An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)
Tomul LXIII, 2017, f. 2

Discrete optimization for ordered weak proximal Kannan contractions

Moosa Gabeleh

Received: 23.VIII.2013 / Last revision: 24.II.2014 / Accepted: 25.III.2014

Abstract Let us consider a non-self mapping $T : A \to B$, where A and B are two nonempty subsets of a partially ordered set that is equipped a metric. A best proximity point $x^\star$ for such a mapping T is a point such that $d(x^\star, Tx^\star) = dist(A, B)$. In this work, we provide different existence results of best proximity points and so, we establish some new fixed point theorems in the setting of partially ordered set.

Keywords best proximity point · fixed point · ordered weak proximal Kannan contraction · partially ordered metric space

Mathematics Subject Classification (2010) 47H10 · 47H09

1 Introduction

Let A and B be two nonempty subsets of a metric space X. A non-self mapping $T : A \to B$ is said to be a *contraction* if there exists a constant $r \in [0, 1)$, such that $d(Tx, Ty) \leq rd(x, y)$, for all $x, y \in A$. The *Banach contraction principle* asserts that if a self-mapping $T : X \to X$ is a contraction and (X, d) is complete, then T has a unique fixed point $x \in X$. This result was extended to other important classes of mappings and has numerous applications (see, e.g., [10] and the references therein).

Fixed point theory of partially ordered metric spaces was initiated by Nieto and Rodriguez-Lopez; see the paper [12], where the authors provided some applications to *ordinary differential equations* as well.

Let $(X, \preceq)$ be a partially ordered set. A self mapping $T : X \to X$ is said to be *monotone nondecreasing* iff $T(x) \preceq T(y)$ whenever $x, y \in X, x \preceq y$. In [12], the authors established the following result which is an extension of Banach contraction principle in partially ordered metric spaces.

Theorem 1.1 ([12]) *Let $(X, \preceq)$ be a partially ordered set and let there exist a metric d in X which makes (X, d) into a complete metric space. Assume that X satisfies the*

Moosa Gabeleh
Department of Mathematics,
Ayatollah Boroujerdi University,
Boroujerd, Iran
E-mail: Gabeleh@abru.ac.ir; gab.moo@gmail.com

condition if a nondecreasing sequence $x_n \to x \in X$, then $x_n \preceq x$, $\forall n$. Let $T : X \to X$ be a monotone and nondecreasing mapping for which there exists $\alpha \in [0, 1[$ such that $d(Tx, Ty) \leq \alpha d(x, y)$ for every $y \preceq x$. If there exists $x_0 \in X$ with $x_0 \preceq T(x_0)$, then T has a fixed point.

Let (X, d) be a metric space. A self-mapping $T : X \to X$ is called *Kannan contraction* if there exists $\alpha \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \alpha[d(x, Tx) + d(y, Ty)], \tag{1.1}$$

for all $x, y \in X$. We know that if X is complete metric space, every Kannan self-mapping defined on X has a unique fixed point (see [9]). Note that, the notion of contraction mappings and Kannan contraction mappings are independent. That is, there exists a contraction mapping, which is not Kannan contraction and a Kannan contraction mapping, which is not a contraction. Therefore, we cannot compare these two class of mappings directly.

In [11], Kikkawa and Suzuki proved the following interesting fixed point theorem, which is a generalization of Kannan's fixed point theorem.

Theorem 1.2 ([11]) *Define a non-increasing function φ from $[0, 1)$ into $(\frac{1}{2}, 1]$ by*

$$\varphi(r) = \begin{cases} 1, & \text{if} \quad 0 \leq r < \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r}, & \text{if} \quad \frac{1}{\sqrt{2}} \leq r < 1. \end{cases}$$

Let (X, d) be a complete metric space and let T be a self-mapping on X. Let $\alpha \in [0, \frac{1}{2})$ and put $r := \frac{\alpha}{1-\alpha} \in [0, 1)$. Assume that

$$\varphi(r)d(x, Tx) \leq d(x, y) \quad \text{implies} \quad d(Tx, Ty) \leq \alpha[d(x, Tx) + d(y, Ty)], \tag{1.2}$$

for all $x, y \in X$. Then T has a unique fixed point z and $\lim_n T^n x = z$ holds for every $x \in X$.

