bundle of E, $R(E)$, admits a $G(\rho_p)-$reduction ([9]). We use this property.

Lemma 1.3 *Let $\xi = (E, \pi, M)$ be a vector bundle with fibre $\mathbb{R}^n$, M be the paracompact base and $GL(n, \mathbb{R})$ be the structure group. Then ξ admits a g.p.f. ρ_p if and only if $GL(n, \mathbb{R})$ admits a $G(\rho_p)-$reduction.*

Proof. Suppose that ξ has a g.p.f., ρ_p. Let $\mathcal{U} = \{U, V, W, ...\}$ be an open covering of M, where each open set is endowed with an adapted frame r_U, r_V, $r_W, ...$ Denote $E_1 = \{r_U(x) \,/\, x \in U,\ U \in \mathcal{U}\}$, $p_1 : E_1 \to M$, $p_1(r_1(x)) = x \in U \subset M$. Let $R(E_1)$ be the frame bundle of adapted frames, $R(E_1) = \left(E_1, p_1, M, G(\rho_p), \widetilde{A}\right)$; here $\widetilde{A}$ is a trivializing atlas of $R(E_1)$. The group $G(\rho_p)$ operates simply transitive on each fiber $p_1^{-1}(x)$, $x \in M$. Now we determine a local product structure A for $R(E_1)$. Let (U, χ_U) be the bundle chart of $R(E)$ defined by $\chi_U : U \times GL(n, \mathbb{R}) \to \pi^{-1}U \subset R(E)$, $\chi_U(x, A) = r_U(x) A$, $x \in U$, $A \in GL(n, \mathbb{R})$. If $U \cap V \neq \varPhi$, we consider the bundle chart (V, χ_V), $\chi_V(x, A) = r_V(x) A$, $x \in V$. For $U \cap V \neq \varPhi$ we have $r_U = r_V S$, where $S \in G(\rho_p)$. We analyze the transition maps for the atlas $\mathcal{B} = \{(U, \chi_U),\ (V, \chi_V), ...\}_{U,V \in \mathcal{U}}$. If $\chi_{U,x}$, $\chi_{V,x}$ denote the restrictions of χ_U, χ_V to $\pi^{-1}(x)$, $x \in U \cap V$, we obtain the maps $g_{VU} : U \cap V \to GL(n, \mathbb{R})$, $g_{VU}(x) A = \chi_{V,x}^{-1} \chi_{U,x}(A) = \chi_{V,x}^{-1}(r_V(x) S A) = SA$, $\forall A \in GL(n, \mathbb{R})$, $S \in G(\rho_p)$. For $A = I_n \in GL(n, \mathbb{R})$ we have $g_{VU}(x) = S \in G(\rho_p)$. Therefore $GL(n, \mathbb{R})$ admits a $G(\rho_p)-$reduction, $R(E_1)$ ([13], [14]).

Conversely, if $GL(n, \mathbb{R})$ admits a $G(\rho_p)-$reduction $\widetilde{R}$, then $R(E)$ has a trivializing atlas $\mathcal{B}_1 = \{(U, \psi_U),\ (V, \psi_V), ...\}_{U,V \in \mathcal{U}}$ and its transition maps $g'_{VU}(x) = \psi_{V,x}^{-1} \psi_{U,x} \in G(\rho_p)$, $\forall x \in U \cap V \neq \varPhi$, $U, V \in \mathcal{U}$. If $r'_U = \begin{pmatrix} \underset{U}{s'_i}, \underset{U}{s'_{\hat{i}}} \end{pmatrix}$, $r'_V = \begin{pmatrix} \underset{V}{s'_i}, \underset{V}{s'_{\hat{i}}} \end{pmatrix}$ are some frames of $\widetilde{R}$, then $\underset{U}{s'_i} = \delta_i^j \underset{V}{s'_j}$, $\underset{U}{s'_{\hat{i}}} = S_{\hat{i}}^a \underset{V}{r'_a}$, where δ_i^j are the Kronecker symbols and $S' = \begin{pmatrix} \delta_i^j & S'^{i}_{\hat{j}} \\ 0 & S'^{\hat{i}}_{\hat{j}} \end{pmatrix} \in G(\rho_p)$. Hence $\underset{U}{s'_i} = \underset{V}{s'_i}$ on $U \cap V \neq \varPhi$. Because U, V are arbitrary, the collection $\left\{ \underset{U}{s'_i} \right\}_{U \in \mathcal{U}}$ defines a g.p.f. of $R(E)$. □

With the above notations we can formulate the following statement.

Theorem 1.1 *Let $\xi = (E, \pi, M)$ be a vector bundle of rank n over a paracompact manifold M. The following statements are equivalent:*

1) ξ is endowed with a global partial frame $\rho_p = (s_1, ..., s_p)$, $1 \leq p < n$;

2) The structure of ξ is given from the isomorphism $\rho : E \simeq (M \times \mathbb{R}^p) \oplus E/E(\rho_p)$;

3) The frame bundle $R(E)$ of ξ is endowed with $G(\rho_p)-$ subbundle $R(\rho_p)$.

