An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)
Tomul LXIII, 2017, f. 2

On some sufficient conditions for L^1-convergence of double sine series

Xhevat Z. Krasniqi

Received: 11.VIII.2013 / Revised: 26.XI.2013 / Accepted: 29.XI.2013

Abstract In this paper we introduce some numerical classes of double sequences. Such classes are used to show some sufficient conditions for L^1-convergence of double sine series. This study partially extends very recent results of Leindler, and particularly those of Zhou, from single to two-dimensional sine series.

Keywords classes of sequences · L^1-convergence · double sine series · uniform convergence · regular convergence · bounded variation

Mathematics Subject Classification (2010) 42A20 · 42A32 · 42B99

1 Introduction

It is well a known fact that if a trigonometric series converges in L^1-metric to a function $f \in L^1$, then it is the Fourier series of the function f. However Riesz [15] gave a counter example showing that in the metric space L^1 the converse statement is not true. This fact motivated various authors to study L^1-convergence of the trigonometric series. During their investigations some authors introduced the so-called modified trigonometric sums. By these new sums they achieved to approximate their limits better than the classical trigonometric series in the sense that they converge in L^1-metric to the sum of the trigonometric series whereas the classical series itself may not.

For some classical and newer attendances pertaining to this topic, we refer the reader to the paper of Tikhonov [13] who has proved interesting theorems regarding to L^1-convergence of trigonometric series. He gave necessary and sufficient conditions under which a trigonometric series converges in L^1-norm, and for more results of such kind of conditions we refer the reader to the references therein.

Xhevat Z. Krasniqi
University of Prishtina,
Department of Mathematics and Computer Sciences,
Avenue Mother Theresa,
10000 Prishtina, Kosovo
E-mail: xhevat.krasniqi@uni-pr.edu; xheki00@hotmail.com

On one hand, in a recent paper, ZHOU defined the notion of Logarithm Rest Bounded Variation Sequence ($LRBVS$, see the definition in [14]) which plays central role in his paper. Among others, he gave necessary and sufficient condition of the L^1-convergence of sine series

$$\sum_{n=1}^{\infty} a_n \sin nx, \tag{1.1}$$

assuming that $\{a_n\} \in LRBVS$, but without prior condition that the sum function of (1.1) is integrable.

In the following we shall use the notion $u \ll w$ ($u \gg w$) at inequalities if there exists a positive constant A such that $u \leq Aw$ ($u \geq Aw$) holds, where A is not necessarily the same at each occurrence. Also we shall denote by $s_n(x)$ the partial sums of order n of the series (1.1).

Very recently LEINDLER [9] defined a certain unification of the logarithm sequence (for suitability of interested reader we suggest to find the definitions of numerical classes $RBVS$ and $\gamma RBVS$ in [7]-[8]).

Now, we recall some definitions of generalizations of decreasing monotonicity related particularly to L^1-convergence of sine series.

A sequence $a := \{a_n\}$ of positive numbers will be called Almost Monotone Sequence, briefly $a \in AMS$, if $a_n \leq K a_m$, for all $n \geq m$, where $K = K(a)$ is a positive constant.

A positive nondecreasing sequence $\alpha := \{\alpha_n\}$ will be called Log-Type Sequence, briefly LTS, if it satisfies the conditions:

$$\alpha_n \to \infty, \quad \alpha_{n^2} \ll \alpha_n, \quad \text{and} \quad |\triangle \alpha_n| \ll \frac{\alpha_n}{n \log n}, \tag{1.2}$$

where $\triangle \alpha_n = \alpha_n - \alpha_{n+1}$.

By means of this definition, Leindler defined two new classes of sequences.

Let $\gamma := \{\gamma_n\}$ be a positive sequence. If

$$\alpha \in LTS \quad \text{and} \quad \left\{\frac{a_n}{\alpha_n}\right\} \in \gamma RBVS, \tag{1.3}$$

then the sequence $a := \{a_n\}$ will be called γ Log-Type Rest Bounded Variation Sequence, in symbol $a \in \gamma LTRBVS$.

If $\gamma_n = a_n/\alpha_n$, then the sequence a will be said simply Log-Type Rest Bounded Variation Sequence, in symbol $a \in LTRBVS$.

In other words, $a \in LTRBVS$ if $\alpha \in LTS$ and $\{\frac{a_n}{\alpha_n}\} \in RBVS$.

Leindler's results pertaining to the above considered topic can be read as follows:

Theorem 1.1 *Let $a \in LTRBVS$, that is, $\{\alpha_n\} \in LTS$ and $\{\frac{a_n}{\alpha_n}\} \in RBVS$. Write $g(x) = \sum_{n=1}^{\infty} a_n \sin nx$ at x, where it converges. Then*

$$\lim_{n\to\infty} \|g - s_n(g)\| = 0 \tag{1.4}$$

if and only if

$$\sum_{n=1}^{\infty} \frac{a_n}{n} < \infty. \tag{1.5}$$

Theorem 1.2 *Let $\{\alpha_n\} \in LTS$ and $\{\gamma_n\} \in AMS$. If $a_n/\alpha_n \in \gamma RBVS$, that is, $a \in \gamma LTRBVS$, and*

$$\sum_{n=1}^{\infty} \frac{\alpha_n \gamma_n}{n} < \infty, \tag{1.6}$$

then (1.4) *holds.*

Theorem 1.3 *Let $\{\alpha_n\} \in LTS$ and $\{\frac{a_n}{\alpha_n}\} \in RBVS$, that is, $a \in LTRBVS$. Then the condition*

$$\sum_{n=2}^{\infty} |\triangle a_n| \log n < \infty \tag{1.7}$$

and condition (1.5) *are equivalent.*

On the other hand, the L^1-convergence of double trigonometric series has been an attractive issue by lots of authors, and still receives a considerable attention. Moreover, most of the results regarding to this topic obtained for single trigonometric series are extended for two or multiple trigonometric series. Just for curiosity of the reader, we remind papers [1]-[6] and [10]-[12], where it is possible to find a good amount of results mentioned above as well as in their references.