Note that the function $\varphi(r)$ defined in Theorem 1.1 is the best constant for every r (see Theorem 2.4 of [11]).

Now, consider the non-self mapping $T : A \to B$, in which A, B are nonempty subsets of a metric space (X, d). Clearly, the fixed point equation $Tx = x$ may not have solution. Hence, it is contemplated to find an approximate $x \in A$ such that the error $d(x, Tx)$ is minimum. A point $x^\star \in A$ is called a *best proximity point* of T if

$$d(x^\star, Tx^\star) = dist(A, B) := \inf\{d(x, y) : (x, y) \in A \times B\}.$$

Best proximity point theory is an interesting subject of optimization theory which recently attracted the attention of many authors (see, for instance, [1,3–7,13,15]).

This paper is basically motivated by the recent paper [14] which the author studied sufficient conditions for the existence of best proximity points in partially ordered metric spaces. We organize it as follows: in Section 2 we introduce some definitions and notations. In Section 3, we establish a best proximity point theorem for *ordered weak proximal Kannan contractions* and we obtain some new fixed point theorems in the setting of partially ordered metric spaces. In Section 4, we conclude a best proximity point result for *ordered weak Kannan contractions* in the setting of uniformly convex Banach spaces equipped a partially ordered relation.

2 Preliminaries

It is assumed throughout this section that $(X, \preceq)$ is a partially ordered set and there exists a metric d on X and let A and B are nonempty subsets of a metric space (X, d). Let us fix the following notations which will be needed throughout this article:

$$\begin{aligned} &d^*(a,b) = d(a,b) - dist(A,B), \quad \forall (a,b) \in A \times B, \\ &A_0 := \{x \in A : d(x,y) = dist(A,B), \quad \text{for some} \quad y \in B\}, \\ &B_0 := \{y \in B : d(x,y) = dist(A,B), \quad \text{for some} \quad x \in A\}. \end{aligned}$$

It is easy to see that if (A, B) is a nonempty, weakly compact and convex pair of subsets of a Banach space X, then A_0 and B_0 are nonempty, closed and convex subsets of X.

Definition 2.1 ([14]) A mapping $T : A \to B$ is said to be a proximally increasing if it satisfies the condition that

$$\begin{cases} x \preceq y, \\ d(u, Tx) = dist(A,B), \\ d(v, Ty) = dist(A,B), \end{cases} \qquad \Rightarrow u \preceq v,$$

for all $x, y, u, v \in A$.

Define a strictly decreasing function θ from $[0, \frac{1}{2})$ onto $(\frac{1}{2}, 1]$ by $\theta(r) = 1 - r$. Let $\alpha \in [0, \frac{1}{2})$ and put $r := \frac{\alpha}{1-\alpha}$.

Definition 2.2 A non-self mapping $T : A \to B$ is said to be an ordered weak proximal Kannan contraction if, for all $u, v, x, y \in A$ with $x \preceq y$, $d(u, Tx) = dist(A, B)$ and $d(v, Ty) = dist(A, B)$, we have

$$\theta(r) d^*(x, Tx) \leq d(x, y) \;\text{ implies }\; d(u, v) \leq \alpha[d^*(x, Tx) + d^*(y, Ty)]. \tag{2.1}$$

Definition 2.3 A non-self mapping $T : A \to B$ is said to be an ordered weak Kannan contraction if, for all $x, y \in A$ with $x \preceq y$ we have

$$\theta(r) d^*(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq \alpha[d^*(x, Tx) + d^*(y, Ty)]. \tag{2.2}$$

Definition 2.4 A non-self mapping $T : A \to B$ is said to be an ordered proximal Kannan contraction if, for all $u, v, x, y \in A$ with $x \preceq y$, $d(u, Tx) = dist(A, B)$ and $d(v, Ty) = dist(A, B)$, we have

$$d(u, v) \leq \alpha[d^*(x, Tx) + d^*(y, Ty)]. \tag{2.3}$$

Definition 2.5 A non-self mapping $T : A \to B$ is said to be an ordered Kannan contraction if, for all $x, y \in A$ with $x \preceq y$

$$d(Tx, Ty) \leq \alpha[d^*(x, Tx) + d^*(y, Ty)]. \tag{2.4}$$

3 Ordered weak proximal Kannan contractions

Let us state our main result of this article.

