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# Direct method and approximation of the reciprocal difference functional equations in various normed spaces

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**Abstract** In this paper, using direct method, we prove the generalized Hyers-Ulam-Rassias stability of the reciprocal difference functional equations in various normed spaces.

**Keywords** Hyers-Ulam-Rassias stability · non-Archimedean normed spaces · random normed spaces · fuzzy normed space

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## 1 Introduction and preliminaries

By a *non-Archimedean field* we mean a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot| : \mathbb{K} \to [0, \infty)$ such that, for all $r, s \in \mathbb{K}$, the following conditions hold:
$(a)$ $|r| = 0$ if and only if $r = 0$;
$(b)$ $|rs| = |r||s|$;
$(c)$ $|r + s| \leq \max\{|r|, |s|\}$.
Clearly, by (b), $|1| = |-1| = 1$ and so, by induction, it follows from (c) that $|n| \leq 1$ for all $n \geq 1$.

**Definition 1.1** Let $X$ be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$.
(1) A function $\|\cdot\| : X \to \mathbb{R}$ is a *non-Archimedean norm* (valuation) if it satisfies the following conditions:
$(a)$ $\|x\| = 0$ if and only if $x = 0$ for all $x \in X$;
$(b)$ $\|rx\| = |r|\|x\|$ for all $r \in \mathbb{K}$ and $x \in X$;
$(c)$ the strong triangle inequality (ultra-metric) holds, that is, $\|x + y\| \leq \max\{\|x\|, \|y\|\}$, for all $x, y \in X$.
(2) The space $(X, \|\cdot\|)$ is called a non-Archimedean normed space.

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Note that $||x_n - x_m|| \leq \max\{||x_{j+1} - x_j|| : m \leq j \leq n-1\}$, for all $m, n \in \mathbb{N}$ with $n > m$.

**Definition 1.2** Let $(X, \|\cdot\|)$ be a non-Archimedean normed space.
($a$) A sequence $\{x_n\}$ is a *Cauchy sequence* in $X$ if $\{x_{n+1} - x_n\}$ converges to zero in $X$.
($b$) The non-Archimedean normed space $(X, \|\cdot\|)$ is said to be *complete* if every Cauchy sequence in $X$ is convergent.

The most important examples of non-Archimedean spaces are $p$-adic numbers. A key property of $p$-adic numbers is that they do not satisfy the Archimedean axiom: for all $x, y > 0$, there exists a positive integer $n$ such that $x < ny$.

*Example 1.1* Fix a prime number $p$. For any nonzero rational number $x$, there exists a unique positive integer $n_x$ such that $x = \frac{a}{b}p^{n_x}$, where $a$ and $b$ are positive integers not divisible by $p$. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x-y|_p$ is denoted by $\mathbb{Q}_p$, which is called the *p-adic number field*. In fact, $\mathbb{Q}_p$ is the set of all formal series $x = \sum_{k \geq n_x}^{\infty} a_k p^k$, where $|a_k| \leq p-1$. The addition and multiplication between any two elements of $\mathbb{Q}_p$ are defined naturally. The norm $|\sum_{k \geq n_x}^{\infty} a_k p^k|_p = p^{-n_x}$ is a non-Archimedean norm on $\mathbb{Q}_p$ and $\mathbb{Q}_p$ is a locally compact filed.

In section 3, we adopt the usual terminology, notions and conventions of the theory of random normed spaces as in [55].

Throughout this paper, let $\triangle^+$ denote the set of all probability distribution functions $F : \mathbb{R} \cup [-\infty, +\infty] \to [0, 1]$ such that $F$ is left-continuous and nondecreasing on $\mathbb{R}$ and $F(0) = 0, F(+\infty) = 1$. It is clear that the set $D^+ = \{F \in \triangle^+ : l^- F(-\infty) = 1\}$, where $l^- f(x) = \lim_{t \to x^-} f(t)$, is a subset of $\triangle^+$. The set $\triangle^+$ is partially ordered by the usual point-wise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. For any $a \geq 0$, the element $H_a(t)$ of $D^+$ is defined by

$$H_a(t) = \begin{cases} 0, & \text{if } t \leq a, \\ 1, & \text{if } t > a. \end{cases}$$

We can easily show that the maximal element in $\triangle^+$ is the distribution function $H_0(t)$.

**Definition 1.3** A function $T : [0, 1]^2 \to [0, 1]$ is a *continuous triangular norm* (briefly, a $t$-norm) if $T$ satisfies the following conditions:
($a$) $T$ is commutative and associative;
($b$) $T$ is continuous;
($c$) $T(x, 1) = x$ for all $x \in [0, 1]$;
($d$) $T(x, y) \leq T(z, w)$ whenever $x \leq z$ and $y \leq w$, for all $x, y, z, w \in [0, 1]$.

Three typical examples of continuous $t$-norms are as follows: $T_P(x, y) = xy$, $T_{max}(x, y) = \max\{a + b - 1, 0\}$, $T_M(x, y) = \min(a, b)$. Recall that, if $T$ is a $t$-norm and $\{x_n\}$ is a sequence in $[0, 1]$, then $T_{i=1}^n x_i$ is defined recursively by $T_{i=1}^1 x_1 = x_1$ and $T_{i=1}^n x_i = T(T_{i=1}^{n-1} x_i, x_n)$ for all $n \geq 2$. $T_{i=n}^{\infty} x_i$ is defined by $T_{i=1}^{\infty} x_{n+i}$.

**Definition 1.4** A *random normed space* is a triple $(X, \mu, T)$, where $X$ is a vector space, $T$ is a continuous $t$-norm and $\mu : X \to D^+$ is a mapping such that the following conditions hold:
$(a)$ $\mu_x(t) = H_0(t)$ for all $t > 0$ if and only if $x = 0$;
$(b)$ $\mu_{\alpha x}(t) = \mu_x\left(\frac{t}{|\alpha|}\right)$ for all $\alpha \in \mathbb{R}$ with $\alpha \neq 0$, $x \in X$ and $t \geq 0$;
$(c)$ $\mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and $t, s \geq 0$.

Every normed space $(X, \|\cdot\|)$ defines a random normed space $(X, \mu, T_M)$, where $\mu_u(t) = \frac{t}{t+\|u\|}$ for all $t > 0$ and $T_M$ is the minimum $t$-norm. This space $X$ is called the *induced random normed space*.

If the $t$-norm $T$ is such that $\sup_{0<a<1} T(a,a) = 1$, then every random normed space $(X, \mu, T)$ is a metrizable linear topological space with the topology $\tau$ (called the $\mu$*-topology* or the $(\epsilon, \delta)$*-topology*, where $\epsilon > 0$ and $\lambda \in (0,1)$) induced by the base $\{U(\epsilon, \lambda)\}$ of neighborhoods of $\theta$, where $U(\epsilon, \lambda) = \{x \in X : \mu_x(\epsilon) > 1 - \lambda\}$.

**Definition 1.5** Let $(X, \mu, T)$ be an random normed space.
$(a)$ A sequence $\{x_n\}$ in $X$ is said to be *convergent* to a point $x \in X$ (write $x_n \to x$ as $n \to \infty$) if $\lim_{n\to\infty} \mu_{x_n - x}(t) = 1$, for all $t > 0$.
$(b)$ A sequence $\{x_n\}$ in $X$ is called a *Cauchy sequence* in $X$ if $\lim_{n\to\infty} \mu_{x_n - x_m}(t) = 1$, for all $t > 0$.
$(c)$ A random normed space $(X, \mu, T)$ is said to be *complete* if every Cauchy sequence in $X$ is convergent.