In order to make an advanced study in this direction, here, we are going to extend Theorem 1.1, Theorem 1.2, and Theorem 1.3 from single to two-dimensional sine series, which is the main purpose of this paper.

To my best knowledge the following definition was introduced for the first time by Bokaev, Mukanov [1].

We say that the sequence $\beta := \{a_{m,n}\}$ belongs to the class $RBVS^2$ if

$$a_{m,n} \to 0 \quad \text{as} \quad m+n \to \infty, \tag{1.8}$$

and

$$\sum_{k=m}^{\infty} \sum_{\ell=n}^{\infty} |\triangle_{1,1} a_{k,\ell}| \leq K(\beta) a_{m,n}, \tag{1.9}$$

where $\triangle_{1,1} a_{k,\ell} = a_{k,\ell} - a_{k+1,\ell} - a_{k,\ell+1} + a_{k+1,\ell+1}$, and $K(\beta)$ is a positive finite constant depending only on β.

A nonnegative bounded sequence $\beta := \{a_{m,n}\}$ is called Logarithm Rest Bounded Variation Sequence, in symbols: $\beta \in LRBVS^2_{M,N}$, if M, N are positive integers and the sequence $\{a_{m,n} \log^{-M} m \log^{-N} n\}$ belongs to $RBVS^2$.

Further, we introduce several new classes of double sequences as follows: For fixed m, n let $\gamma_1 := \{\gamma_{m,n}\}$ be a double nonnegative sequence. We say that the sequence $\beta := \{a_{m,n}\}$ belongs to the class $\gamma_1 RBVS^2$ if $a_{m,n} \to 0$ as $m+n \to \infty$, and

$$\sum_{k=m}^{\infty} \sum_{\ell=n}^{\infty} |\triangle_{1,1} a_{k,\ell}| \leq K(\gamma_1) \gamma_{m,n}. \tag{1.10}$$

It is obvious that for $\gamma_1 = \beta$ then $\gamma_1 RBVS^2 \equiv RBVS^2$, and condition (1.10) clearly implies conditions

$$\sum_{k=m}^{\infty} |\triangle_{1,0} a_{k,\ell}| \leq K(\gamma_1) \gamma_{m,n} \tag{1.11}$$

and

$$\sum_{\ell=n}^{\infty}|\triangle_{0,1}a_{k,\ell}|\leq K(\gamma_1)\gamma_{m,n}, \tag{1.12}$$

where $\triangle_{1,0}a_{k,\ell}=a_{k,\ell}-a_{k+1,\ell}$, $\triangle_{0,1}a_{k,\ell}=a_{k,\ell}-a_{k,\ell+1}$.

A positive nondecreasing double sequence $\alpha_1:=\{\alpha_{m,n}\}$ will be called Log-Type Sequence, briefly $\alpha_1\in LTS^2$, if it satisfies the conditions:

$$\alpha_{m,n}\to\infty \quad \text{as} \quad m+n\to\infty, \tag{1.13}$$

$$\alpha_{m^2,n}\ll\alpha_{m,n}, \quad \alpha_{m,n^2}\ll\alpha_{m,n}, \tag{1.14}$$

and

$$\begin{aligned}|\triangle_{0,1}\alpha_{m,n}|&\ll\frac{\alpha_{m,n}}{n\log n},\\ |\triangle_{1,0}\alpha_{m,n}|&\ll\frac{\alpha_{m,n}}{m\log m},\\ |\triangle_{1,1}\alpha_{m,n}|&\ll\frac{\alpha_{m,n}}{mn\log m\log n}.\end{aligned} \tag{1.15}$$

If $\{\alpha_{m,n}\}\in LTS^2$ and $\{\frac{a_{m,n}}{\alpha_{m,n}}\}\in\gamma_1RBVS^2$, then the sequence $\{a_{m,n}\}$ will be called γ_1 Log-Type Rest Bounded Variation Sequence, in symbol $\beta\in\gamma_1LTRBVS^2$.

If $\gamma_{m,n}=a_{m,n}/\alpha_{m,n}$, then the sequence β will be said simply Log-Type Rest Bounded Variation Sequence, in symbol $\beta\in LTRBVS^2$.

In other words, $\beta\in LTRBVS$ if $\alpha_1\in LTS^2$ and $\{\frac{a_{m,n}}{\alpha_{m,n}}\}\in RBVS^2$.

Obviously, the following embedding relations follow:

$$LRBVS^2_{M,N}\subset LTRBVS^2.$$

Let us suppose that $g_2(x,y)$ is a function, periodic with period 2π in each variable, and integrable in Lebesgue's sense, briefly denoted $g_2\in L^1$. The L^1-norm of a function $g_2(x,y)$ is defined by

$$\|g_2\|:=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}|g_2(x,y)|dxdy,$$

and for double sine series

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{m,n}\sin mx\sin ny \tag{1.16}$$

its partial sum are defined by

$$s_{m,n}(x,y):=\sum_{k=1}^{m}\sum_{\ell=1}^{n}a_{m,n}\sin mx\sin ny.$$

In favor of the reader we shall recall a basic notion about double sequences.

A double sequence $\{w_{m,n}\}$ is called a Cauchy double sequence if for every $\varepsilon>0$, there is $(m_0,n_0)\in\mathbb{N}\times\mathbb{N}$ such that $|w_{m,n}-w_{p,q}|<\varepsilon$, for all $(m,n),(p,q)\geq(m_0,n_0)$.

Now we shall pass to the main results given in next section.