Theorem 3.1 *Let X be a nonempty set such that $(X, \preceq)$ is a partially ordered set and (X, d) is a complete metric space. Suppose that (A, B) is a nonempty pair of subsets of X such that A_0 is nonempty and closed. Let $T : A \to B$ be a non-self mapping satisfies the following conditions.*

(i) T is a proximally increasing and ordered weak proximal Kannan contraction and $T(A_0) \subseteq B_0$.

(ii) There exist elements $x_0, x_1 \in A_0$ such that $d(x_1, Tx_0) = dist(A, B)$ and $x_0 \preceq x_1$.

(iii) If $\{x_n\}$ is an increasing sequence of elements in A converging to x, then $x_n \preceq x$, for all n.

Then T has a best proximity point.

Proof. By condition (ii) there exist $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and $d(x_1, Tx_0) = dist(A, B)$. Since $Tx_1 \in B_0$, there exists $x_2 \in A_0$ such that $d(x_2, Tx_1) = dist(A, B)$. By the fact that T is proximally increasing we conclude that $x_1 \preceq x_2$. Continuing this process, we can find a sequence $\{x_n\}$ in A_0 such that

$$d(x_{n+1}, Tx_n) = dist(A, B), \quad x_n \preceq x_{n+1}, \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{3.1}$$

For all $n \in \mathbb{N}$ we have $d(x_n, Tx_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n) = d(x_n, x_{n+1}) + dist(A, B)$. Note that $\theta(r) \leq 1$, thus $x_n \preceq x_{n+1}$, $d(x_{n+1}, Tx_n) = d(x_{n+2}, Tx_{n+1}) = dist(A, B)$ and $\theta(r)d^*(x_n, Tx_n) \leq d(x_n, x_{n+1})$. By the fact that T is ordered weak proximal Kannan contraction, $d(x_{n+1}, x_{n+2}) \leq \alpha[d^*(x_n, Tx_n) + d^*(x_{n+1}, Tx_{n+1})] \leq \alpha[d(x_n, x_{n+1}) + d^*(x_{n+1}, Tx_n) + d(x_{n+1}, x_{n+2}) + d^*(x_{n+2}, Tx_{n+1})] = \alpha[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})]$. So,

$$d(x_{n+1}, x_{n+2}) \leq \frac{\alpha}{1-\alpha} d(x_n, x_{n+1}) = r d(x_n, x_{n+1}), \ \forall n \in \mathbb{N}.$$

By induction we obtain $d(x_n, x_{n+1}) \leq r^n d(x_0, x_1)$, which implies that $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) \leq \sum_{n=1}^{\infty} r^n d(x_0, x_1) < \infty$. Thus, $\{x_n\}$ is a Cauchy and increasing sequence in A_0. Since A_0 is closed and X is complete, we deduce that $\{x_n\}$ is a convergent sequence. Let $x^\star \in A_0$ be such that $x_n \to x^\star$. It follows from the condition (iii) that $x_n \preceq x^\star$ for each $n \in \mathbb{N}$. We now assert that

$$d^*(x^\star, Tx) \leq \alpha d(x^\star, x), \quad \forall x \in A_0 \quad \text{with} \quad x_n \preceq x \precneqq x^\star, \forall n \in \mathbb{N}. \tag{3.2}$$

Let $x \in A_0$ be such that $x_n \preceq x$, for all $n \in \mathbb{N}$ and $x \precneqq x^\star$. Since $T(A_0) \subseteq B_0$, there exists $y \in A_0$ such that $d(y, Tx) = dist(A, B)$. By the fact that $x_n \to x^\star$, there exists $N_1 \in \mathbb{N}$ such that $d(x_n, x^\star) \leq \frac{1}{3} d(x, x^\star), \forall n \geq N_1$. We now have

$$\begin{aligned}
\theta(r)d^*(x_n, Tx_n) &\leq d^*(x_n, Tx_n) \\
&\leq d(x_n, x^\star) + d(x^\star, x_{n+1}) + d^*(x_{n+1}, Tx_n) \\
&= d(x_n, x^\star) + d(x^\star, x_{n+1}) \leq \frac{2}{3} d(x, x^\star) \\
&= d(x, x^\star) - \frac{1}{3} d(x, x^\star) \leq d(x, x^\star) - d(x_n, x^\star) \leq d(x_n, x).
\end{aligned}$$