**Theorem 1.6** *If $(X, \mu, T)$ be a random normed space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$.*

**Definition 1.7** Let $X$ be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a fuzzy norm on $X$ if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,
$(N1)$ $N(x,t) = 0$ for $t \leq 0$;
$(N2)$ $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;
$(N3)$ $N(cx,t) = N\left(x, \frac{t}{|c|}\right)$ if $c \neq 0$;
$(N4)$ $N(x+y, c+t) \geq \min\{N(x,s), N(y,t)\}$;
$(N5)$ $N(x,.)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$;
$(N6)$ for $x \neq 0$, $N(x,.)$ is continuous on $\mathbb{R}$.

The pair $(X, N)$ is called a fuzzy normed vector space.

*Example 1.2* Let $(X, \|.\|)$ be a normed linear space and $\alpha, \beta > 0$. Then

$$N(x,t) = \begin{cases} \dfrac{\alpha t}{\alpha t + \beta\|x\|}, & t > 0, x \in X \\ 0, & t \leq 0, x \in X \end{cases}$$

is a fuzzy norm on $X$.

**Definition 1.8** Let $(X, N)$ be a fuzzy normed vector space. A sequence $\{x_n\}$ in $X$ is said to be convergent or converge if there exists an $x \in X$ such that $\lim_{t\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, $x$ is called the limit of the sequence $\{x_n\}$ in $X$ and we denote it by $N - \lim_{t\to\infty} x_n = x$.

**Definition 1.9** Let $(X, N)$ be a fuzzy normed vector space. A sequence $\{x_n\}$ in $X$ is called Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \epsilon$.

It is well known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces $X$ and $Y$ is continuous at a point $x \in X$ if for each sequence $\{x_n\}$ converging to $x_0 \in X$, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be continuous on $X$.

**Definition 1.10** Let $X$ be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on $X$ if $d$ satisfies the following conditions:
$(a)$ $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;
$(b)$ $d(x, y) = d(y, x)$ for all $x, y \in X$;
$(c)$ $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

**Theorem 1.11** *Let $(X, d)$ be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in X$, either $d(J^n x, J^{n+1} x) = \infty$ for all nonnegative integers $n$ or there exists a positive integer $n_0$ such that*:
$(a)$ $d(J^n x, J^{n+1} x) < \infty$*, for all $n_0 \geq n_0$;*
$(b)$ *the sequence $\{J^n x\}$ converges to a fixed point $y^*$ of $J$;*
$(c)$ *$y^*$ is the unique fixed point of $J$ in the set $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$;*
$(d)$ $d(y, y^*) \leq \frac{d(y,Jy)}{1-L}$*, for all $y \in Y$.*

A classical question in the theory of functional equations is the following: *When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?*. If the problem accepts a solution, we say that the equation is *stable*. The first stability problem concerning group homomorphisms was raised by Ulam [57] in 1940. In the next year, Hyers [24] gave a positive answer to the above question for additive groups under the assumption that the groups are Banach spaces. In 1978, Rassias [43] proved a generalization of Hyers' theorem for linear mappings.

This new concept is known as generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations. Furthermore, in 1994, a generalization of Rassias' theorem was obtained by Găvruta [22] by replacing the bound $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$.

In 1983, a generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [56] for mappings $f : X \to Y$, where $X$ is a normed space and $Y$ is a Banach space. In 1984, Cholewa [12] noticed that the theorem of Skof is still true if the relevant domain $X$ is replaced by an Abelian group and, in 2002, Czerwik [14] proved the generalized Hyers-Ulam stability of the quadratic functional equation. The reader is referred to ([1]-[8], [12]-[14], [16]-[22], [24]-[25], [32]-[35], [37]-[54]) and references therein for detailed information on stability of functional equations.

In 1897, Hensel [23] has introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications (see [15, 27, 30, 31, 36]).

Katsaras [26] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view (see [21], [29], [37]).

In particular, Bag and Samanta [9], following Cheng and Mordeson [11], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Karmosil and Michalek type [28]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [10].

In this paper, using the fixed point and direct methods, we prove the Hyers-Ulam-Rassias stability of the following reciprocal difference functional equation:

$$f\left(\frac{x+y+z}{3}\right) = f(x+y+z) + \frac{2f(x)f(y)f(z)}{f(x)f(y)+f(y)f(z)+f(z)f(z)} \tag{1.1}$$

in various normed spaces.

## 2 Non-Archimedean stability of reciprocal difference functional equation (1.1)

In this section, using direct method, we prove the Hyers-Ulam-Rassias stability of the reciprocal difference mapping (1.1) in non-Archimedean normed spaces.

**Theorem 2.1** *Let $G$ be an additive semigroup and that $X$ is a complete non-Archimedean normed space. Assume that $\zeta : G^3 \to [0, +\infty)$ is a function such that*

$$\lim_{n\to\infty} \frac{1}{|3|^n} \zeta\left(\frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n}\right) = 0, \tag{2.1}$$

*for all $x, y, z \in G$. Suppose that, for any $x \in G$, the limit*

$$\pounds(x) = \lim_{n\to\infty} \max_{0\le k<n} \frac{1}{|3|^{k+1}} \zeta\left(\frac{x}{3^{k+1}}, \frac{x}{3^{k+1}}, \frac{x}{3^{k+1}}\right), \tag{2.2}$$

*exists and $f : G \to X$ be a mapping satisfying*

$$\left\| f\left(\frac{x+y+z}{3}\right) - f(x+y+z) - \frac{2f(x)f(y)f(z)}{f(x)f(y)+f(y)f(z)+f(z)f(z)} \right\|_X \le \zeta(x, y, z), \tag{2.3}$$

*for all $x, y, z \in G$. Then the limit $A(x) := \lim_{n\to\infty} \frac{1}{3^n} f\left(\frac{x}{3^n}\right)$ exists for all $x \in G$ and defines a reciprocal difference mapping $A : G \to X$ satisfies* (1.1) *and the following inequality*

$$\|f(x) - A(x)\|_X \le |3| \pounds(x). \tag{2.4}$$

*Moreover, if*

$$\lim_{j\to\infty} \lim_{n\to\infty} \max_{j\le k<n+j} \frac{1}{|3|^{k+1}} \zeta\left(\frac{x}{3^{k+1}}, \frac{y}{3^{k+1}}, \frac{z}{3^{k+1}}\right) = 0$$

*then $A$ is the unique reciprocal difference satisfying* (2.4).