Proof. If $\rho_1 = (s_1, ..., s_p)$ is g.p.f. of ξ, using Lemma 1.2, it follows that 1)⇒2). Now we consider the canonical base (e_i) of $\mathbb{R}^p$ and let $r = \{(x, e_i) \,/\, x \in M\}$ be a global frame of $\theta^p = (M \times \mathbb{R}^p \to M)$. Then $\rho^{-1}(r)$ is a g.p.f. of ξ. If $r' = \{(x, e'_i) / \ x \in M\}$ is another global frame of θ^p, $\rho^{-1}(r')$ and $\rho^{-1}(r)$ define two p.t.v.b. isomorphic. Now one remarks that Lemma 1.3. indicates the equivalence 1)⇔3), q.e.d. □

This theorem suggests a method of construction for p.t.s. in certain conditions (see Section 2).

A vector bundle in which one, hence all, of these conditions holds is called a p.t.v.b.

Proposition 1.1 *Let $\xi = (E, \pi, M)$ be a vector bundle with the base M a paracompact manifold and $\xi^* = (E^*, \pi^*, M)$ be the dual of ξ. Then ξ^* is partial trivial iff ξ is partial trivial.*

Proof. Let $\{g_{UV}\}_{U,V\in\mathcal{U}} \in GL(n, \mathbb{R})$ be the transition maps of ξ. The transition maps of ξ^* are $\left\{{}^t g_{UV}^{-1}\right\}$ (i. e. the transpose inverse of g_{UV}, when the latter is interpreted as a non-singular matrix). Now we can use Lemma 1.3 and the proof is complete. □

2 On the constructions of partial trivial vector bundles

We begin with some preliminary results concerning the construction of partial trivial structures. These constructions can be realized by operations with bundles and homotopy classes.

Lemma 2.1 *Let $\xi = (E, \pi, M)$ be a vector bundle over M and $f : M' \to M$ be a map of differentiable manifolds. Each partial trivial structure of ξ induces a partial trivial structure in the induced bundle $f^*\xi$ of ξ under f.*

Proof. Consider a g.p.f. $\rho_p = (s_1, ..., s_p)$ of ξ and let $\sigma_i : M' \to f^*\xi$, $\sigma_i(x') = (x', s_i(f(x')))$, $i = 1, 2, ..., p$, $x' \in M'$ be the sections of induced bundle $f^*\xi = (f^*E, f^*\pi, M')$. The sections $\sigma_1, ..., \sigma_p$ are independent at each point $x' \in M'$:

$$\left\{\sigma_1(x') = \left(x', s_1\left(f(x')\right)\right), ..., \sigma_p(x') = \left(x', s_p\left(f(x')\right)\right),\ x' \in M'\right\}$$

is a linearly independent set of vectors. Since $f^*(\xi; \rho_p) = (f^*\xi, f^*\rho_p)$ where $f^*\rho_p = (\sigma_1, ..., \sigma_p)$, the result follows. □

Lemm 2.2 *Let $G_p(\mathbb{R}^s)$ be the Grassmann manifold of $p-$dimensional subspace of $\mathbb{R}^s$, $p \le s \le +\infty$. For each map $f : M \to G_p(\mathbb{R}^s)$ there is a partial trivial vector bundle over M.*

Proof. **Method 1.** For each $s \in \{1, 2, ...\}$ consider the product bundle $\theta_1^s = (G_p(\mathbb{R}^s) \times \mathbb{R}^s, p_1, G_p(\mathbb{R}^s))$ over $G_p(\mathbb{R}^s)$ and projection p_1. Let $\gamma_p^s = \left(E\left(\gamma_p^s\right), p^*, G_p(\mathbb{R}^s)\right)$ be the canonical vector bundle over $G_p(\mathbb{R}^s)$, $E\left(\gamma_p^s\right) = \{(V, x) \in G_p(\mathbb{R}^s) \times \mathbb{R}^s /\ x \in V\}$ be the total space of γ_p^s, and p^* be its projection. Then $\theta_1^s \oplus \gamma_p^s$ is a p.t.v.b. with the trivial component θ_1^s and "geometrical obstruction" γ_p^s. Via Lemma 2.1, $f^*\left(\theta_1^s \oplus \gamma_p^s\right)$ is a p.t.v.b. over M.