Thereby, for each $n \geq N_1$ $x_n \preceq x$, $d(x_{n+1}, Tx_n) = d(y, Tx) = dist(A, B)$ and $\theta(r)d^*(x_n, Tx_n) \leq d(x_n, x)$. Since T is ordered weak proximal Kannan contraction,

$$d(x_{n+1}, y) \leq \alpha[d^*(x_n, Tx_n) + d^*(x, Tx)] \leq \alpha[d(x_n, x_{n+1}) + d^*(x, Tx)].$$

Hence,

$$\begin{aligned} d(x^\star, Tx) &= \lim_{n\to\infty} d(x_{n+1}, Tx) \\ &\leq \lim_{n\to\infty} [d(x_{n+1}, y) + d(y, Tx)] \\ &\leq \lim_{n\to\infty} [\alpha d(x_n, x_{n+1}) + \alpha d^*(x, Tx) + d(y, Tx)] \\ &\leq \lim_{n\to\infty} [\alpha r^n d(x_0, x_1) + \alpha d^*(x, Tx) + dist(A, B)] \\ &= \alpha d^*(x, Tx) + dist(A, B). \end{aligned}$$

Therefore,

$$d^*(x^\star, Tx) \leq \alpha d^*(x, Tx), \quad \forall x \in A_0, \quad \text{with} \quad x_n \preceq x \precneqq x^\star, \forall n \in \mathbb{N}.$$

That is, (3.2) holds. We now have $d^*(x_n, Tx_n) \leq d(x_n, x^\star) + d^*(x^\star, Tx_n) \leq d(x_n, x^\star) + \alpha d^*(x_n, Tx_n)$. Thus,

$$\theta(r)d^*(x_n, Tx_n) = (1 - r)d^*(x_n, Tx_n) \leq (1 - \alpha)d^*(x_n, Tx_n) \leq d(x_n, x^\star). \tag{3.3}$$

Besides, since $x^\star \in A_0$ and $T(A_0) \subseteq B_0$, there exists $y^\star \in A_0$ such that $d(y^\star, Tx^\star) = dist(A, B)$. So, we have $x_n \preceq x^\star$, $d(x_{n+1}, Tx_n) = d(y^\star, Tx^\star) = dist(A, B)$ and $\theta(r)d^*(x_n, Tx_n) \leq d(x_n, x^\star)$, which concludes that

$$\begin{aligned} d(x_{n+1}, y^\star) &\leq \alpha[d^*(x_n, Tx_n) + d^*(x^\star, Tx^\star)] \\ &\leq \alpha[d(x_n, x_{n+1}) + d^*(x_{n+1}, Tx_n) + d^*(x^\star, Tx^\star)]. \end{aligned}$$

Letting $n \to \infty$ in previous relation, we obtain

$$d(x^\star, y^\star) \leq \alpha d^*(x^\star, Tx^\star) \leq \alpha[d(x^\star, y^\star) + d^*(y^\star, Tx^\star)] = \alpha d(x^\star, y^\star),$$

which implies that $x^\star = y^\star$. Hence, $x^\star$ is a best proximity point of the non-self mapping T and this completes the proof. □

The next results conclude from Theorem 3.1, immediately.

Corollary 3.2 *Let X be a nonempty set such that $(X, \preceq)$ is a partially ordered set and (X, d) is a complete metric space. Suppose that (A, B) is a nonempty pair of subsets of X such that A_0 is nonempty and closed. Let $T : A \to B$ be a non-self mapping satisfies the following conditions.*

(*i*) *T is a proximally increasing and ordered proximal Kannan contraction and $T(A_0) \subseteq B_0$.*

(*ii*) *There exist elements $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = dist(A, B) \quad \text{and} \quad x_0 \preceq x_1.$$

(*iii*) *If $\{x_n\}$ is an increasing sequence of elements in A converging to x, then $x_n \preceq x$, for all n.*

Then T has a best proximity point.