*Proof.* Putting $x = y = z = \frac{x}{3}$ in (2.3), we get

$$\left\| \frac{1}{3} f\left(\frac{x}{3}\right) - f(x) \right\|_X \leq \zeta\left(\frac{x}{3}, \frac{x}{3}, \frac{x}{3}\right), \tag{2.5}$$

for all $x \in G$. Replacing $x$ by $\frac{x}{3^n}$ in (2.5), we obtain

$$\left\| \frac{f\left(\frac{x}{3^{n+1}}\right)}{3^{n+1}} - \frac{1}{3^n} f\left(\frac{x}{3^n}\right) \right\|_X \leq \frac{1}{|3|^n} \zeta\left(\frac{x}{3^{n+1}}, \frac{x}{3^{n+1}}, \frac{x}{3^{n+1}}\right). \tag{2.6}$$

Thus, it follows from (2.1) and (2.6) that the sequence $\left\{\frac{1}{3^n} f\left(\frac{x}{3^n}\right)\right\}_{n\geq 1}$ is a Cauchy sequence. Since $X$ is complete, it follows that $\left\{\frac{1}{3^n} f\left(\frac{x}{3^n}\right)\right\}_{n\geq 1}$ is convergent. Set

$$A(x) := \lim_{n\to\infty} \frac{1}{3^n} f\left(\frac{x}{3^n}\right). \tag{2.7}$$

By induction on $n$, one can show that

$$\left\| \frac{f\left(\frac{x}{3^n}\right)}{3^n} - f(x) \right\|_X \leq \max\left\{ \frac{\zeta\left(\frac{x}{3^{k+1}}, \frac{x}{3^{k+1}}, \frac{x}{3^{k+1}}\right)}{|3|^k}; 0 \leq k < n \right\}, \tag{2.8}$$

for all $n \geq 1$ and $x \in G$. By taking $n \to \infty$ in (2.8) and using (2.2), one obtains (2.4). By (2.1), (2.3) and (2.7), we get

$$\begin{aligned}
&\left\| A\left(\frac{x+y+z}{3}\right) - A(x+y+z) - \frac{2A(x)A(y)A(z)}{A(x)A(y)+A(y)A(z)+A(z)A(z)} \right\|_X \\
&= \lim_{n\to\infty} \frac{\left\| f\left(\frac{x+y+z}{3^{n+1}}\right) - f\left(\frac{x+y+z}{3^n}\right) - \frac{2f\left(\frac{x}{3^n}\right)f\left(\frac{y}{3^n}\right)f\left(\frac{z}{3^n}\right)}{f\left(\frac{x}{3^n}\right)f\left(\frac{y}{3^n}\right)+f\left(\frac{y}{3^n}\right)f\left(\frac{z}{3^n}\right)+f\left(\frac{z}{3^n}\right)f\left(\frac{x}{3^n}\right)} \right\|_X}{|3|^n} \\
&\leq \lim_{n\to\infty} \frac{1}{|3|^n} \zeta\left(\frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n}\right) = 0,
\end{aligned}$$

for all $x, y, z \in G$ . So

$$A\left(\frac{x+y+z}{3}\right) = A(x+y+z) + \frac{2A(x)A(y)A(z)}{A(x)A(y)+A(y)A(z)+A(z)A(z)},$$

for all $x, y, z \in G$. Hence $A : G \to X$ is a reciprocal difference mapping.

To prove the uniqueness property of $A$, let $L$ be another mapping satisfying (2.4). Then we have

$$\begin{aligned}
\left\| A(x) - L(x) \right\|_X &= \lim_{j\to\infty} \frac{1}{|3|^j} \left\| A\left(\frac{x}{3^j}\right) - L\left(\frac{x}{3^j}\right) \right\|_X \\
&\leq \lim_{j\to\infty} \frac{\max\left\{ \left\| A\left(\frac{x}{3^j}\right) - f\left(\frac{x}{3^j}\right) \right\|_X, \left\| f\left(\frac{x}{3^j}\right) - L\left(\frac{x}{3^j}\right) \right\|_X \right\}}{|3|^j} \\
&\leq \lim_{j\to\infty} \lim_{n\to\infty} \max_{j\leq k<n+j} \frac{1}{|3|^k} \zeta\left(\frac{x}{3^{k+1}}, \frac{x}{3^{k+1}}, \frac{x}{3^{k+1}}\right) = 0,
\end{aligned}$$

for all $x \in G$. Therefore, $A = L$. This completes the proof. $\square$

**Corollary 2.2** *Let $\xi : [0,\infty) \to [0,\infty)$ be a function satisfying*

$$\xi\left(\frac{t}{|3|}\right) \leq \xi\left(\frac{1}{|3|}\right)\xi(t), \quad \xi\left(\frac{1}{|3|}\right) < |3|,$$

*for all $t \geq 0$. Assume that $\kappa > 0$ and $f : G \to X$ is a mapping such that*

$$\left\| f\left(\frac{x+y+z}{3}\right) - f(x+y+z) - \frac{2f(x)f(y)f(z)}{f(x)f(y)+f(y)f(z)+f(z)f(z)} \right\|_X \\ \leq \kappa\left(\xi(|x|)+\xi(|y|)+\xi(|z|)\right),$$

*for all $x,y,z \in G$. Then there exists a unique reciprocal difference mapping $A : G \to X$ such that $\|f(x) - A(x)\|_X \leq 3|3|\kappa\xi(|x|)$.*

*Proof.* If we define $\zeta : G^3 \to [0,\infty)$ by $\zeta(x,y,z) := \kappa(\xi(|x|)+\xi(|y|)+\xi(|z|))$, then we have

$$\lim_{n\to\infty} \frac{1}{|3|^n}\zeta\left(\frac{x}{3^n},\frac{y}{3^n},\frac{z}{3^n}\right) = \lim_{n\to\infty}\left[\frac{1}{|3|}\xi\left(\frac{1}{3}\right)\right]^n \zeta(x,y,z) = 0,$$

for all $x,y,z \in G$. On the other hand, it follows that

$$\pounds(x) = \lim_{n\to\infty}\max_{0\leq k<n}\frac{1}{|3|^{k+1}}\zeta\left(\frac{x}{3^{k+1}},\frac{x}{3^{k+1}},\frac{x}{3^{k+1}}\right) = \frac{1}{|3|}\zeta\left(\frac{x}{3},\frac{x}{3},\frac{x}{3}\right) = 3\kappa\xi(|x|),$$

exists for all $x \in G$. Also, we have

$$\lim_{j\to\infty}\lim_{n\to\infty}\max_{j\leq k<n+j}\frac{\zeta\left(\frac{x}{3^{k+1}},\frac{x}{3^{k+1}},\frac{x}{3^{k+1}}\right)}{|3|^{k+1}} = \lim_{j\to\infty}\frac{\zeta\left(\frac{x}{3^{j+1}},\frac{x}{3^{j+1}},\frac{x}{3^{j+1}}\right)}{|3|^{j+1}} = 0.$$

Applying Theorem 2.1, we get the desired results. □

**Theorem 2.3** *Let $G$ be an additive semigroup and that $X$ is a complete non-Archimedean normed space. Assume that $\zeta : G^3 \to [0,+\infty)$ is a function such that*

$$\lim_{n\to\infty}|3|^n\zeta\Big(3^n x, 3^n y, 3^n z\Big) = 0, \tag{2.9}$$

*for all $x,y,z \in G$. Suppose that, for any $x \in G$, the limit*

$$\pounds(x) = \lim_{n\to\infty}\max_{0\leq k<n}|3|^k\zeta\left(3^k x, 3^k x, 3^k x\right), \tag{2.10}$$

*exists and $f : G \to X$ is a mapping satisfying* (2.3). *Then the limit $A(x) := \lim_{n\to\infty} 3^n f(3^n x)$ exists for all $x \in G$ and*

$$\|f(x) - A(x)\|_X \leq |3|\pounds(x), \tag{2.11}$$

*for all $x \in G$. Moreover, if*

$$\lim_{j\to\infty}\lim_{n\to\infty}\max_{j\leq k<n+j}|3|^k\zeta\left(3^k x, 3^k x, 3^k x\right) = 0,$$

*then $A$ is the unique mapping satisfying* (2.11).