Method 2. Let $I : G_p(\mathbb{R}^s) \hookrightarrow G_{p+t}\left(\mathbb{R}^{s+t}\right)$ denote the inclusion map and $\gamma_{p+t}^{s+t}|_{G_p(\mathbb{R}^s)}$ be the canonical vector bundle of base $G_{p+t}\left(\mathbb{R}^{s+t}\right)$ ([11, p. 22]). The restriction $\gamma_{p+t}^{s+t}|_{G_p(\mathbb{R}^s)}$ is a vector bundle isomorphic to $\theta_1^s \oplus \gamma_p^s$. Now we use the map $h = If : M \to G_{p+t}\left(\mathbb{R}^{s+t}\right)$ to obtain the induced bundle $h^*\left(\gamma_{p+t}^{s+t}\right) = (If)^*\left(\gamma_{p+t}^{s+t}\right) = f^*\left(\theta_1^s \oplus \gamma_p^s\right)$. This is a p.t.v.b. The proof is complete. □

This result constitutes the base for Example 3 below.

Lemma 2.3 *Let $\xi^n = (E, \pi, M)$ be a vector bundle over a paracompact manifold M. If the classifying map of ξ^n, $\gamma : M \to G_p \times G_{n-p}$ is of form $\gamma = (\gamma_0, \gamma_1)$ where γ_0 is a constant map, then ξ^n is a p.t.v.b.*

Proof. Since component γ_0 of γ is a constant map, the induced vector bundle under γ_0 is constant. The result follows. □

Proposition 2.1 *Let $\xi^n = (E, \pi, M)$ be a vector bundle over a paracompact manifold M, with the fibre $\mathbb{R}^n$. If the canonical vector bundle γ_n^s is a partial trivial vector bundle, then ξ^n is partial trivial too.*

Proof. Let $\sigma_p = \left(\sigma_1', ..., \sigma_p'\right)$ be a global partial frame which defines a p.t.s. of γ_n^s, $n \leq s \leq +\infty$ and $(u_1, f_1) : f_1^* (\gamma_n^s) \to \gamma_n^s$, $u_1 (x, y) = y$, $(x, y) \in E (f_1^* (\gamma_n^s))$ (the total space of $f_1^* (\gamma_n^s)$) is a canonical morphism of induced bundle $f_1^* (\gamma_n^s)$ under the map $f_1 : M \to G_n (\mathbb{R}^s)$, $f_1 (x) = g_E \left(\pi^{-1} (x)\right) \in G_n (\mathbb{R}^s)$; g_E is Gauss map of ξ^n. Then the sections of $f_1^* (\gamma_n^s)$, $s_i : M \to f_1^* (\gamma_n^s)$, $s_i (x) = (x, \sigma_i (f (x)))$ such that $u_1 s_i (x) = \sigma_i' (f_1 (x))$, $x \in M$, $i = 1, ..., p$ define a partial trivial structure in $f_1^* (\gamma_n^s)$. Since M is a paracompact manifold, ξ^n is isomorphic with $f_1^* (\gamma_n^s)$ ([9, p. 30]). □

Remark 2.1 This proposition also results by Lemma 2.1, but we have preferred a direct proof.

Corollary (of Theorem 1.1). *The structure of γ_n^s endowed with a p.t.s. σ_p is given by the isomorphism $\gamma_n^s \simeq (G_n (\mathbb{R}^s) \times \mathbb{R}^p) \oplus E (\gamma_n^s) / E (\sigma_p)$, where $E (\sigma_p)$ is the subspace of total space $E (\gamma_n^s)$ generated by global partial frame $\sigma_p = \left(\sigma_1', ..., \sigma_p'\right)$ of γ_n^s.*

This result follows from Theorem 1.1 and the definition of (γ_n^s, σ_p).

Construction of partial trivial structures for vector bundles

In this section we give a response to the question: when and how it is possible to obtain a partial trivial structure for a vector bundle? From the isomorphism $E \simeq (M \times \mathbb{R}^p) \oplus E | E (\rho_p)$ (Theorem 1.1) we obtain the following methods for the construction of partial trivial structures for vector bundles.

Case I. Description of construction of partial trivial structures using direct sum.

Given a real vector bundle $\xi^n = (E, \pi, M)$ with the fibre $\mathbb{R}^n$, we consider the product bundle $\theta^s = (M \times \mathbb{R}^s \to M)$. Then the direct sum $\eta^{n,s} = \theta^s \oplus \xi^n$ is for each s, a p.t.v.b. over M. Here ξ^n is a "geometrical obstruction" to triviality of $\eta^{n,s}$. In particular, if ξ^n admits a vector subbundle $\xi_1^{n-p} = (E_1, \pi_1, M)$, then $\eta_1 = \theta_1^p \oplus \xi_1^{n-p}$ is a p.t.v.b., where $\theta_1^p = (M \times \mathbb{R}^p \to M)$. If ξ^n and η_1 are isomorphic, then ξ^n is a p.t.v.b. We can express the following result: For each vector bundle ξ^n there exists an infinite set of p.t.v.b. that have ξ^n as "geometrical obstruction" to triviality.