2 Main results

Our main result is the following theorem:

Theorem 2.1 *Let $\beta \in LTRBVS^2$, that is, $\{\alpha_{m,n}\} \in LTS^2$ and $\{\frac{a_{m,n}}{\alpha_{m,n}}\} \in RBVS^2$. Write $g_2(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n} \sin mx \sin ny$ at (x,y), where it converges. If*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_{m,n}}{mn} < \infty, \tag{2.1}$$

then

$$\lim_{m+n\to\infty} \|g_2 - s_{m,n}(g_2)\| = 0. \tag{2.2}$$

Proof. Let $(m,n) \geq (r,s)$, that is, $m > r$ and $n > s$. Then we can write

$$\begin{aligned} s_{m,n}(x,y) - s_{r,s}(x,y) &= \sum_{k=1}^{r}\sum_{\ell=s+1}^{n}(\cdot) + \sum_{k=r+1}^{m}\sum_{\ell=1}^{s}(\cdot) + \sum_{k=r+1}^{m}\sum_{\ell=s+1}^{n}(\cdot) \\ &:= \lambda_{m,n}^{(1)}(x,y) + \lambda_{m,n}^{(2)}(x,y) + \lambda_{m,n}^{(3)}(x,y). \end{aligned} \tag{2.3}$$

Using summation by parts four times we get

$$\begin{aligned} \lambda_{m,n}^{(3)}(x,y) &= \sum_{k=r+1}^{m}\sum_{\ell=s+1}^{n} \frac{a_{k,\ell}}{\alpha_{k,\ell}} \alpha_{k,\ell} \sin kx \sin \ell y \\ &= \sum_{k=r+1}^{m-1}\sum_{\ell=s+1}^{n-1} \triangle_{1,1}\left(\frac{a_{k,\ell}}{\alpha_{k,\ell}}\right) \sum_{i=1}^{k}\sum_{j=1}^{\ell} \alpha_{i,j} \sin ix \sin jy \\ &+ \sum_{\ell=s+1}^{n-1} \triangle_{0,1}\left(\frac{a_{m,\ell}}{\alpha_{m,\ell}}\right) \sum_{i=1}^{m}\sum_{j=1}^{\ell} \alpha_{i,j} \sin ix \sin jy \\ &- \sum_{\ell=s+1}^{n-1} \triangle_{0,1}\left(\frac{a_{r+1,\ell}}{\alpha_{r+1,\ell}}\right) \sum_{i=1}^{r}\sum_{j=1}^{\ell} \alpha_{i,j} \sin ix \sin jy \\ &+ \sum_{k=r+1}^{m-1} \triangle_{1,0}\left(\frac{a_{k,n}}{\alpha_{k,n}}\right) \sum_{i=1}^{k}\sum_{j=1}^{n} \alpha_{i,j} \sin ix \sin jy \\ &- \sum_{k=r+1}^{m-1} \triangle_{1,0}\left(\frac{a_{k,s+1}}{\alpha_{k,s+1}}\right) \sum_{i=1}^{k}\sum_{j=1}^{s} \alpha_{i,j} \sin ix \sin jy \end{aligned}$$

$$
\begin{aligned}
&+\frac{a_{m,n}}{\alpha_{m,n}}\sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_{i,j}\sin ix\sin jy-\frac{a_{r+1,n}}{\alpha_{r+1,n}}\sum_{i=1}^{r}\sum_{j=1}^{n}\alpha_{i,j}\sin ix\sin jy\\
&-\frac{a_{m,s+1}}{\alpha_{m,s+1}}\sum_{i=1}^{m}\sum_{j=1}^{s}\alpha_{i,j}\sin ix\sin jy\\
&+\frac{a_{r+1,s+1}}{\alpha_{r+1,s+1}}\sum_{i=1}^{r}\sum_{j=1}^{s}\alpha_{i,j}\sin ix\sin jy:=\sum_{p=1}^{9}J_p(x,y).
\end{aligned}
$$

Since after applying summation by parts three times we obtain

$$
\begin{aligned}
&\sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_{i,j}\sin ix\sin jy\\
&\quad=\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}\triangle_{1,1}\alpha_{i,j}\widetilde{D}_i(x)\widetilde{D}_j(y)+\sum_{j=1}^{n-1}\triangle_{0,1}\alpha_{m,j}\widetilde{D}_m(x)\widetilde{D}_j(y)\\
&\quad+\sum_{i=1}^{m-1}\triangle_{1,0}\alpha_{i,n}\widetilde{D}_i(x)\widetilde{D}_n(y)+\alpha_{m,n}\widetilde{D}_m(x)\widetilde{D}_n(y),
\end{aligned}
$$

then we estimate as follows

$$
\begin{aligned}
&\int_0^{2\pi}\!\!\int_0^{2\pi}\left|\sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_{i,j}\sin ix\sin jy\right|dxdy\\
&\quad\leq\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}|\triangle_{1,1}\alpha_{i,j}|\int_0^{2\pi}\!\!\int_0^{2\pi}|\widetilde{D}_i(x)\widetilde{D}_j(y)|dxdy\\
&\quad+\sum_{j=1}^{n-1}|\triangle_{0,1}\alpha_{m,j}|\int_0^{2\pi}\!\!\int_0^{2\pi}|\widetilde{D}_m(x)\widetilde{D}_j(y)|dxdy\\
&\quad+\sum_{i=1}^{m-1}|\triangle_{1,0}\alpha_{i,n}|\int_0^{2\pi}\!\!\int_0^{2\pi}|\widetilde{D}_i(x)\widetilde{D}_n(y)|dxdy\\
&\quad+|\alpha_{m,n}|\int_0^{2\pi}\!\!\int_0^{2\pi}|\widetilde{D}_m(x)\widetilde{D}_n(y)|dxdy\ll\alpha_{m,n}\log m\log n.
\end{aligned}
$$

Now we denote

$$
R_{k,n}^{(1,0)}:=\sum_{\mu=k}^{\infty}\left|\triangle_{1,0}\left(\frac{a_{\mu,n}}{\alpha_{\mu,n}}\right)\right|,\qquad R_{m,\ell}^{(0,1)}:=\sum_{\nu=\ell}^{\infty}\left|\triangle_{0,1}\left(\frac{a_{m,\nu}}{\alpha_{m,\nu}}\right)\right|
$$

and

$$
R_{k,\ell}^{(1,1)}:=\sum_{\mu=k}^{\infty}\sum_{\nu=\ell}^{\infty}\left|\triangle_{1,1}\left(\frac{a_{\mu,\nu}}{\alpha_{\mu,\nu}}\right)\right|.
$$