*Proof.* It follows from (2.5), we get

$$\|f(x)-3f(3x)\|_X \leq |3|\zeta(x,x,x), \tag{2.12}$$

for all $x\in G$. Replacing $x$ by $3^n x$ in (2.12), we obtain

$$\left\|3^n f(3^n x)-3^{n+1}f(3^{n+1}x)\right\|_X \leq |3|^{n+1}\zeta\left(3^n x,3^n x,3^n x\right). \tag{2.13}$$

Thus it follows from (2.9) and (2.13) that the sequence $\{3^n f(3^n x)\}_{n\geq 1}$ is convergent. Set $A(x):=\lim_{n\to\infty}3^n f(3^n x)$. On the other hand, it follows from (2.13) that

$$\begin{aligned}\|3^p f(3^p x)-3^q f(3^q x)\| &= \left\|\sum_{k=p}^{q-1}3^{k+1}f(3^{k+1}x)-3^k f(3^k x)\right\|_X\\ &\leq \max_{p\leq k<q-1}\left\{\left\|3^{k+1}f(3^{k+1}x)-3^k f(3^k x)\right\|_X\right\}\\ &\leq |3|\max_{p\leq k<q}|3|^k\zeta(3^k x,3^k x,3^k x),\end{aligned}$$

for all $x\in G$ and $p,q\geq 0$ with $q>p\geq 0$. Letting $p=0$, taking $q\to\infty$ in the last inequality and using (2.10), we obtain (2.11).

The rest of the proof is similar to the proof of Theorem 2.1. □

**Corollary 2.4** *Let $\xi:[0,\infty)\to[0,\infty)$ be a function satisfying $\xi\left(|3|\,t\right)\leq\xi\left(|3|\right)\xi(t)$, $\xi\left(|3|\right)<\frac{1}{|3|}$, for all $t\geq 0$. Let $\kappa>0$ and $f:G\to X$ is a mapping satisfying*

$$\begin{aligned}&\left\|f\left(\frac{x+y+z}{3}\right)-f(x+y+z)-\frac{2f(x)f(y)f(z)}{f(x)f(y)+f(y)f(z)+f(z)f(z)}\right\|_X\\ &\leq\kappa\left(\xi(|x|)\cdot\xi(|y|)\cdot\xi(|z|)\right),\end{aligned}$$

*for all $x,y,z\in G$. Then there exists a unique reciprocal difference mapping $A:G\to X$ such that $\|f(x)-A(x)\|\leq|3|\kappa\xi(|x|)^3$.*

*Proof.* If we define $\zeta:G^3\to[0,\infty)$ by $\zeta(x,y,z):=\kappa\left(\xi(|x|)\cdot\xi(|y|)\cdot\xi(|z|)\right)$ and applying Theorem 2.3, then we get the desired results. □

## 3 Random stability of the reciprocal difference mapping (1.1)

Throughout this section, using direct method we prove the Hyers-Ulam-Rassias stability of the reciprocal difference mapping (1.1) in random normed space.

**Theorem 3.1** *Let $X$ be a real linear space, $(Z,\mu',\min)$ be a random normed space and $\varphi:X^3\to Z$ be a function such that there exists $0<\alpha<3$ such that*

$$\mu'_{\varphi(\frac{x}{3},\frac{y}{3},\frac{z}{3})}(t)\geq\mu'_{\varphi(x,y,z)}\left(\frac{t}{\alpha}\right), \tag{3.1}$$

*for all $x, y, z \in X$ and $t > 0$ and $\lim_{n\to\infty} \mu'_{\varphi(\frac{x}{3^n},\frac{y}{3^n},\frac{z}{3^n})}(3^n t) = 1$ for all $x, y, z \in X$ and $t > 0$. Let $(Y, \mu, \min)$ be a complete random normed space. Let $f : X \to Y$ be a mapping satisfying*

$$\mu_{f\left(\frac{x+y+z}{3}\right)-f(x+y+z)-\frac{2f(x)f(y)f(z)}{f(x)f(y)+f(y)f(z)+f(z)f(z)}}(t) \geq \mu'_{\varphi(x,y,z)}(t), \tag{3.2}$$

*for all $x, y, z \in X$ and $t > 0$. Then the limit $A(x) = \lim_{n\to\infty} \frac{1}{3^n} f\left(\frac{x}{3^n}\right)$ exists for all $x \in X$ and defines a unique reciprocal difference mapping $A : X \to Y$ satisfies (1.1) and the following inequality*

$$\mu_{f(x)-A(x)}(t) \geq \mu'_{\varphi(x,x,x)}\left(\frac{(3-\alpha)t}{3\alpha}\right), \tag{3.3}$$

*for all $x \in X$ and $t > 0$.*

*Proof.* Putting $x = y = z = \frac{x}{3}$ in (3.2), we have

$$\mu_{\frac{f(\frac{x}{3})}{3}-f(x)}(t) \geq \mu'_{\varphi\left(\frac{x}{3},\frac{x}{3},\frac{x}{3}\right)}(t) \geq \mu'_{\varphi(x,x,x)}\left(\frac{t}{\alpha}\right), \tag{3.4}$$

for all $x \in X$. Replacing $x$ by $\frac{x}{3^n}$ in (3.4) and using (3.1), we obtain

$$\mu_{\frac{f\left(\frac{x}{3^{n+1}}\right)}{3^{n+1}}-\frac{f\left(\frac{x}{3^n}\right)}{3^n}}(t) \geq \mu'_{\varphi\left(\frac{x}{3^n},\frac{x}{3^n},\frac{x}{3^n}\right)}\left(\frac{3^n t}{\alpha}\right) \geq \mu'_{\varphi(x,x,x)}\left(\frac{3^n t}{\alpha^{n+1}}\right)$$

and so

$$\begin{aligned}
\mu_{\frac{f(\frac{x}{3^n})}{3^n}-f(x)}\left(\sum_{k=0}^{n-1}\frac{\alpha^{k+1}t}{3^k}\right) &= \mu_{\sum_{k=0}^{n-1}\frac{f\left(\frac{x}{3^{k+1}}\right)}{3^{k+1}}-\frac{f\left(\frac{x}{3^k}\right)}{3^k}}\left(\sum_{k=0}^{n-1}\frac{\alpha^{k+1}t}{3^k}\right) \\
&\geq T_{k=0}^{n-1}\left(\mu_{\frac{f\left(\frac{x}{3^{k+1}}\right)}{3^{k+1}}-\frac{f\left(\frac{x}{3^k}\right)}{3^k}}\left(\frac{\alpha^{k+1}t}{3^k}\right)\right) \\
&\geq T_{k=0}^{n-1}\left(\mu'_{\varphi(x,x,x)}(t)\right) = \mu'_{\varphi(x,x,x)}(t).
\end{aligned}$$

This implies that

$$\mu_{\frac{f(\frac{x}{3^n})}{3^n}-f(x)}(t) \geq \mu'_{\varphi(x,x,x)}\left(\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^{k+1}}{3^k}}\right). \tag{3.5}$$

Replacing $x$ by $\frac{x}{2^p}$ in (3.5), we obtain

$$\mu_{\frac{f\left(\frac{x}{3^{n+p}}\right)}{3^{n+p}}-\frac{f\left(\frac{x}{3^p}\right)}{3^p}}(t) \geq \mu'_{\varphi(x,x,x)}\left(\frac{t}{\sum_{k=p}^{n+p-1}\frac{\alpha^{k+1}}{3^k}}\right). \tag{3.6}$$