The following facts are necessary at the beginning of second case. Let $Vect_n (M)$ be the set of isomorphism classes of vector bundles of rank n, over M and $[M, G_n]$ be the set of homotopy classes $[f]$ of maps $f : M \to G_n$. There exists a one-to-one correspondence $Vect_n (M) \simeq [M, G_n]$ ([11], [8]). We denote by $\{\xi_1\}$ the isomorphism class of $\xi_1 \in Vect_n (M)$. The isomorphism class $\{f^* \xi_1\}$ of the vector bundle $f^* (\xi_1)$,

$\xi_1 = (E_1, p_1, G_n)$ is independent of $f \in [f]$ and $\xi_1 \in \{\xi_1\}$. Now we can formulate the following case.

Case II. Construction of partial trivial structures in a vector bundle over a paracompact manifold.

Consider $\xi^n = (E, \pi, M)$ a vector bundle over a paracompact manifold M, where E is the total space of ξ^n and π is its projection. Since M is a paracompact manifold, there is a Gauss map $\widetilde{g}_E : E \to \mathbb{R}^\infty$ of ξ^n [11, p. 30]. Moreover, there exists a vector bundle morphism $(u, g_E) : \xi^n \to \gamma_n^s$, $n \le s \le \infty$, $\widetilde{g}_E = qu$, $q : E(\gamma_n^s) \to \mathbb{R}^s$, $q(V, x) = x$; γ_n^s is canonical vector bundle over $G_n(\mathbb{R}^s)$,

$$E(\gamma_n^s) = \{(V, x) \in G_n(\mathbb{R}^s) \times \mathbb{R}^s / \ x \in V\}$$

is the total space of γ_n^s. The morphism (u, g_E) is an isomorphism when restricted to any fibre of ξ^n and $g_E : M \to G_n(\mathbb{R}^s)$, $g_E(x) = \widetilde{g}_E(\pi^{-1}(x))$, $u(y) = (g_E(\pi(y)), \widetilde{g}_E(y))$, $y \in E$. We consider the Grassmann varieties $G_p = G_p(\mathbb{R}^\infty)$, $G_{n-p} = G_{n-p}(\mathbb{R}^\infty)$, and the product manifold $G_p \times G_{n-p}$. Let $K : G_p \times G_{n-p} \to G_n$ be the map induced by the direct sum maps $G_p(\mathbb{R}^s) \times G_{n-p}\left(\mathbb{R}^{s'}\right) \to G_n\left(\mathbb{R}^{s+s'}\right)$, sending $(\alpha, \beta) \to \alpha \oplus \beta$. If g_E admits a lift $g = (g_0, g_1)$ in the following commutative diagram homotopy (Fig. 1), where $g_0 : M \to G_p$ is a constant map, then we obtain the induced vector bundle $g^*\left(\gamma_p^s \times \gamma_{n-p}^{s'}\right) = g_0^*\left(\gamma_p^s\right) \oplus g_1^*\left(\gamma_{n-p}^{s'}\right)$, $n \le s, s' \le +\infty$ (Lemma 2.3).

$$\begin{tikzcd} & G_p X G_{n-p} \arrow[d] \\ M \arrow[ur, dashed, "{[g]=[(g_0,g_1)]}"] \arrow[r, "{[g_E]}"'] & G_n \end{tikzcd}$$

Fig. 1

The morphism u admits a representation $u = f_{\gamma_n} v$, where $v : \xi^n \to g^*\left(\gamma_p^s \times \gamma_{n-p}^{s'}\right)$ is an isomorphism of vector bundles over M and $f_{\gamma_n} : g^*\left(\gamma_p^s \times \gamma_{n-p}^{s'}\right) \to \gamma_p^s \times \gamma_{n-p}^{s'}$, $f_{\gamma_n}(x, y) = y$ is the canonical morphism [9, p. 27]. Hence there exists the isomorphism $\xi^n \simeq g_0^*\left(\gamma_p^s\right) \oplus g_1^*\left(\gamma_{n-p}^{s'}\right)$. Therefore ξ^n is a p.t.v.b. with the partial trivial structure defined by $v^{-1}\left(g_0^*\left(\gamma_p^s\right)\right)$, where v^{-1} is the inverse of v. Now g_1 can be interpreted as a classifying map of $E/E(\rho_p)$, where $E(\rho_p) = v^{-1}\left(g_0^*\left(\gamma_p^s\right)\right)$.

Conversely, let $\xi^n = (E, \pi, M)$ be a p.t.v.b. with the base M a paracompact manifold. Via Theorem 1.1, $E \simeq (M \times \mathbb{R}^p) \oplus E/E(\rho_p)$, $E(\rho_p) \simeq \theta^p$. Let $g_0' : M \to G_p(\mathbb{R}^s)$ be a classifying map of θ^p and $g_0 : M \to G_p(\mathbb{R}^s)$ be a constant map. Then $g_0^*(\gamma_p) \simeq \theta^p$. From the bijection $Vect_p(M) \simeq [M, G_p]$, we obtain $[g_0] \leftrightarrow [\theta^p] \leftrightarrow [g_0']$ for homotopy classes. Therefore the class of map which classifies θ^p contains a constant map. If the map $g_1 : M \to G_{n-p}$ classifies $E/E(\rho_p)$, then (g_0, g_1) classifies the bundle ξ^n. Now considering the isomorphism classes of vector bundles and homotopy classes of maps, we obtain the following result.