Thus, using the above estimate we obtain

$$\int_0^{2\pi}\int_0^{2\pi}\left|J_6(x,y)\right|dxdy \ll \frac{a_{m,n}}{\alpha_{m,n}}\alpha_{m,n}\log m\log n = a_{m,n}\log m\log n,$$
$$\int_0^{2\pi}\int_0^{2\pi}\left|J_7(x,y)\right|dxdy \ll \frac{a_{r+1,n}}{\alpha_{r+1,n}}\alpha_{r,n}\log r\log n \ll a_{r+1,n}\log m\log n,$$
$$\int_0^{2\pi}\int_0^{2\pi}\left|J_8(x,y)\right|dxdy \ll \frac{a_{m,s+1}}{\alpha_{m,s+1}}\alpha_{m,s+1}\log m\log s \ll a_{m,s+1}\log m\log n,$$
$$\int_0^{2\pi}\int_0^{2\pi}\left|J_9(x,y)\right|dxdy \ll \frac{a_{r+1,s+1}}{\alpha_{r+1,s+1}}\alpha_{r+1,s+1}\log r\log s \ll a_{r+1,s+1}\log m\log n.$$

Then we proceed as follows

$$\begin{aligned}
\int_0^{2\pi}\int_0^{2\pi}\left|J_4(x,y)\right|dxdy &\ll \sum_{k=r+1}^{m-1}\left|\triangle_{1,0}\left(\frac{a_{k,n}}{\alpha_{k,n}}\right)\right|\alpha_{k,n}\log k\log n\\
&\ll \sum_{k=r+1}^{m-1}\left(R^{(1,0)}_{k,n}-R^{(1,0)}_{k+1,n}\right)\alpha_{k,n}\log k\log n\\
&\ll |R^{(1,0)}_{r+1,n}\alpha_{r+1,n}\log(r+1)-R^{(1,0)}_{m,n}\alpha_{m,n}\log m\\
&+\sum_{k=r+1}^{m-1}R^{(1,0)}_{k+1,n}\left(\alpha_{k+1,n}\log(k+1)-\alpha_{k,n}\log k\right)\log n|\\
&\ll a_{r+1,n}\log(r+1)\log n + a_{m,n}\log m\log n\\
&+\sum_{k=r+1}^{m-1}\frac{a_{k+1,n}}{\alpha_{k+1,n}}\left|\triangle_{1,0}\alpha_{k,n}\log(k+1)-\alpha_{k,n}\log\left(1+\frac{1}{k}\right)\right|\log n\\
&\ll \left\{a_{r+1,n}\log(r+1)+a_{m,n}\log m+\sum_{k=r+1}^{m-1}\frac{a_{k+1,n}}{k}\right\}\log n,
\end{aligned}$$

and with the same reasoning we have obtained

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{2\pi}\left|J_5(x,y)\right|dxdy\\
&\qquad\ll \left\{a_{r+1,s+1}\log(r+1)+a_{m,s+1}\log m+\sum_{k=r+1}^{m-1}\frac{a_{k+1,s+1}}{k}\right\}\log s,\\
&\int_0^{2\pi}\int_0^{2\pi}\left|J_2(x,y)\right|dxdy\\
&\qquad\ll \left\{a_{m,s+1}\log(s+1)+a_{m,n}\log n+\sum_{\ell=s+1}^{n-1}\frac{a_{m,\ell+1}}{\ell}\right\}\log m,
\end{aligned}$$

and

$$\int_0^{2\pi}\int_0^{2\pi}\big|J_3(x,y)\big|dxdy$$
$$\ll \left\{a_{r+1,s+1}\log(s+1)+a_{r+1,n}\log n+\sum_{\ell=s+1}^{n-1}\frac{a_{r+1,s+1}}{\ell}\right\}\log r.$$

The estimating of the integral $\int_0^{2\pi}\int_0^{2\pi}\big|J_1(x,y)\big|dxdy$ is more complicated. Therefore for convenience of the reader we will sketch it in details. Indeed, we have

$$\int_0^{2\pi}\int_0^{2\pi}\big|J_1(x,y)\big|dxdy \ll \sum_{k=r+1}^{m-1}\sum_{\ell=s+1}^{n-1}\left|\triangle_{1,1}\left(\frac{a_{k,\ell}}{\alpha_{k,\ell}}\right)\right|\alpha_{k,\ell}\log k\log\ell$$
$$\ll \sum_{k=r+1}^{m-1}\sum_{\ell=s+1}^{n-1}\left(R^{(1,1)}_{k,\ell}-R^{(1,1)}_{k+1,\ell}-R^{(1,1)}_{k,\ell+1}+R^{(1,1)}_{k+1,\ell+1}\right)\alpha_{k,\ell}\log k\log\ell$$
$$\ll \sum_{k=r+1}^{m-1}\left(\sum_{\ell=s+1}^{n-1}\triangle_{0,1}\left(\triangle_{1,0}R^{(1,1)}_{k,\ell}\right)\alpha_{k,\ell}\log\ell\right)\log k$$
$$= \sum_{k=r+1}^{m-1}\left\{\triangle_{1,0}R^{(1,1)}_{k,s+1}\alpha_{k,s+1}\log(s+1)-\triangle_{1,0}R^{(1,1)}_{k,n}\alpha_{k,n}\log n\right.$$
$$\left.+\sum_{\ell=s+1}^{n-1}\triangle_{1,0}R^{(1,1)}_{k,\ell+1}\left(\alpha_{k,\ell+1}\log(\ell+1)-\alpha_{k,\ell}\log\ell\right)\right\}\log k$$
$$= \log(s+1)\sum_{k=r+1}^{m-1}\triangle_{1,0}R^{(1,1)}_{k,s+1}\alpha_{k,s+1}\log k-\log n\sum_{k=r+1}^{m-1}\triangle_{1,0}R^{(1,1)}_{k,n}\alpha_{k,n}\log k$$
$$+\sum_{\ell=s+1}^{n-1}\log(\ell+1)\left\{\sum_{k=r+1}^{m-1}\triangle_{1,0}\left(R^{(1,1)}_{k,\ell+1}\right)\alpha_{k,\ell+1}\log k\right\}$$
$$-\sum_{\ell=s+1}^{n-1}\log\ell\left\{\sum_{k=r+1}^{m-1}\triangle_{1,0}\left(R^{(1,1)}_{k,\ell+1}\right)\alpha_{k,\ell}\log k\right\}$$
$$= \log(s+1)\left\{R^{(1,1)}_{r+1,s+1}\alpha_{r+1,s+1}\log(r+1)-R^{(1,1)}_{m,s+1}\alpha_{m,s+1}\log m\right.$$
$$\left.+\sum_{k=r+1}^{m-1}R^{(1,1)}_{k+1,s+1}\left[\alpha_{k+1,s+1}\log(k+1)-\alpha_{k,s+1}\log k\right]\right\}$$
$$-\log n\left\{R^{(1,1)}_{r+1,n}\alpha_{r+1,n}\log(r+1)-R^{(1,1)}_{m,n}\alpha_{m,n}\log m\right.$$
$$\left.+\sum_{k=r+1}^{m-1}R^{(1,1)}_{k+1,n}\left[\alpha_{k+1,n}\log(k+1)-\alpha_{k,n}\log k\right]\right\}$$