Since $\lim_{p,n\to\infty}\mu'_{\varphi(x,x,x)}(\frac{t}{\sum_{k=p}^{n+p-1}\frac{\alpha^{k+1}}{3^k}}) = 1$, it follows that $\{\frac{f(\frac{x}{3^n})}{3^n}\}_{n=1}^{\infty}$ is a Cauchy sequence in a complete RNS $(Y,\mu,\min)$ and so there exists a point $A(x)\in Y$ such that $\lim_{n\to\infty}\frac{f(\frac{x}{3^n})}{3^n} = A(x)$. Fix $x\in X$ and put $p=0$ in (3.6) and so, for any $\epsilon>0$,

$$\begin{aligned}\mu_{A(x)-f(x)}(t+\epsilon) &\geq T\left(\mu_{A(x)-\frac{f(\frac{x}{3^n})}{3^n}}(\epsilon),\mu_{\frac{f(\frac{x}{3^n})}{3^n}-f(x)}(t)\right)\\ &\geq T\left(\mu_{A(x)-\frac{f(\frac{x}{3^n})}{3^n}}(\epsilon),\mu'_{\varphi(x,x,x)}\left(\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^{k+1}}{3^k}}\right)\right). \qquad (3.7)\end{aligned}$$

Taking $n\to\infty$ in (3.7), we get

$$\mu_{A(x)-f(x)}(t+\epsilon)\geq\mu'_{\varphi(x,x,x)}\left(\frac{(3-\alpha)t}{3\alpha}\right). \qquad (3.8)$$

Since $\epsilon$ is arbitrary, by taking $\epsilon\to 0$ in (3.8), we get $\mu_{A(x)-f(x)}(t)\geq\mu'_{\varphi(x,x,x)}(\frac{(3-\alpha)t}{3\alpha})$. Replacing $x,y$ and $z$ by $\frac{x}{3^n}$, $\frac{y}{3^n}$ and $\frac{z}{3^n}$ in (3.2), respectively, we get

$$\mu_{\frac{f(\frac{x+y+z}{3^{n+1}})-f(\frac{x+y+z}{3^n})-\frac{2f(\frac{x}{3^n})f(\frac{y}{3^n})f(\frac{z}{3^n})}{f(\frac{x}{3^n})f(\frac{y}{3^n})+f(\frac{y}{3^n})f(\frac{z}{3^n})+f(\frac{z}{3^n})f(\frac{x}{3^n})}}{3^n}}(t)\geq\mu'_{\varphi(\frac{x}{3^n},\frac{y}{3^n},\frac{z}{3^n})}(3^n t),$$

for all $x,y,z\in X$ and $t>0$. Since $\lim_{n\to\infty}\mu'_{\varphi(\frac{x}{3^n},\frac{y}{3^n},\frac{z}{3^n})}(3^n t)=1$, we conclude that $A$ satisfies (1.1).

To prove the uniqueness of the reciprocal difference mapping $A$, assume that there exists another reciprocal difference mapping $L:X\to Y$ which satisfies (3.3). Then we have

$$\begin{aligned}\mu_{A(x)-L(x)}(t) &= \lim_{n\to\infty}\mu_{\frac{A(\frac{x}{3^n})}{3^n}-\frac{L(\frac{x}{3^n})}{3^n}}(t)\\ &\geq\lim_{n\to\infty}\min\left\{\mu_{\frac{A(\frac{x}{3^n})}{3^n}-\frac{f(\frac{x}{3^n})}{3^n}}\left(\frac{t}{2}\right),\mu_{\frac{L(\frac{x}{3^n})}{3^n}-\frac{f(\frac{x}{3^n})}{3^n}}\left(\frac{t}{2}\right)\right\}\\ &\geq\lim_{n\to\infty}\mu'_{\varphi(\frac{x}{3^n},\frac{x}{3^n},\frac{x}{3^n})}\left(\frac{3^n(3-\alpha)t}{6\alpha}\right)\\ &\geq\lim_{n\to\infty}\mu'_{\varphi(x,x,x)}\left(\frac{3^n(3-\alpha)t}{6\alpha^{n+1}}\right).\end{aligned}$$

Since $\lim_{n\to\infty}\mu'_{\varphi(x,x,x)}\left(\frac{3^n(3-\alpha)t}{6\alpha^{n+1}}\right)=1$.Therefore, it follows that $\mu_{A(x)-L(x)}(t)=1$, for all $t>0$ and so $A(x)=L(x)$. This completes the proof. □

**Corollary 3.2** *Let $X$ be a real normed linear space, $(Z,\mu',\min)$ be an RNS and $(Y,\mu,\min)$ be a complete random normed space. Let $r$ be a positive real number with $0<r<1$ , $z_0\in Z$ and $f:X\to Y$ be a mapping satisfying*

$$\mu_{f(\frac{x+y+z}{3})-f(x+y+z)-\frac{2f(x)f(y)f(z)}{f(x)f(y)+f(y)f(z)+f(z)f(x)}}(t)\geq\mu'_{(\|x\|^r+\|y\|^r+\|z\|^r)z_0}(t), \qquad (3.9)$$

*for all $x, y, z \in X$ and $t > 0$. Then the limit $A(x) = \lim_{n\to\infty} \frac{f(\frac{x}{3^n})}{3^n}$ exists for all $x \in X$ and defines a unique reciprocal difference mapping $A : X \to Y$ such that $\mu_{f(x)-A(x)}(t) \geq \mu'_{\|x\|^p z_0}(\frac{(3-3^r)t}{3^{r+2}})$, for all $x \in X$ and $t > 0$.*

*Proof.* Let $\alpha = 3^r$ and $\varphi : X^3 \to Z$ be a mapping defined by $\varphi(x, y, z) = (\|x\|^r + \|y\|^r + \|z\|^r)z_0$. Then, applying Theorem 3.1, the conclusion follows. $\square$

**Corollary 3.3** *Let $X$ be a real linear space, $(Z, \mu', \min)$ be a random normed space and $(Y, \mu, \min)$ be a complete random normed space. Let $\delta \geq 0$ ,$z_0 \in Z$ and $f : X \to Y$ be a mapping satisfying*

$$\mu_{f(\frac{x+y+z}{3})-f(x+y+z)-\frac{2f(x)f(y)f(z)}{f(x)f(y)+f(y)f(z)+f(z)f(z)}}(t) \geq \mu'_{\delta z_0}(t), \tag{3.10}$$

*for all $x, y, z \in X$ and $t > 0$. Then the limit $A(x) = \lim_{n\to\infty} \frac{f(\frac{x}{3^n})}{3^n}$ exists for all $x \in X$ and defines a unique reciprocal difference mapping $A : X \to Y$ such that $\mu_{f(x)-A(x)}(t) \geq \mu'_{\delta z_0}\left(\frac{2t}{3}\right)$, for all $x \in X$ and $t > 0$.*

*Proof.* Let $\alpha = 1$ and $\varphi : X^3 \to Z$ be a mapping defined by $\varphi(x, y, z) = \delta z_0$. Then, applying Theorem 3.1, the conclusion follows. $\square$