Theorem 2.1 *Let $V_n^p(M)$ be the set of isomorphism classes of partial trivial bundles $\xi^n = (E,\pi,M)$ over a paracompact manifold M, $n = rank\xi^n$, where ξ^n is endowed with a partial trivial structure ρ_p. Then there exists a bijection from $V_n^p(M)$ to the set of homotopy classes of homotopy lifts $[g] = [(g_0, g_1)]$ of $[g_E]$, where $g_0 : M \to G_p$ is a constant map.*

For the following construction we use the methods from cases I and II.

Case III. Let $\theta_1^p = (G_p(\mathbb{R}^s) \times \mathbb{R}^p \to G_p(\mathbb{R}^s))$, $p < s \leq +\infty$ be a product bundle over $G_p(\mathbb{R}^s)$. Consider a vector bundle $\xi^n = (E,\pi,M)$ over a paracompact manifold M, and g_E a classifying map of ξ^n. Denote by $g = (g_0, g_1)$ a lift of g_E in commutative diagram Fig. 1, where g_0, g_1 are not constant maps. Then $g_0^*(\theta_1^p)$ is a trivial vector bundle (see Lemmas 2.1 and 2.2). In this situation the vector bundles ξ^n and $g_E^*\left(\theta_1^p \times \gamma_{n-p}^{s'}\right) = g_0^*(\theta_1^p) \oplus g_1^*\left(\gamma_{n-p}^{s'}\right)$ are isomorphic over M (see the proof of case II). Hence ξ^n is a p.t.v.b.

We can formulate now the following result.

Theorem 2.2 *Let $\xi^n = (E,\pi,M)$ be a vector bundle over a paracompact manifold M, and $[g] = [(g_0, g_1)]$ be a lift of g_E, where g_0, g_1 are or not constant maps. The trivial structure of θ_1^p induces a partial trivial structure in ξ^n.*

Case IV. This case is defined by the condition: the classifying map g_E of ξ^n (with the base M paracompact) admits a lift $[g] = [(g_0, g_1)]$, where g_0, g_1 are constant. Under this condition there exist the isomorphisms (see the proof of case II):

$$\xi^n \simeq g^*(\gamma_p \times \gamma_{n-p}) = g_0^*(\gamma_p) \oplus g_1^*(\gamma_{n-p}) \simeq (M \times \mathbb{R}^p) \oplus \left(M \times \mathbb{R}^{n-p}\right) \simeq M \times \mathbb{R}^n.$$

Therefore, ξ^n is a trivial vector bundle if $p = n$.

For completion we consider the following excepted case too.

Case V:$p = n$. This remarkable case is described by the well-known isomorphism:

$$\xi^n \simeq (M \times \mathbb{R}^n) \oplus 0 \simeq M \times \mathbb{R}^n,$$

where $0 = (M, 1_M, M)$ is the null vector bundle over M. This result is also described in the case IV using the classifying maps.

From the precedent results we have the following conclusion: A partial trivial structure for a vector bundle $\xi^n = (E,\pi,M)$ over a paracompact manifold M can be realized by direct sum of vector bundles and a lift $[g] = [(g_0, g_1)]$ of Gauss map $[g_E]$ of ξ^n in diagram Fig.1, where g_0 is a constant map. These constructions guarantee the existence of a large class of partial trivial vector bundles.

Examples 1) a) It is well-known that only S^1, S^3, S^7 are parallelizable spheres. We can construct some partial trivial structures on the spheres S^n $(n \neq 1,3,7)$. If $\rho_{\nu(n)} = \left(X_1, ..., X_{\nu(n)}\right)$ is a set of $\nu(n)$ independent regular vector fields on S^n, $\rho_{\nu(n)}$ defines a partial trivial structure in tangent bundle $T(S^n)$.

b) Let $\theta^s = (S^n \times \mathbb{R}^s, \widetilde{p}, S^n)$ be the product bundle of base S^n. The restriction $\xi_1 = T(S^{n+s})|_{S^n}$ is isomorphic to $T(S^n) \oplus \theta^s$. Therefore ξ_1 is a p.t.v.b. over S^n.