The following corollary is a generalized version of Theorem 1.2 in partially ordered metric spaces.

Corollary 3.3 *Let X be a nonempty set such that $(X, \preceq)$ is a partially ordered set and (X, d) is a complete metric space. Assume that X satisfies the condition*

$$\text{if a nondecreasing sequence } x_n \to x \in X, \text{ then } x_n \preceq x \ \forall n.$$

Let $T : X \to X$ be a monotone and nondecreasing mapping for which

$$\theta(r)d(x, Tx) \leq d(x, y) \quad \text{implies} \quad d(Tx, Ty) \leq \alpha[d(x, Tx) + d(y, Ty)],$$

for every $y \preceq x$. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has a fixed point.

Finally, we deduce the Kannan version of Theorem 1.1 as below.

Corollary 3.4 *Let X be a nonempty set such that $(X, \preceq)$ is a partially ordered set and (X, d) is a complete metric space. Assume that X satisfies the condition*

$$\text{if a nondecreasing sequence } x_n \to x \in X, \text{ then } x_n \preceq x \ \forall n.$$

Let $T : X \to X$ be a monotone and nondecreasing mapping for which

$$d(Tx, Ty) \leq \alpha[d(x, Tx) + d(y, Ty)],$$

for every $y \preceq x$. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has a fixed point.

Example 3.1 Suppose that $X = \mathbb{R}$ with the usual metric and usual ordered relation "$\leq$". Suppose that $A := [0, 2] \cup \{5\}$ and $B := [3, 4]$. Then A and B are nonempty closed subsets of X and $A_0 = \{2, 5\}$ and $B_0 = \{3, 4\}$. Note that $dist(A, B) = 1$. Let $T : A \to B$ be a mapping defined as

$$T(x) = \begin{cases} \frac{7}{2}, & \text{if} \quad x = 0, \\ 4, & \text{if} \quad x \neq 0. \end{cases}$$

Note that $d(u, Tx) = dist(A, B)$ holds for $u = 5$ and for all $x \in A - \{0\}$. Thereby, T is both proximally increasing and ordered weak proximal Kannan contraction. Now, Theorem 3.1 ensures the existence of a best proximity point for the mapping which is a point $x^\star = 5$.

Example 3.2 Let X be the complex valued continuous functions space on $[0, 1]$, where $\mathbb{C}$ is equipped with $\|.\|_1$ norm. Then for each $f \in X$ we have $f = f_1 + if_2$, where f_1 and f_2 are continuous real valued functions on $[0, 1]$. Now let i_d denote the identity mapping defined on $[0, 1]$. Consider the following usual order on X, $f \preceq g \Leftrightarrow f_1(x) \leq g_1(x)$ and $f_2(x) \leq g_2(x), \forall x \in [0, 1]$, where $f = f_1 + if_2$ and $g = g_1 + ig_2$. Suppose that

$$A := \{f = f_1 + if_2 : 0 = f_1(0) \leq f_1(x) \leq f_1(1) = 1, \ f_2(x) = 0, \forall x \in [0, 1]\},$$
$$B := \{g = g_1 + ig_2 : g_1(x) = x, \ 0 = g_2(0) \leq g_2(x) \leq g_2(1) = 1, \forall x \in [0, 1]\}.$$

Note that (A, B) is a closed pair of subsets of the complete metric space X. Also, for each $(f, g) \in A \times B$ we have

$$\begin{aligned}\|f - g\| &= \sup_{0\leq x\leq 1} \|f_1(x) - (x + ig_2(x))\|_1 \\ &= \sup_{0\leq x\leq 1} (|f_1(x) - x| + |g_2(x)|) = \sup_{0\leq x\leq 1} |f_1(x) - x| + \sup_{0\leq x\leq 1} |g(x)| \geq 1.\end{aligned}$$