**Theorem 3.4** *Let $X$ be a real linear space, $(Z, \mu', \min)$ be a random normed space and $\varphi : X^3 \to Z$ be a function such that there exists $0 < \alpha < \frac{1}{3}$ such that $\mu'_{\varphi(3x,3y,3z)}(t) \geq \mu'_{\alpha\varphi(x,y,z)}(t)$ and $\lim_{n\to\infty} \mu'_{\varphi(3^n x,3^n y,3^n z)}\left(\frac{t}{3^n}\right) = 1$, for all $x, y, z \in X$ and $t > 0$. Let $(Y, \mu, \min)$ be a complete random normed space. If $f : X \to Y$ be a mapping satisfying (3.2). Then the limit $A(x) = \lim_{n\to\infty} 3^n f(3^n x)$ exists for all $x \in X$ and defines a unique reciprocal difference mapping $A : X \to Y$ such that*

$$\mu_{f(x)-A(x)}(t) \geq \mu'_{\varphi(x,x,x)}\left(\frac{(1-3\alpha)t}{3}\right), \tag{3.11}$$

*for all $x \in X$ and $t > 0$.*

*Proof.* It follows from (3.4) that

$$\mu_{3f(3x)-f(x)}(t) \geq \mu'_{\varphi(x,x,x)}\left(\frac{t}{3}\right). \tag{3.12}$$

Replacing $x$ by $3^n x$ in (3.12), we obtain that $\mu_{3^{n+1}f(3^{n+1}x)-3^n f(3^n x)}(t) \geq \mu'_{\varphi(3^n x,3^n x,3^n x)}(\frac{t}{3^{n+1}}) \geq \mu_{\varphi(x,2x,x)}(\frac{t}{3^{n+1}\alpha^n})$. The rest of the proof is similar to the proof of Theorem 3.1. $\square$

**Corollary 3.5** *Let $X$ be a real normed linear space, $(Z, \mu', \min)$ be a random normed space and $(Y, \mu, \min)$ be a complete random normed space. Let $r$ be a positive real number with $r > 1$ , $z_0 \in Z$ and $f : X \to Y$ be a mapping satisfying (3.9). Then the limit $A(x) = \lim_{n\to\infty} 3^n f(3^n x)$ exists for all $x \in X$ and defines a unique reciprocal difference mapping $A : X \to Y$ such that $\mu_{f(x)-A(x)}(t) \geq \mu'_{\|x\|^p z_0}(\frac{(3^r-3)t}{3^{r+2}})$, for all $x \in X$ and $t > 0$.*

*Proof.* Let $\alpha = 3^{-r}$ and $\varphi : X^3 \to Z$ be a mapping defined by $\varphi(x, y, z) = (\|x\|^r + \|y\|^r + \|z\|^r)z_0$. Then, applying Theorem 3.4, the conclusion follows. $\square$

**Corollary 3.6** *Let $X$ be a real normed linear space, $(Z, \mu', \min)$ be a random normed space and $(Y, \mu, \min)$ be a complete random normed space. Let $\delta > 0$ , $z_0 \in Z$ and $f : X \to Y$ be a mapping satisfying* (3.10)*. Then the limit $A(x) = \lim_{n\to\infty} 3^n f(3^n x)$ exists for all $x \in X$ and defines a unique reciprocal difference mapping $A : X \to Y$ such that and $\mu_{f(x)-A(x)}(t) \geq \mu'_{\delta z_0}(\frac{2t}{9})$, for all $x \in X$ and $t > 0$.*

*Proof.* Let $\alpha = \frac{1}{9}$ and $\varphi : X^3 \to Z$ be a mapping defined by $\varphi(x, y, z) = \delta z_0$. Then, applying Theorem 3.4, the conclusion follows. $\square$

## 4 Fuzzy stability of reciprocal difference mapping (1.1)

In this section, using direct method, we prove the Hyers-Ulam-Rassias stability of functional equation (1.1) in fuzzy Banach spaces. Throughout this section, we assume that $X$ is a linear space, $(Y, N)$ is a fuzzy Banach space and $(Z, N')$ is a fuzzy normed space. Moreover, we assume that $N(x, .)$ is a left continuous function on $\mathbb{R}$.

**Theorem 4.1** *Assume that a mapping $f : X \to Y$ satisfies the inequality*

$$N\left(f\left(\frac{x+y+z}{3}\right) - f(x+y+z) - \frac{2f(x)f(y)f(z)}{f(x)f(y)+f(y)f(z)+f(z)f(z)}, t\right) \geq N'(\varphi(x, y, z), t), \quad (4.1)$$

*for all $x, y, z \in X$, $t > 0$ and $\varphi : X^3 \to Z$ is a mapping for which there is a constant $r \in \mathbb{R}$ satisfying $0 < |r| < 3$ such that*

$$N'\left(\varphi\left(\frac{x}{3}, \frac{y}{3}, \frac{z}{3}\right), t\right) \geq N'\left(\varphi(x, y, z), \frac{t}{|r|}\right), \quad (4.2)$$

*for all $x, y, z \in X$ and all $t > 0$. Then we can find a unique reciprocal difference mapping $A : X \to Y$ satisfying* (1.1) *and the inequality*

$$N(f(x) - A(x), t) \geq N'\left(\frac{3|r|\varphi(x, x, x)}{3 - |r|}, t\right), \quad (4.3)$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* It follows from (4.2) that

$$N'\left(\varphi\left(\frac{x}{3^j}, \frac{y}{3^j}, \frac{z}{3^j}\right), t\right) \geq N'\left(\varphi(x, y, z), \frac{t}{|r|^j}\right). \quad (4.4)$$

So

$$N'\left(\varphi\left(\frac{x}{3^j}, \frac{y}{3^j}, \frac{z}{3^j}\right), |r|^j t\right) \geq N'\left(\varphi(x, y, z), t\right),$$

for all $x, y, z \in X$ and all $t > 0$.

Substituting $x=y=z=\frac{x}{3}$ in (4.1), we obtain

$$N\left(\frac{f\left(\frac{x}{3}\right)}{3}-f(x),t\right)\geq N'\left(\varphi\left(\frac{x}{3},\frac{x}{3},\frac{x}{3}\right),t\right), \tag{4.5}$$

for all $x\in X$ and all $t>0$. Replacing $x$ by $\frac{x}{3^j}$ in (4.5), we have

$$\begin{aligned}N\left(\frac{f\left(\frac{x}{3^{j+1}}\right)}{3^{j+1}}-\frac{f\left(\frac{x}{3^j}\right)}{3^j},\frac{t}{3^j}\right)&\geq N'\left(\varphi\left(\frac{x}{3^{j+1}},\frac{x}{3^{j+1}},\frac{x}{3^{j+1}}\right),t\right)\\&\geq N'\left(\varphi\left(x,x,x\right),\frac{t}{|r|^{j+1}}\right),\end{aligned} \tag{4.6}$$

for all $x\in X$, all $t>0$ and any integer $j\geq 0$. So

$$\begin{aligned}&N\left(f(x)-\frac{f\left(\frac{x}{3^n}\right)}{3^n},\sum_{j=0}^{n-1}\frac{|r|^{j+1}t}{3^j}\right)\\&=N\left(\sum_{j=0}^{n-1}\left[\frac{f\left(\frac{x}{3^{j+1}}\right)}{3^{j+1}}-\frac{f\left(\frac{x}{3^j}\right)}{3^j}\right],\sum_{j=0}^{n-1}\frac{|r|^{j+1}t}{3^j}\right)\\&\geq\min_{0\leq j\leq n-1}\left\{N\left(\frac{f\left(\frac{x}{3^{j+1}}\right)}{3^{j+1}}-\frac{f\left(\frac{x}{3^j}\right)}{3^j},\frac{|r|^{j+1}t}{3^j}\right)\right\}\\&\geq N'(\varphi(x,x,x),t)\end{aligned}$$