$$
\begin{aligned}
&+\sum_{\ell=s+1}^{n-1}\log(\ell+1)\Big\{R^{(1,1)}_{r+1,\ell+1}\alpha_{r+1,\ell+1}\log(r+1)-R^{(1,1)}_{m,\ell+1}\alpha_{m,\ell+1}\log m\\
&+\sum_{k=r+1}^{m-1}R^{(1,1)}_{k+1,\ell+1}\left[\alpha_{k+1,\ell+1}\log(k+1)-\alpha_{k,\ell+1}\log k\right]\Big\}\\
&-\sum_{\ell=s+1}^{n-1}\log\ell\Big\{R^{(1,1)}_{r+1,\ell+1}\alpha_{r+1,\ell}\log(r+1)-R^{(1,1)}_{m,\ell+1}\alpha_{m,\ell}\log m\\
&+\sum_{k=r+1}^{m-1}R^{(1,1)}_{k+1,\ell+1}\left[\alpha_{k+1,\ell}\log(k+1)-\alpha_{k,\ell}\log k\right]\Big\}\\
=&\Big|R^{(1,1)}_{r+1,s+1}\alpha_{r+1,s+1}\log(r+1)\log(s+1)-R^{(1,1)}_{m,s+1}\alpha_{m,s+1}\log m\log(s+1)\\
&+\sum_{k=r+1}^{m-1}R^{(1,1)}_{k+1,s+1}\left[\alpha_{k+1,s+1}\log(k+1)-\alpha_{k,s+1}\log k\right]\log(s+1)\\
&-R^{(1,1)}_{r+1,n}\alpha_{r+1,n}\log(r+1)\log n+R^{(1,1)}_{m,n}\alpha_{m,n}\log m\log n\\
&+\sum_{k=r+1}^{m-1}R^{(1,1)}_{k+1,n}\left[\alpha_{k+1,n}\log(k+1)-\alpha_{k,n}\log k\right]\log n\\
&+\sum_{\ell=s+1}^{n-1}R^{(1,1)}_{r+1,\ell+1}\left[\alpha_{r+1,\ell+1}\log(\ell+1)-\alpha_{r+1,\ell}\log\ell\right]\log(r+1)\\
&+\sum_{\ell=s+1}^{n-1}R^{(1,1)}_{m,\ell+1}\left[\alpha_{m,\ell}\log\ell-\alpha_{m,\ell+1}\log(\ell+1)\right]\log m\\
&+\sum_{k=r+1}^{m-1}\sum_{\ell=s+1}^{n-1}R^{(1,1)}_{k+1,\ell+1}\big[\alpha_{k+1,\ell+1}\log(k+1)\log(\ell+1)\\
&-\alpha_{k+1,\ell}\log(k+1)\log\ell-\alpha_{k,\ell+1}\log k\log(\ell+1)+\alpha_{k,\ell}\log k\log\ell\big]\Big|\\
=&\Big|R^{(1,1)}_{r+1,s+1}\alpha_{r+1,s+1}\log(r+1)\log(s+1)-R^{(1,1)}_{m,s+1}\alpha_{m,s+1}\log m\log(s+1)\\
&-R^{(1,1)}_{r+1,n}\alpha_{r+1,n}\log(r+1)\log n+R^{(1,1)}_{m,n}\alpha_{m,n}\log m\log n\\
&+\sum_{k=r+1}^{m-1}R^{(1,1)}_{k+1,s+1}\left[\triangle_{1,0}\alpha_{k,s+1}\log(k+1)-\alpha_{k,s+1}\log\left(1+\frac{1}{k}\right)\right]\log(s+1)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=r+1}^{m-1} R^{(1,1)}_{k+1,n}\left[\triangle_{1,0}\alpha_{k,n}\log(k+1)-\alpha_{k,n}\log\left(1+\frac{1}{k}\right)\right]\log n\\
&+\sum_{\ell=s+1}^{n-1} R^{(1,1)}_{r+1,\ell+1}\left[\triangle_{0,1}\alpha_{r+1,\ell}\log(\ell+1)-\alpha_{r+1,\ell}\log\left(1+\frac{1}{\ell}\right)\right]\log(r+1)\\
&+\sum_{\ell=s+1}^{n-1} R^{(1,1)}_{m,\ell+1}\left[\triangle_{0,1}\alpha_{m,\ell}\log(\ell+1)-\alpha_{m,\ell}\log\left(1+\frac{1}{\ell}\right)\right]\log m\\
&+\sum_{k=r+1}^{m-1}\sum_{\ell=s+1}^{n-1} R^{(1,1)}_{k+1,\ell+1}\Bigg[\triangle_{1,1}\alpha_{k,\ell}\log(k+1)\log(\ell+1)\\
&-\triangle_{1,0}\alpha_{k,\ell}\log(k+1)\log\left(1+\frac{1}{\ell}\right)\\
&-\triangle_{0,1}\alpha_{k,\ell}\log\left(1+\frac{1}{k}\right)\log(\ell+1)+\alpha_{k,\ell}\log\left(1+\frac{1}{k}\right)\log\left(1+\frac{1}{\ell}\right)\Bigg]\Bigg|\\
\ll\, & a_{r+1,s+1}\log(r+1)\log(s+1)+a_{m,s+1}\log m\log(s+1)\\
&+a_{r+1,n}\log(r+1)\log n+a_{m,n}\log m\log n\\
&+\sum_{k=r+1}^{m-1}\frac{a_{k+1,s+1}}{\alpha_{k+1,s+1}}\left[|\triangle_{1,0}\alpha_{k,s+1}|\log(k+1)+\frac{\alpha_{k,s+1}}{k}\right]\log(s+1)\\
&+\sum_{k=r+1}^{m-1}\frac{a_{k+1,n}}{\alpha_{k+1,n}}\left[|\triangle_{1,0}\alpha_{k,n}|\log(k+1)+\frac{\alpha_{k,n}}{k}\right]\log n\\
&+\sum_{\ell=s+1}^{n-1}\frac{a_{r+1,\ell+1}}{\alpha_{r+1,\ell+1}}\left[|\triangle_{0,1}\alpha_{r+1,\ell}|\log(\ell+1)+\frac{\alpha_{r+1,\ell}}{\ell}\right]\log(r+1)\\
&+\sum_{\ell=s+1}^{n-1}\frac{a_{m,\ell+1}}{\alpha_{m,\ell+1}}\left[|\triangle_{0,1}\alpha_{m,\ell}|\log(\ell+1)+\frac{\alpha_{m,\ell}}{\ell}\right]\log m\\
&+\sum_{k=r+1}^{m-1}\sum_{\ell=s+1}^{n-1}\frac{a_{k+1,\ell+1}}{\alpha_{k+1,\ell+1}}\Bigg[|\triangle_{1,1}\alpha_{k,\ell}|\log(k+1)\log(\ell+1)\\
&+\frac{|\triangle_{1,0}\alpha_{k,\ell}|}{\ell}\log(k+1)+\frac{|\triangle_{0,1}\alpha_{k,\ell}|}{k}\log(\ell+1)+\frac{\alpha_{k,\ell}}{k\ell}\Bigg]\\
\ll\, & a_{r+1,s+1}\log(r+1)\log(s+1)+a_{m,s+1}\log m\log(s+1)\\
&+a_{r+1,n}\log(r+1)\log n+a_{m,n}\log m\log n\\
&+\sum_{k=r+1}^{m-1}\frac{a_{k+1,s+1}}{k}\log(s+1)+\sum_{k=r+1}^{m-1}\frac{a_{k+1,n}}{k}\log n\\
&+\sum_{\ell=s+1}^{n-1}\frac{a_{r+1,\ell+1}}{\ell}\log(r+1)+\sum_{\ell=s+1}^{n-1}\frac{a_{m,\ell+1}}{\ell}\log m+\sum_{k=r+1}^{m-1}\sum_{\ell=s+1}^{n-1}\frac{a_{k+1,\ell+1}}{k\ell}.
\end{aligned}
$$