This implies that $dist(A, B) \geq 1$. Besides, for a pair $(i_d, i_d + ii_d) \in A \times B$ we have $\|i_d - (i_d + ii_d)\| = \sup_{0\leq x\leq 1} \|ii_d(x)\|_1 = \sup_{0\leq x\leq 1} |i_d(x)| = 1$, which implies that $dist(A, B) = 1$. Moreover, we note that $A_0 = \{i_d\}$ and $B_0 = B$. Now, define the mapping $T : A \to B$ by $T(f_1 + if_2) = i_d + if_1$, for all $f_1 + if_2 \in A$. Then T is both proximally increasing and ordered weak proximal Kannan contraction. Also, A satisfies the condition (iii) of Theorem 3.1. Hence, the existence of a best proximity point is deduced from Theorem 3.1.

4 Ordered weak Kannan contractions

Here, we recall a geometric notion on a nonempty pair of subsets of a metric space (X, d).

Definition 4.1 ([8]) Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. The pair (A, B) is said to have WP-property if and only if

$$\begin{cases} d(x_1, y_1) = dist(A, B) \\ d(x_2, y_2) = dist(A, B) \end{cases} \Rightarrow d(x_1, x_2) \leq d(y_1, y_2),$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

It was announced in [2] that every nonempty, bounded, closed and convex pair of subsets of a uniformly convex Banach space X has the WP-property. Also, it is clear that if (A, B) be a nonempty pair of subsets of a metric space (X, d) such that $dist(A, B) = 0$ then (A, B) has the WP-property.

Here, we state the relationship between ordered weak proximal Kannan contractions and ordered weak Kannan contractions.

Proposition 4.2 *Let X be a nonempty set such that $(X, \preceq)$ is a partially ordered set and (X, d) is a complete metric space. Suppose that (A, B) is a nonempty pair of subsets of X such that A_0 is nonempty and (A, B) has the WP-property. Assume that $T : A \to B$ is a mapping such that $T(A_0) \subseteq B_0$.*

(i) If T is ordered weak Kannan contraction, then T is ordered weak proximal Kannan contraction.

(ii) If T is ordered weak proximal Kannan contraction and moreover, (B, A) has the WP- property, then $T|_{A_0}$ is ordered weak Kannan contraction.

Proof. (i) Let T be an ordered weak Kannan contraction mapping and $u, v, x, y \in A$ be such that

$$\begin{cases} x \preceq y, \\ d(u, Tx) = dist(A, B), \\ d(v, Ty) = dist(A, B), \end{cases} \quad \text{and} \quad \theta(r)d^*(x, Tx) \leq d(x, y).$$

Since (A, B) has the WP-property and T is ordered weak Kannan contraction,

$$d(u, v) \leq d(Tx, Ty) \leq \alpha[d^*(x, Tx) + d^*(y, Ty)],$$

which implies that T is ordered weak proximal Kannan contraction.

(ii) Suppose that T is ordered weak proximal Kannan contraction and $x, y \in A_0$ be such that $x \preceq y, \eta(r)d^*(x, Tx) \leq d(x, y)$. Since $T(A_0) \subseteq B_0$, there exist $u, v \in A_0$ such that

$$\begin{cases} d(u, Tx) = dist(A, B) \\ d(v, Ty) = dist(A, B). \end{cases}$$

In view of the fact that T is ordered weak proximal Kannan contraction and both (A, B) and (B, A) have the WP-property, we must have $d(Tx, Ty) = d(u, v) \leq \alpha[d^*(x, Tx) + d^*(y, Ty)]$, which concludes that $T|_{A_0}$ is ordered weak Kannan contraction. □

The following result is obtained, immediately.

Theorem 4.3 *Let $(X, \preceq)$ be a partially ordered set and let there exist a metric d in X which makes (X, d) into a complete metric space. Suppose that (A, B) is a nonempty closed pair of subsets of X such that (A, B) has the WP-property. Assume that A_0 is nonempty and closed. Let $T : A \to B$ be a non-self mapping satisfies the following conditions.*

(i) T is a proximally increasing and ordered weak Kannan contraction and $T(A_0) \subseteq B_0$.
(ii) There exist elements $x_0, x_1 \in A_0$ such that $d(x_1, Tx_0) = dist(A, B)$ and $x_0 \preceq x_1$.
(iii) If $\{x_n\}$ is an increasing sequence of elements in A converging to x, then $x_n \preceq x$, for all n.