which yields

$$\begin{aligned}N\left(\frac{f\left(\frac{x}{3^{n+p}}\right)}{3^{n+p}}-\frac{f\left(\frac{x}{3^p}\right)}{3^p},\sum_{j=0}^{n-1}\frac{|r|^{j+1}t}{3^{j+p}}\right)&\geq N'\left(\varphi\left(\frac{x}{3^p},\frac{x}{3^p},\frac{x}{3^p}\right),t\right)\\&\geq N'\left(\varphi(x,x,x),\frac{t}{|r|^p}\right),\end{aligned}$$

for all $x\in X$, $t>0$ and any integers $n>0$, $p\geq 0$. So

$$N\left(\frac{f\left(\frac{x}{3^{n+p}}\right)}{3^{n+p}}-\frac{f\left(\frac{x}{3^p}\right)}{3^p},\sum_{j=0}^{n-1}\frac{|r|^{j+p+1}t}{3^{j+p}}\right)\geq N'(\varphi(x,x,x),t),$$

for all $x\in X$, $t>0$ and any integers $n>0$, $p\geq 0$. Hence one obtains

$$N\left(\frac{f\left(\frac{x}{3^{n+p}}\right)}{3^{n+p}}-\frac{f\left(\frac{x}{3^p}\right)}{3^p},t\right)\geq N'\left(\varphi(x,x,x),\frac{t}{\sum_{j=0}^{n-1}\frac{|r|^{j+p+1}}{3^{j+p}}}\right), \tag{4.7}$$

for all $x\in X$, $t>0$ and any integers $n>0$, $p\geq 0$. Since, the series $\sum_{j=0}^{\infty}\frac{|r|^j}{3^j}$ is convergent, we see by taking the limit $p\to\infty$ in the last inequality that a sequence

$\{\frac{f(\frac{x}{3^n})}{3^n}\}$ is a Cauchy sequence in the fuzzy Banach space $(Y, N)$ and so it converges in $Y$. Therefore a mapping $A : X \to Y$ defined by $A(x) := N - \lim_{n\to\infty} \frac{f(\frac{x}{3^n})}{3^n}$ is well defined for all $x \in X$. It means that

$$\lim_{n\to\infty} N\left(A(x) - \frac{f\left(\frac{x}{3^n}\right)}{3^n}, t\right) = 1, \tag{4.8}$$

for all $x \in X$ and all $t > 0$. In addition, it follows from (4.7) that

$$N\left(f(x) - \frac{f\left(\frac{x}{3^n}\right)}{3^n}, t\right) \geq N'\left(\varphi(x, x, x), \frac{t}{\sum_{j=0}^{n-1} \frac{|r|^{j+1}}{3^j}}\right),$$

for all $x \in X$ and all $t > 0$. So

$$\begin{aligned}
&N(f(x) - A(x), t) \\
&\geq \min\left\{N\left(f(x) - \frac{f\left(\frac{x}{3^n}\right)}{3^n}, (1-\epsilon)t\right), N\left(A(x) - \frac{f\left(\frac{x}{3^n}\right)}{3^n}, \epsilon t\right)\right\} \\
&\geq N'\left(\varphi(x, x, x), \frac{t}{\sum_{j=0}^{n-1} \frac{|r|^{j+1}}{3^j}}\right) \geq N'\left(\varphi(x, x, x), \frac{(3-|r|)\epsilon t}{3|r|}\right)
\end{aligned}$$

for sufficiently large $n$ and for all $x \in X$, $t > 0$ and $\epsilon$ with $0 < \epsilon < 1$. Since $\epsilon$ is arbitrary and $N'$ is left continuous, we obtain

$$N(f(x) - A(x), t) \geq N'\left(\varphi(x, x, x), \frac{(3-|r|)t}{3|r|}\right),$$

for all $x \in X$ and $t > 0$. It follows from (4.1) that

$$\begin{aligned}
&N\Bigg(\frac{1}{3^n} f\left(\frac{x+y+z}{3^{n+1}}\right) - \frac{1}{3^n} f\left(\frac{x+y+z}{3^n}\right) \\
&- \frac{2f\left(\frac{x}{3^n}\right) f\left(\frac{y}{3^n}\right) f\left(\frac{z}{3^n}\right)}{3^n[f\left(\frac{x}{3^n}\right) f\left(\frac{y}{3^n}\right) + f\left(\frac{y}{3^n}\right) f\left(\frac{z}{3^n}\right) + f\left(\frac{z}{3^n}\right) f\left(\frac{x}{3^n}\right)]}, t\Bigg) \\
&\geq N'\left(\varphi\left(\frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n}\right), 3^n t\right) \geq N'\left(\varphi(x, y, z), \frac{3^n t}{|r|^n}\right),
\end{aligned}$$

for all $x, y, z \in X$, $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} N'\left(\varphi(x, y, z), \frac{3^n t}{|r|^n}\right) = 1$ and so

$$\begin{aligned}
&N\Bigg(\frac{1}{3^n} f\left(\frac{x+y+z}{3^{n+1}}\right) - \frac{1}{3^n} f\left(\frac{x+y+z}{3^n}\right) \\
&- \frac{2f\left(\frac{x}{3^n}\right) f\left(\frac{y}{3^n}\right) f\left(\frac{z}{3^n}\right)}{3^n[f\left(\frac{x}{3^n}\right) f\left(\frac{y}{3^n}\right) + f\left(\frac{y}{3^n}\right) f\left(\frac{z}{3^n}\right) + f\left(\frac{z}{3^n}\right) f\left(\frac{x}{3^n}\right)]}, t\Bigg) \to 1,
\end{aligned}$$

for all $x, y, z \in X$ and all $t > 0$. Therefore, we obtain in view of (4.8)

$$
\begin{aligned}
&N\left(A\left(\frac{x+y+z}{3}\right) - A(x+y+z) - \frac{2A(x)A(y)A(z)}{A(x)A(y)+A(y)A(z)+A(z)A(z)}, t\right) \\
&\geq \min\Bigg\{N\Bigg(A\left(\frac{x+y+z}{3}\right) - A(x+y+z) - \frac{2A(x)A(y)A(z)}{A(x)A(y)+A(y)A(z)+A(z)A(z)} \\
&- \frac{1}{3^n} f\left(\frac{x+y+z}{3^{n+1}}\right) - \frac{1}{3^n} f\left(\frac{x+y+z}{3^n}\right) \\
&- \frac{2f\left(\frac{x}{3^n}\right)f\left(\frac{y}{3^n}\right)f\left(\frac{z}{3^n}\right)}{3^n[f\left(\frac{x}{3^n}\right)f\left(\frac{y}{3^n}\right)+f\left(\frac{y}{3^n}\right)f\left(\frac{z}{3^n}\right)+f\left(\frac{z}{3^n}\right)f\left(\frac{x}{3^n}\right)]}, \frac{t}{2}\Bigg), \\
&N\Bigg(\frac{1}{3^n} f\left(\frac{x+y+z}{3^{n+1}}\right) - \frac{1}{3^n} f\left(\frac{x+y+z}{3^n}\right) \\
&- \frac{2f\left(\frac{x}{3^n}\right)f\left(\frac{y}{3^n}\right)f\left(\frac{z}{3^n}\right)}{3^n[f\left(\frac{x}{3^n}\right)f\left(\frac{y}{3^n}\right)+f\left(\frac{y}{3^n}\right)f\left(\frac{z}{3^n}\right)+f\left(\frac{z}{3^n}\right)f\left(\frac{x}{3^n}\right)]}, \frac{t}{2}\Bigg)\Bigg\} \\
&= N\Bigg(\frac{1}{3^n} f\left(\frac{x+y+z}{3^{n+1}}\right) - \frac{1}{3^n} f\left(\frac{x+y+z}{3^n}\right) \\
&- \frac{2f\left(\frac{x}{3^n}\right)f\left(\frac{y}{3^n}\right)f\left(\frac{z}{3^n}\right)}{3^n[f\left(\frac{x}{3^n}\right)f\left(\frac{y}{3^n}\right)+f\left(\frac{y}{3^n}\right)f\left(\frac{z}{3^n}\right)+f\left(\frac{z}{3^n}\right)f\left(\frac{x}{3^n}\right)]}, \frac{t}{2}\Bigg) \\
&\geq N'\left(\varphi(x,y,z), \frac{3^n t}{2|r|^n}\right) \to 1 \quad \text{as } n \to \infty
\end{aligned}
$$