Since $\{\frac{a_{k,\ell}}{\alpha_{k,\ell}}\} \in RBVS^2$, then for $\sqrt{r} \le k \le r$, $\sqrt{s} \le \ell \le s$, we have

$$\frac{a_{k,\ell}}{\alpha_{k,\ell}} \gg \sum_{\mu=k}^{\infty}\sum_{\nu=\ell}^{\infty}\left|\triangle_{1,1}\left(\frac{a_{\mu,\nu}}{\alpha_{\mu,\nu}}\right)\right| \gg \sum_{\mu=r}^{\infty}\sum_{\nu=s}^{\infty}\left|\triangle_{1,1}\left(\frac{a_{\mu,\nu}}{\alpha_{\mu,\nu}}\right)\right| \gg \frac{a_{r,s}}{\alpha_{r,s}}$$

From (1.14) we can write

$$\alpha_{r,s} = \alpha_{\sqrt{r}^2,s} \ll \alpha_{\sqrt{r},\sqrt{s}^2} \ll \alpha_{\sqrt{r},\sqrt{s}} \ll \alpha_{k,\ell},$$

thus we obtain

$$a_{k,\ell} \gg a_{r,s} \quad \text{for all} \quad \sqrt{r} \le k \le r, \quad \sqrt{s} \le \ell \le s. \tag{2.4}$$

Now we will show that $a_{m,n}\log m \log n \to 0$, as $m+n\to\infty$. Indeed, by (2.1) and (2.4) we have that

$$a_{m,n}\log m\log n \ll a_{m,n}\sum_{k=\sqrt{m}}^{m}\sum_{\ell=\sqrt{n}}^{n}\frac{1}{k\ell} \ll \sum_{k=\sqrt{m}}^{m}\sum_{\ell=\sqrt{n}}^{n}\frac{a_{k,\ell}}{k\ell} \ll \sum_{k=\sqrt{m}}^{\infty}\sum_{\ell=\sqrt{n}}^{\infty}\frac{a_{k,\ell}}{k\ell} \to 0,$$

as $m+n\to\infty$.

Moreover,

$$\sum_{\ell=s+1}^{n-1}\frac{a_{m,\ell+1}}{\ell}\log m \ll \sum_{k=\sqrt{m}}^{m}\sum_{\ell=s+1}^{n-1}\frac{a_{k,\ell+1}}{k\ell} \ll \sum_{k=\sqrt{m}}^{\infty}\sum_{\ell=s+1}^{n-1}\frac{a_{k,\ell+1}}{k\ell} \to 0,$$

as $m\to\infty$, uniformly in n, and similarly

$$\sum_{k=r+1}^{m-1}\frac{a_{k+1,n}}{k}\log n \ll \sum_{k=r+1}^{m-1}\sum_{\ell=\sqrt{n}}^{n}\frac{a_{k+1,\ell}}{k\ell} \ll \sum_{k=r+1}^{m-1}\sum_{\ell=\sqrt{n}}^{\infty}\frac{a_{k+1,\ell}}{k\ell} \to 0,$$

as $n\to\infty$, uniformly in m.