2) Let X be a regular vector field on a manifold M. The subbundle $[X]$ generated by X is trivial. $[X]$ defines an $1-$dimensional parallelizable foliation $\mathcal{F}_1$ of M. Therefore, we obtain a partial trivial structure and a $\mathcal{F}_1-$flat structure on M ([4]).

a) The structure of TM is given by the following isomorphism: $TM \simeq (M \times \mathbb{R}) \oplus TM/[X]$.

b) It is well-known that a manifold M admits a regular vector field X iff Euler-Poincaré characteristic of M is null, $\chi(M) = 0$. Then, from a) we get: $TM \simeq (M \times \mathbb{R}) \oplus TM/[X]$ iff $\chi(M) = 0$. Here it appears a relation between the annulment of a topological invariant $\chi(M)$ and the structure of TM when M admits a regular vector field.

3) Let $h : G_n(\mathbb{R}^s) \to G_{n+t}(\mathbb{R}^{s+t})$ be the map $h(V) = V \oplus W$, where W is the subspace with bases $(e_{s+1}, ..., e_{s+t})$ in $\mathbb{R}^{s+t}$. The restriction $\gamma_{n+t}^{s+t}|_{G_n(\mathbb{R}^s)} \simeq \gamma_n^s \oplus \theta^t$, where $\theta^t = \left(G_n(\mathbb{R}^s) \times \mathbb{R}^t \to G_n(\mathbb{R}^s)\right)$ [9, p. 22].

a) Let M be a manifold and $f : M \to G_n(\mathbb{R}^s)$ be a map. Then the induced vector bundle $f^*\left(\gamma_n^s \oplus \theta^t\right)$ is a p.t.v.b. over M.

b) If M is a paracompact manifold and $\xi^n = (E, \pi, M)$ is a vector bundle over M, there exists a gauss map $g : E \to \mathbb{R}^\infty$, $g = qu$, where $(u, f_1) : \xi^n \to \gamma_n^s$, $n \leq s \leq +\infty$, is a vector bundle morphism that is an isomorphism on fibres [9, p. 29]. Here $q : E(\gamma_n^s) \to \mathbb{R}^s \subset \mathbb{R}^\infty$, $n < s \leq +\infty$, $q(V, y) = y$, $\forall (V, y) \in G_n(\mathbb{R}^s) \times \mathbb{R}^t$, $y \in V$, $n = \dim V$. The induced bundle $f_1^*\left(\gamma_n^s \oplus \theta^t\right)$ is a p.t.v.b. over M. This example is a completion for Lemma 2.3. The bundle $\gamma_n^s \oplus \theta^t$ can be also obtained directly (see case I).

4) Let M be a $\mathcal{F}-$foliated manifold, $p = \dim \mathcal{F}$. If $\mathcal{F}$ is a parallelizable foliation, let $\rho = (X_1, ..., X_p)$ be a global tangent frame to $\mathcal{F}$.Then the tangent bundle TM of M is a p.t.v.b. (M is a partial parallelizable manifold) [4] . If $p = m = \dim M$ and $\mathcal{F}$ is the tangential foliation of M, then TM is trivial vector bundle and M is a parallelizable manifold.

5) We consider a partial parallelizable (or parallelizable) manifold M and let N be an arbitrary manifold. Then $T(M \times N) \simeq TM \oplus TN$ is a partial trivial vector bundle and hence $M \times N$ is a partial parallelizable manifold.

6) Let $P = (E, \pi, M, G)$ be a principal $G-$bundle over M with the total space E. Since the vertical foliation V of P is parallelizable, we obtain the isomorphism of vector bundle $TE \simeq (M \times \mathbb{R}^p) \oplus TE/TV$, where $p = \dim V$ and TV denotes tangent bundle of V. TE is a p.t.v.b.

3 Pontryagin classes of a partial trivial vector bundle

In this section we consider the Pontryagin algebra of a p.t.v.b. ξ and its total class $p(\xi)$. We use as definition of Pontryagin classes their expressions in terms of curvature of a connection ([10] , [12]).

3.1 Preliminary considerations

Let $gl(n; \mathbb{R})$ be the algebra of general linear group $GL(n; \mathbb{R})$. The set

$$S = \left\{ \alpha = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} \in gl(n; \mathbb{R}), \ A \in gl(n-p; \mathbb{R}) \right\}$$

is a subalgebra of $gl(n; \mathbb{R})$. The map $u : gl(n-p; \mathbb{R}) \to S$, $u(A) = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}$ is an isomorphism of $\mathbb{R}-$algebras. Hence $gl(n-p; \mathbb{R})$ coincides, up to isomorphism with algebra S.