which implies

$$
A\left(\frac{x+y+z}{3}\right) - A(x+y+z) = \frac{2A(x)A(y)A(z)}{A(x)A(y)+A(y)A(z)+A(z)A(z)},
$$

for all $x, y, z \in X$. Thus $A : X \to Y$ is a mapping satisfying the equation (1.1) and the inequality (4.3).

To prove the uniqueness, let there is another mapping $L : X \to Y$ which satisfies the inequality (4.3). Then, for all $x \in X$, we have

$$
\begin{aligned}
&N(A(x) - L(x), t) = N\left(\frac{1}{3^n} A\left(\frac{x}{3^n}\right) - \frac{1}{3^n} L\left(\frac{x}{3^n}\right), t\right) \\
&\geq \min\left\{N\left(\frac{1}{3^n} A\left(\frac{x}{3^n}\right) - \frac{1}{3^n} f\left(\frac{x}{3^n}\right), \frac{t}{2}\right), N\left(\frac{1}{3^n} f\left(\frac{x}{3^n}\right) - \frac{1}{3^n} L\left(\frac{x}{3^n}\right), \frac{t}{2}\right)\right\} \\
&\geq N'\left(\varphi\left(\frac{x}{3^n}, \frac{x}{3^n}, \frac{x}{3^n}\right), \frac{3^n(3-|r|)t}{6|r|}\right) \\
&\geq N'\left(\varphi(x,x,x), \frac{3^n(3-|r|)t}{6|r|^{n+1}}\right) \to 1 \quad \text{as } n \to \infty,
\end{aligned}
$$

for all $t > 0$. Therefore $A(x) = L(x)$ for all $x \in X$, this completes the proof. □

**Corollary 4.2** *Let $X$ be a normed spaces and that $(\mathbb{R}, N')$ a fuzzy Banach space. Assume that there exist real numbers $\theta \geq 0$ and $0 < p < 1$ such that a mapping $f : X \to Y$ satisfying the following inequality*

$$N\left(f\left(\frac{x+y+z}{3}\right) - f(x+y+z) - \frac{2f(x)f(y)f(z)}{f(x)f(y)+f(y)f(z)+f(z)f(z)}, t\right) \geq N'\left(\theta\left(\|x\|^p + \|y\|^p + \|z\|^p\right), t\right),$$

*for all $x, y, z \in X$ and $t > 0$. Then there is a unique reciprocal difference mapping $A : X \to Y$ that satisfying (1.1) and the inequality*

$$N(f(x) - A(x), t) \geq N'\left(\theta\|x\|^p, \frac{t}{18}\right).$$

*Proof.* Let $\varphi(x, y, z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ and $|r| = 2$. Apply Theorem 4.1, we get desired results. $\square$

**Theorem 4.3** *Assume that a mapping $f : X \to Y$ satisfies the inequality (4.1) and $\varphi : X^3 \to Z$ is a mapping for which there is a constant $r \in \mathbb{R}$ satisfying $0 < |r| < \frac{1}{3}$ such that*

$$N'\left(\varphi(x, y, z), |r|t\right) \geq N'\left(\varphi\left(\frac{x}{3}, \frac{y}{3}, \frac{z}{3}\right), t\right), \tag{4.9}$$

*for all $x, y, z \in X$ and all $t > 0$. Then we can find a unique reciprocal difference mapping $A : X \to Y$ that satisfying (1.1) and the following inequality*

$$N(f(x) - A(x), t) \geq N'\left(\frac{3\varphi(x, x, x)}{1 - 3|r|}, t\right), \tag{4.10}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* It follows from (4.5) that

$$N\left(3f(3x) - f(x), 3t\right) \geq N'(\varphi(x, x, x), t), \tag{4.11}$$

for all $x \in X$ and all $t > 0$. Replacing $x$ by $3^n x$ in (4.11), we obtain

$$\begin{aligned} N\left(3^{n+1}f(3^{n+1}x) - 3^n f(3^n x), 3^{n+1}t\right) &\geq N'(\varphi(3^n x, 3^n x, 3^n x), t) \\ &\geq N'\left(\varphi(x, x, x), \frac{t}{|r|^n}\right). \end{aligned} \tag{4.12}$$

So

$$N\left(3^{n+1}f(3^{n+1}x) - 3^n f(3^n x), 3^{n+1}|r|^n t\right) \geq N'\left(\varphi(x, x, x), t\right), \tag{4.13}$$

for all $x \in X$ and all $t > 0$. Proceeding as in the proof of Theorem 4.1, we obtain that

$$N\left(f(x) - 3^n f(3^n x), \sum_{j=0}^{n-1} 3^{n+1}|r|^n t\right) \geq N'(\varphi(x, x, x), t),$$

for all $x \in X$, all $t > 0$ and any integer $n > 0$. So

$$\begin{aligned} N\left(f(x) - 3^n f(3^n x), t\right) &\geq N'\left(\varphi(x,x,x), \frac{t}{\sum_{j=0}^{n-1} 3^{n+1}|r|^n}\right) \\ &\geq N'\left(\varphi(x,x,x), \frac{(1-3|r|)t}{3}\right). \end{aligned} \tag{4.14}$$

The rest of the proof is similar to the proof of Theorem 4.1. $\square$

**Corollary 4.4** *Let $X$ be a normed spaces and that $(\mathbb{R}, N')$ a fuzzy Banach space. Assume that there exist real numbers $\theta \geq 0$ and $0 < p = p_1 + p_2 + p_3 < 1$ such that a mapping $f : X \to Y$ satisfying the following inequality*

$$\begin{aligned} N\left(f\left(\frac{x+y+z}{3}\right) - f(x+y+z) - \frac{2f(x)f(y)f(z)}{f(x)f(y)+f(y)f(z)+f(z)f(z)}, t\right) \\ \geq N'\left(\theta\left(\|x\|^{p_1} \cdot \|y\|^{p_2} \cdot \|z\|^{p_3}\right), t\right), \end{aligned}$$

*for all $x, y, z \in X$ and $t > 0$. Then there is a unique reciprocal difference mapping $A : X \to Y$ that satisfying* (1.1) *and the inequality*

$$N(f(x) - A(x), t) \geq N'\left(\theta\|x\|^p, \frac{t}{12}\right).$$

*Proof.* Let $\varphi(x, y, z) := \theta\left(\|x\|^{p_1} \cdot \|y\|^{p_2} \cdot \|z\|^{p_3}\right)$ and $|r| = \frac{1}{4}$. Applying Theorem 4.3, we get desired results. $\square$

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