So, we have proved that all of integrals

$$\int_0^{2\pi}\int_0^{2\pi}\left|J_p(x,y)\right|dxdy \to 0, \quad \text{for} \quad p=1,2,\ldots,9,$$

and therefore

$$\int_0^{2\pi}\int_0^{2\pi}\left|\lambda_{m,n}^{(3)}(x,y)\right|dxdy \to 0. \tag{2.5}$$

In very same reasoning we have obtained that

$$\int_0^{2\pi}\int_0^{2\pi}\left|\lambda_{m,n}^{(q)}(x,y)\right|dxdy \to 0, \quad q=1,2. \tag{2.6}$$

Finally, putting the estimates (2.5) and (2.6) to (2.3), and applying Cauchy's criterion we completely prove the statement of the theorem. $\square$

If we take $\alpha_{m,n} = a_{m,n} \log^{-M} m \log^{-N} n$, then the following corollary holds true, which is an extension of one partial result proved in [14] from single to the two-dimensional case.

Corollary 2.2 *Let* $\{a_{m,n} \log^{-M} m \log^{-N} n\} \in LRBVS^2_{M,N}$. *Write*

$$g_2(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n} \sin mx \sin ny$$

at (x,y), *where it converges. If*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_{m,n}}{mn} < \infty,$$

then

$$\lim_{m+n\to\infty} \|g_2 - s_{m,n}(g_2)\| = 0.$$

Theorem 2.1 has a further generalization as follows.

Theorem 2.3 *Let* $\{\gamma_{m,n}\} \in AMS$ *with respect to* m *and* n *respectively, and* $\{\alpha_{m,n}\} \in LTS^2$. *If* $\{a_{m,n}\} \in \gamma LTRBVS^2$, *and*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \alpha_{m,n}\gamma_{m,n}(mn)^{-1} < \infty, \tag{2.7}$$

then $\lim_{m+n\to\infty} \|g_2 - s_{m,n}(g_2)\| = 0$.

Proof. We shall use all estimates of integrals $\int_0^{2\pi}\int_0^{2\pi}\left|J_q(x,y)\right|dxdy$, $q = 1,2,\ldots,9$, obtained in the proof of the Theorem 2.1. But in this case we use the hypothesis $\{\frac{a_{m,n}}{\alpha_{m,n}}\} \in \gamma RBVS^2$ which implies that

$$\frac{a_{k,n}}{\alpha_{k,n}} \le R^{(1,0)}_{k,n} \ll \gamma_{m,\ell}, \quad \frac{a_{m,\ell}}{\alpha_{m,\ell}} \le R^{(0,1)}_{m,\ell} \ll \gamma_{m,n}, \quad \frac{a_{m,n}}{\alpha_{m,n}} \le R^{(1,1)}_{m,n} \ll \gamma_{m,n}$$

and hence

$$a_{k,n}\ll\alpha_{k,n}\gamma_{m,n}, a_{m,\ell}\ll\alpha_{m,\ell}\gamma_{m,n}, a_{m,n}\ll\alpha_{m,n}\gamma_{m,n}, m,n = 1,2,\ldots. \tag{2.8}$$

Using these estimates we obtain

$$\int_0^{2\pi}\int_0^{2\pi}\left|J_6(x,y)\right|dxdy \ll \alpha_{m,n}\gamma_{m,n} \log m \log n,$$
$$\int_0^{2\pi}\int_0^{2\pi}\left|J_7(x,y)\right|dxdy \ll \alpha_{r+1,n}\gamma_{m,n} \log m \log n,$$
$$\int_0^{2\pi}\int_0^{2\pi}\left|J_8(x,y)\right|dxdy \ll \alpha_{m,s+1}\gamma_{m,n} \log m \log n,$$

$$\int_0^{2\pi}\int_0^{2\pi}\big|J_9(x,y)\big|dxdy \ll \alpha_{r+1,s+1}\gamma_{m,n}\log m\log n,$$

$$\int_0^{2\pi}\int_0^{2\pi}\big|J_4(x,y)\big|dxdy$$
$$\ll \left\{\alpha_{r+1,n}\log(r+1)+\alpha_{m,n}\log m+\sum_{k=r+1}^{m-1}\frac{\alpha_{k+1,n}}{k}\right\}\gamma_{m,n}\log n,$$

$$\int_0^{2\pi}\int_0^{2\pi}\big|J_5(x,y)\big|dxdy$$
$$\ll \left\{\alpha_{r+1,s+1}\log(r+1)+\alpha_{m,s+1}\log m+\sum_{k=r+1}^{m-1}\frac{\alpha_{k+1,s+1}}{k}\right\}\gamma_{m,n}\log s,$$

$$\int_0^{2\pi}\int_0^{2\pi}\big|J_2(x,y)\big|dxdy$$
$$\ll \left\{\alpha_{m,s+1}\log(s+1)+\alpha_{m,n}\log n+\sum_{\ell=s+1}^{n-1}\frac{\alpha_{m,\ell+1}}{\ell}\right\}\gamma_{m,n}\log m,$$

$$\int_0^{2\pi}\int_0^{2\pi}\big|J_3(x,y)\big|dxdy$$
$$\ll \left\{\alpha_{r+1,s+1}\log(s+1)+\alpha_{r+1,n}\log n+\sum_{\ell=s+1}^{n-1}\frac{\alpha_{r+1,s+1}}{\ell}\right\}\gamma_{m,n}\log r,$$

and

$$\int_0^{2\pi}\int_0^{2\pi}\big|J_1(x,y)\big|dxdy$$
$$\ll \Big\{\alpha_{r+1,s+1}\log(r+1)\log(s+1)+\alpha_{m,s+1}\log m\log(s+1)$$
$$+\alpha_{r+1,n}\log(r+1)\log n+\alpha_{m,n}\log m\log n$$
$$+\sum_{k=r+1}^{m-1}\frac{\alpha_{k+1,s+1}}{k}\log(s+1)+\sum_{k=r+1}^{m-1}\frac{\alpha_{k+1,n}}{k}\log n$$
$$+\sum_{\ell=s+1}^{n-1}\frac{\alpha_{r+1,\ell+1}}{\ell}\log(r+1)+\sum_{\ell=s+1}^{n-1}\frac{\alpha_{m,\ell+1}}{\ell}\log m$$
$$+\sum_{k=r+1}^{m-1}\sum_{\ell=s+1}^{n-1}\frac{\alpha_{k+1,\ell+1}}{k\ell}\Big\}\gamma_{m,n}.$$