Then T has a best proximity point.

Finally, we state the next best proximity point theorems in the setting of uniformly convex Banach spaces equipped a partially ordered relation.

Theorem 4.4 *Let (A, B) be a nonempty, bounded, closed and convex pair of subsets of a uniformly convex Banach space X and let "$\preceq$" be a partially ordered relation on X. Let $T : A \to B$ be a non-self mapping satisfies the following conditions.*

(i) T is a proximally increasing and ordered weak Kannan contraction and $T(A_0) \subseteq B_0$.
(ii) There exist elements $x_0, x_1 \in A_0$ such that $d(x_1, Tx_0) = dist(A, B)$ and $x_0 \preceq x_1$.
(iii) If $\{x_n\}$ is an increasing sequence of elements in A converging to x, then $x_n \preceq x$, for all n.

Then T has a best proximity point.

Theorem 4.5 *Let (A, B) be a nonempty, bounded, closed and convex pair of subsets of a uniformly convex Banach space X and let "$\preceq$" be a partially ordered relation on X. Let $T : A \to B$ be a non-self mapping satisfies the following conditions.*

(i) T is a proximally increasing and ordered Kannan contraction and $T(A_0) \subseteq B_0$.
(ii) There exist elements $x_0, x_1 \in A_0$ such that $d(x_1, Tx_0) = dist(A, B)$ and $x_0 \preceq x_1$.
(iii) If $\{x_n\}$ is an increasing sequence of elements in A converging to x, then $x_n \preceq x$, for all n.

Then T has a best proximity point.

Acknowledgements The author would like to thank the reviewer for his/her valuable suggestions.

References

1. Abkar, A.; Gabeleh, M. – *Best proximity points for cyclic mappings in ordered metric spaces*, J. Optim. Theory Appl., 151 (2011), 418–424.
2. Abkar, A.; Gabeleh, M. – *Global optimal solutions of noncyclic mappings in metric spaces*, J. Optim. Theory Appl., 153 (2012), 298–305.
3. Al-Thagafi, M.A.; Shahzad, N. – *Convergence and existence results for best proximity points*, Nonlinear Anal., 70 (2009), 3665–3671.
4. Amini-Harandi, A. – *Best proximity point theorems for cyclic strongly quasi-contraction mappings*, J. Global Optim., 56 (2013), 1667–1674.
5. Di Bari, C.; Suzuki, T.; Vetro, C. – *Best proximity points for cyclic Meir-Keeler contractions*, Nonlinear Anal., 69 (2008), 3790–3794.
6. Eldred, A.A.; Veeramani, P. – *Existence and convergence of best proximity points*, J. Math. Anal. Appl., 323 (2006), 1001–1006.
7. Espínola, R. – *A new approach to relatively nonexpansive mappings*, Proc. Amer. Math. Soc., 136 (2008), 1987–1995.
8. Gabeleh, M. – *Global optimal solutions of non-self mappings*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., 75 (2013), 67–74.
9. Kannan, R. – *Some results on fixed points. II*, Amer. Math. Monthly, 76 (1969), 405–408.
10. Khamsi, M.A.; Kirk, W.A. – *An introduction to metric spaces and fixed point theory*, Pure and Applied Mathematics (New York), Wiley–Interscience, New York, 2001.
11. Kikkawa, M.; Suzuki, T. – *Some similarity between contractions and Kannan mappings*, Fixed Point Theory Appl., 2008, Art. ID 649749, 8 pp.
12. Nieto, J.J.; Rodríguez-López, R. – *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order, 22 (2005), 223–239 (2006).
13. Petric, M.A. – *Best proximity point theorems for weak cyclic Kannan contractions*, Filomat, 25 (2011), 145–154.
14. Sadiq Basha, S. – *Discrete optimization in partially ordered sets*, J. Global Optim., 54 (2012), 511–517.
15. Sanhan, W.; Mongkolkeha, C.; Kumam, P. – *Generalized proximal ψ-contraction mappings and best proximity points*, Abstr. Appl. Anal., 2012, Art. ID 896912, 19 pp.