Let $\widetilde{f}_k,\ k=1,2,...,n-p$ be the symmetric polynomials of $gl\left(n-p;\mathbb{R}\right)$ and f_k be the restrictions to S of generators of symmetric algebra of $gl\left(n;\mathbb{R}\right)$. We have

$$\det\left(I_n+t\alpha\right)=\det I_p\cdot\det\left(I_{n-p}+tA\right)=\det\left(I_{n-p}+tA\right)$$

or, equivalently

$$\sum_{k=1}^{n-p}f_k\left(\alpha\right)=\sum_{k=1}^{n-p}\widetilde{f}_k\left(A\right),\ \text{where}\ \alpha=\begin{pmatrix}0&0\\0&A\end{pmatrix}\in S.$$

We denote by Σ_k the invariant polynomials of $gl\left(n;\mathbb{R}\right)$. Then

$$\Sigma_k\left(\alpha\right)=trace\left(\alpha^k\right)=trace\begin{pmatrix}0&0\\0&A^k\end{pmatrix}=\Sigma_k\left(A,\right)$$

where $\alpha\in S,\ A\in gl\left(n-p;\mathbb{R}\right)$. Hence, if $\overline{\Sigma}_k$ denotes the invariant polynomial of degree k of $gl\left(n-p;\mathbb{R}\right)$, we have the following result.

Lemma 3.1. *There exists the bijections: 1)* $f_k\left(\alpha\right)\leftrightarrow\widetilde{f}_k\left(A\right)$; *2)* $\Sigma_k\left(\alpha\right)\leftrightarrow\overline{\Sigma}_k\left(A\right)$, $k=1,2,...,n-p,\ \alpha\in S,\ A\in gl\left(n-p;\mathbb{R}\right)$.

3.2 Description of Pontryagin algebra of a partial trivial vector bundle

Theorem 3.1. *Let* $\xi=\left(E,\pi,M\right)$ *be a vector bundle endowed with a partial trivial structure* $\rho_p=\left(s_1,...,s_p\right)$. *Then*
a) $Pont^*\left(E\right)\simeq Pont^*\left(E/E\left(\rho_p\right)\right)$, *where* $\simeq$ *denotes an isomorphism of real algebras;*
b) $p\left(E\right)=p\left(E/E\left(\rho_p\right)\right)$.

Proof. We use the following properties of Pontryagin classes. The classes $p_0\left(E\right)=1$, $p_1\left(E\right)$, ..., $p_{[n/2]}\left(E\right)$, $n=rank\xi$, generate Pontryagin algebra $Pont^*\left(E\right)$. The classes $p_k\left(E\right),\ k=0,1,...,[n/2]$ only depend on isomorphism class $\{\xi\}$ of ξ. They do not depend on the connection of ξ that have been used to represent $p_k\left(E\right)$.

Let $s=\left(\underset{U}{s_1},...,\underset{U}{s_p};s_{p+1},...s_n\right)=\left(\underset{U}{s_i},s_{\widehat{i}}\right)$ $\left(i=1,2,...,p;\ \widehat{i}=p+1,...,n\right)$ be an adapted local frame of ξ (i. e. $\underset{U}{s_i}=s_i|_U,\ U\subset M$ is an open set). Since $E\left(\rho_p\right)$ is a trivial bundle, we can define the connection $\nabla=D_1\oplus D_2$ in E, where D_1 is a flat connection and D_2 is a Riemann connection. With respect to this basis, we write the connections equations:

$$\begin{cases}D_1s_i=\overset{1}{\omega}{}_i^j\otimes s_j=0,\ \text{i. e.}\ \overset{1}{\omega}{}_i^j=0,\ i,j=1,...,p\\ D_2s_{\widehat{i}}=\overset{2}{\omega}{}_{\widehat{i}}^{\widehat{j}}\otimes s_{\widehat{j}},\ \widehat{i},\widehat{j}=p+1,...,n.\end{cases}$$

The curvature forms have the properties: $\overset{1}{\Omega}{}_i^j=0$ and $\overset{2}{\Omega}{}_{\widehat{i}}^{\widehat{j}}=-\overset{2}{\Omega}{}_{\widehat{j}}^{\widehat{i}}$. Then the connection and curvature forms of ∇ can be expressed by matrices

$$\omega=\begin{pmatrix}0&0\\0&\overset{2}{\omega}\end{pmatrix},\ \Omega=\begin{pmatrix}0&0\\0&\overset{2}{\Omega}\end{pmatrix},\overset{2}{\omega}=\begin{pmatrix}\overset{2}{\omega}{}_{\widehat{i}}^{\widehat{j}}\end{pmatrix},\overset{2}{\Omega}=\begin{pmatrix}\overset{2}{\Omega}{}_{\widehat{i}}^{\widehat{j}}\end{pmatrix}.$$

Hence we obtain $p(E(\rho_p)) = 1, p(E) = 1+\sum_{a=1}^{n} p_a(E), p_a(E) = [f_a(\Omega)] = \left[\widetilde{f}_{\widehat{k}}\left(\overset{2}{\Omega}\right)\right] \in H^{4k}(M;\mathbb{R}), p(E) = p(E(\rho_p))\, p(E|E(\rho_p)) = p(E|E(\rho_p))$. This verifies statement b).