Next, based on assumptions of the theorem, (1.13) and (1.14) we have

$$\alpha_{m,n}\gamma_{m,n}\log m\log n \ll \alpha_{m,n}\gamma_{m,n}\sum_{k=\sqrt{m}}^{m}\sum_{\ell=\sqrt{n}}^{n}\frac{1}{k\ell}\ll\sum_{k=\sqrt{m}}^{\infty}\sum_{\ell=\sqrt{n}}^{\infty}\frac{\alpha_{k,\ell}\gamma_{k,\ell}}{k\ell}\to 0,$$

$m+n \to \infty$, where we have used the estimate $\gamma_{m,n} \ll \gamma_{k,\ell}$ which follows from $\{\gamma_{m,n}\} \in AMS$ with respect to each m and n. Therefore, using this and (2.7) we have that

$$\int_0^{2\pi}\int_0^{2\pi} \big|J_p(x,y)\big| dxdy \to 0, \quad \text{for} \quad p = 1, 2, \ldots, 9,$$

and thus

$$\int_0^{2\pi}\int_0^{2\pi} \big|\lambda^{(3)}_{m,n}(x,y)\big| dxdy \to 0. \tag{2.9}$$

Similarly, we have obtained

$$\int_0^{2\pi}\int_0^{2\pi} \big|\lambda^{(q)}_{m,n}(x,y)\big| dxdy \to 0, \quad q = 1, 2. \tag{2.10}$$

Finally, for completing the proof of this theorem we do the same reasoning as in the proof of the Theorem 2.1. □

Remark 2.1 Note that, in particular case, when $\gamma_{m,n} = \frac{a_{m,n}}{\alpha_{m,n}}$ then (2.7) implies (2.1). Thus Theorem 2.3 is a generalization of Theorem 2.1.

The next theorem gives some equivalent conditions.

Theorem 2.4 *Let $\beta \in LTRBVS^2$, that is,*

$$\{\alpha_{m,n}\} \in LTS^2 \quad \textit{and} \quad \left\{\frac{a_{m,n}}{\alpha_{m,n}}\right\} \in RBVS^2.$$

Then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_{m,n}}{mn} < \infty \Leftrightarrow \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |\triangle_{1,1} a_{m,n}| \log(m+1)\log(n+1) < \infty. \tag{2.11}$$

Proof. For the proof of this theorem one has to use some parts of the proof of Theorem 2.1, and also has to repeat the same technique, because of the same assumptions. For this we shall omit the details of the proof. □

In the end of this paper, we give the following remark.

Remark 2.2 In the special case when $\alpha_{m,n} = \log^M m \log^N n$, Theorem 2.4 reduces to the two dimensional version of the Theorem 4 of [14].

Acknowledgements The author wishes to thank the anonymous referee for her/his suggestions which definitely improved the final form of this paper.

References

1. Bokayev, N.A.; Mukanov, Zh.B. – *Weighted integrability of double trigonometric series and of double series with respect to multiplicative systems with coefficients of class $R_0^+BVS^2$*, Translation of Mat. Zametki, 91 (2012), 617–620. Math. Notes, 91 (2012), 575–578.
2. Chen, C.P. – *Integrability and L-convergence of multiple trigonometric series*, Bull. Austral. Math. Soc., 49 (1994), 333–339.
3. Chen, C.P.; Wu, H.C.; Móricz, F. - *Pointwise convergence of multiple trigonometric series*, J. Math. Anal. Appl., 185 (1994), 629–646.
4. Chen, C.P.; Chauang, Y.W. – *L^1-convergence of double Fourier series*, Chinese J. Math., 19 (1991), 391-410.
5. Kaur, K.; Bhatia, S.S.; Ram, B. – *L^1-convergence of complex double Fourier series*, Proc. Indian Acad. Sci. Math. Sci., 113 (2003), 355–363.
6. Kaur, J.; Bhatia, S.S. – *Integrability and L^1-convergence of double cosine trigonometric series*, Anal. Theory Appl., 27 (2011), 32–39.
7. Leindler, L. – *On the degree of approximation of continuous functions*, Acta Math. Hungar., 104 (2004), 105–113.
8. Leindler, L. – *Embedding results regarding strong approximation of sine series*, Acta. Sci. Math. (Szeged), 71 (2005), 91–103.
9. Leindler, L. – *On L^1-convergence of sine series*, Anal. Math., 38 (2012), 123–133.
10. Móricz, F. – *On the integrability and L^1- convergence of double trigonometric series*, Studia Math., 98 (1991), 203–225.
11. Móricz, F. – *Convergence and integrability of double trigonometric series with coefficients of bounded variation*, Proc. Amer. Math. Soc., 102 (1988), 633–640.
12. Móricz, F. – *On the integrability and L^1-convergence of double trigonometric series. II*, Acta Math. Hungar., 69 (1995), 99–110.
13. Tikhonov, S. – *On L^1-convergence of Fourier series*, J. Math. Anal. Appl., 347 (2008), 416–427.
14. Zhou, S.P. – *What condition can correctly generalizes monotonicity in L^1-convergence of sine series?*, Science China (Sci. Sin. Math.), 40 (2010), 801–812 (in Chinese).
15. Zygmund, A. – *Trigonometric Series*, 2nd ed. Vols. I, II, Cambridge University Press, New York, 1959.