Considering the map $\varphi(p_k(E)) = p_k(E|E(\rho_p)), k = 1, 2, ..., n - p$, we can extend φ through linearity to a map $\widetilde{\varphi} : Pont^*(E) \to Pont^*(E|E(\rho_p))$ as follows. If $P = \lambda_{a_1...a_t}(p_1(E))^{a_1} ... (p_t(E))^{a_t}$ is an arbitrary element of $Pont^*(E)$, then:

$$\widetilde{\varphi}(P) = \lambda_{a_1...a_t}(\varphi(p_1(E)))^{a_1} ... (\varphi(p_t(E)))^{a_t} =$$

$$= \lambda_{\widehat{i}_1...\widehat{i}_t}(p_1(E|E(\rho_p)))^{\widehat{i}_1} ... (p_t(E|E(\rho_p)))^{\widehat{i}_t} \in Pont^*(E|E(\rho_p)).$$

The map $\widetilde{\varphi}$ is an isomorphism between $Pont^*(E)$ and $Pont^*(E|E(\rho_p))$. The proof is complete. □

4 An application of partial trivial structures to partial parallelizable manifolds

In this section we consider the special case of partial trivial structure in tangent bundle TM of a differentiable manifold M. The manifold M is called partial parallelizable if its tangent bundle is endowed with a partial trivial structure. We give a partial trivial structure in TM through the existence of a global partial frame $\rho_p = (X_1, ..., X_p)$, where $X_1, ..., X_p \in \Gamma(TM)$, $1 \leq p < m = \dim M$. We see that for $\xi = TM$ the conditions from Theorem 1.1 are equivalent to the partial parallelizability of M. Let $E(\rho_p)$ be the subbundle of TM generated by ρ_p. We analyze only the situation when $E(\rho_p)$ is an involutive subbundle. Denote $\mathcal{F}_p$ the foliation defined by $E(\rho_p)$. Let $F \in \mathcal{F}_p$ be an arbitrary leave and $i_F : F \to M$ be the canonical inclusion. We suppose that the leaves of $\mathcal{F}_p$ are imbedded leaves. In this situation i_F is an imbedding and the topology of F coincides with the topology induced by that of M. Now, consider the maximal integral manifold $\widetilde{M}$ of $E(\rho_p)$. Then the map $i : \widetilde{M} \to M$, $i = \{i_F\}_{F\in\mathcal{F}_p}$, induces a morphism of vector bundles $(Ti, i) : T\widetilde{M} \to TM$, where Ti denotes the tangent map of i. By condition of constant rank for (Ti, i), there is the following exact sequence of vector bundles over $\widetilde{M}$:

$$0 \to T\widetilde{M} \xrightarrow{v} i^*(TM) \xrightarrow{f_{TM}} i^*(TM)/v\left(T\widetilde{M}\right) \to 0.$$

Here $0 = \left(\widetilde{M}, 1_{\widetilde{M}}, \widetilde{M}\right)$, $v(X_x) = (x, (Ti)_x(X_x)) \in i^*(TM)$, $f_{TM}(x, y) = [x, y] =$ the class of (x, y), $x \in \widetilde{M}$, $X_x \in T_x\left(\widetilde{M}\right)$, $y \in T_{i(x)}M$. If M is a paracompact manifold and $\widetilde{M}$ is a paracompact space, then this sequence splits. Therefore, we obtain the following result.

Lemma 4.1. *Let M be a paracompact manifold, $\rho_p = (X_1, ..., X_p)$ be a global partial frame of TM and $E(\rho_p)$ be the subbundle of TM generated by ρ_p. If $E(\rho_p)$ is involutive bundle, let $\mathcal{F}_p$ be the foliation defined by $E(\rho_p)$ and $\widetilde{M}$ the maximal integral manifold* *of* $E(\rho_p)$. *If in addition the leaves of* $\mathcal{F}_p$ *are imbedded, there is an isomorphism of vector bundle over* $\widetilde{M}$:

$$i^*(TM) \simeq T\widetilde{M} \oplus i^*(TM)/v\left(T\widetilde{M}\right).$$

Now we observe that $T\widetilde{M}$ is trivial vector bundle ($\rho_p = (X_1, ..., X_p)$ is a global frame of $T\widetilde{M}$). Consequently, $T\widetilde{M}$ and $\widetilde{M} \times \mathbb{R}^p \to \widetilde{M}$ are two isomorphic vector bundles over $\widetilde{M}$. Hence we obtain the following

Theorem 4.1. *Under the conditions of Lemma* 4.1, *the structure of induced vector bundle* $i^*(TM)$ *is given by the isomorphism* $i^*(TM) \simeq \left(\widetilde{M} \times \mathbb{R}^p\right) \oplus i^*(TM)/v\left(T\widetilde{M}\right)$, *of vector bundles over the paracompact manifold* $\widetilde{M}$.

This result realizes a connection between the partial trivial structures in tangent bundle TM and $\mathcal{F}_p-$flat structures from [4].

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