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∞-harmonic morphisms and ∞-harmonic functions

Ye-Lin Ou

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Abstract The existence of the smooth solutions of the ∞-Laplace equation is rare. In this note, we use the ideas of ∞-harmonic morphisms (maps that preserve solutions of ∞-Laplace equations), ∞-harmonic coordinates, and directional angles to construct many new families of smooth solutions of ∞-Laplace equations in space forms. We also determined all rotationally symmetric ∞-harmonic functions on such spaces and give explicit expressions of the ∞-Laplace operator on a Riemannian manifolds in local coordinates and with respect to local orthonormal frames.

Keywords infinity-harmonic morphisms · ∞-Laplace equations · infinity-harmonic coordinates · rotationally symmetric solutions · space forms

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1 Infinity-harmonic maps and the ∞-Laplace equations

Without otherwise statement all objects in this paper including manifolds, metrics, maps, and vector fields are supposed to be smooth. Solutions of PDEs are referred to strong or classical solutions. An ∞-harmonic map is a map between Riemannian manifolds $\varphi : (M, g) \longrightarrow (N, h)$ whose gradient of the energy density lies in the kernel of its differential, i.e., φ is a solution of the PDEs (see [33])

$$\tau_\infty(\varphi) := \frac{1}{2}\mathrm{d}\varphi(\mathrm{grad}\,|\mathrm{d}\varphi|^2) = 0, \tag{1.1}$$

where $|\mathrm{d}\varphi|^2 = \mathrm{Trace}_g\varphi^*h$ is the energy density of φ.

Maps of constant energy density are clearly ∞-harmonic maps. These include familiar and important maps like totally geodesic maps, isometric immersions, Riemannian submersions, and eigenmaps between spheres. Examples of ∞-harmonic maps with nonconstant energy density include projections of multiply warped products (e.g.,

Ye-Lin Ou
Department of Mathematics,
Texas A&M University-Commerce,
Commerce, TX 75429, USA
E-mail: yelin.ou@tamuc.edu

the projection of the generalized Kasner space-times), equator maps, and radial projections (see [33] for details). For some classifications of ∞-harmonic maps between Riemannian manifolds see [36] and [37].

A very important special case is when the map is a real-valued function $u:(M,g) \longrightarrow \mathbb{R}$. In this case, the ∞-harmonic map equation reduces to a single PDE called the ∞-Laplace equation,

$$\Delta_\infty u := \frac{1}{2} g(\operatorname{grad} u, \operatorname{grad} |\operatorname{grad} u|^2) = 0. \tag{1.2}$$

When the manifold (M,g) is a Euclidean domain, the ∞-Laplace equation takes the form $\Delta_\infty u := \frac{1}{2}\langle \nabla u, \nabla |\nabla u|^2\rangle = \sum_{i,j=1}^m u_{ij}u_iu_j = 0$, where $u : \Omega \subset \mathbb{R}^m \longrightarrow \mathbb{R}$, $u_i = \frac{\partial u}{\partial x^i}$ and $u_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j}$. It was first discovered and studied by Aronsson in his study of "optimal" Lipschitz extension of functions in the late 1960s (see [1], [2]).

The following interesting links of the ∞-Laplace equations partially explain why this nonlinear and highly degenerate elliptic PDE has been so fascinating.

1. The ∞-Laplace equation is solved by any solution of the Eikonal equation (or Hamilton-Jacobi equation by some authors): $|\nabla u|^2 = \text{constant}$.

2. The ∞-Laplace equation is also solved by any harmonic function with minimal graph as we can see from the minimal graph equation (see [30]):

$$(1 + |\nabla u|^2)\Delta u + \sum_{i,j=1}^m u_iu_ju_{ij} = 0,$$

3. The ∞-Laplace equation can be obtained as the Euler-Lagrange equation of the L^∞ variational problem of minimizing $E_\infty(u) = \operatorname{ess\,sup}_\Omega |\mathrm{d}\, u|$ among all Lipschitz continuous functions u with given boundary values on $\partial\Omega$ (see [3], [6], and [7] and the references therein for more detailed background).

4. The ∞-Laplace equation can be viewed (see [1]) as the formal limit, as $p \to \infty$, of p-Laplace equation $\Delta_p\, u := |\nabla u|^{p-2}\,(\Delta\, u + \frac{p-2}{|\nabla u|^2}\Delta_\infty\, u) = 0$.

The solutions of the ∞-Laplace equation are called *∞-harmonic functions* which have the following interpretations:

Lemma 1.1 (see [29]) *Let $u : (M^m, g) \longrightarrow \mathbb{R}$ be a function. Then the following conditions are equivalent:*

(1) *u is an ∞-harmonic function, i.e., $\Delta_\infty u = 0$,*
(2) *u is horizontally homothetic;*
(3) *∇u is perpendicular to $\nabla|\nabla u|^2$;*
(4) $\mathrm{Hess}_u(\nabla u, \nabla u) = 0$*;*
(5) *$|\nabla u|^2$ is constant along any integral curve of ∇u.*

Recently, much work has been done in the study of the ∞-Laplace equation after the work of Crandall and Lions (see e.g. [16]) on the theory of viscosity solutions for fully nonlinear problems. Many important results have been achieved and published in, e.g., [3, 5–10, 14–17, 19–23, 25–28].

On the other hand, the ∞-Laplace equation has been found to have some very interesting applications in areas such as image processing (see, e.g., [12], [35]), mass transfer problems (see, e.g., [19]), and the study of shape metamorphism (see, e.g., [13]).

The existence of smooth solutions of the ∞-Laplace equation is rare as remarked in [18] that " ... in general the $\Delta_\infty u = 0$ does not admit smooth solutions". In fact, all the known examples of ∞-harmonic functions on Euclidean domains in $\mathbb{R}^m$ seem to come from the following families (see e.g., [23] and [24]):

1. Partial distance functions: $u(x_1, \ldots, x_m) = \alpha\sqrt{x_1^2 + \ldots + x_k^2} + \beta,\ 1 \le k \le m.$

2. Affine functions: $u(x_1, \ldots, x_m) = \alpha\langle \mathbf{x}, \mathbf{a}\rangle + \beta.$

3. Weighted 4/3-maximum distance functions: $u(x_1, \ldots, x_m) = \alpha_1|x_1|^{4/3} + \ldots + \alpha_m|x_m|^{4/3} + \beta$, with $\alpha_1^3 + \ldots + \alpha_m^3 = 0$. In particular, $u(x.y) = |x|^{4/3} - |y|^{4/3}$ is infinity-harmonic function in xy-plane. Note that this function is $C^{1,1/3}_{\rm loc}(\mathbb{R}^2)$, not a smooth function on the whole $\mathbb{R}^2$.

4. Spherical angles $\theta(x, y, z) = \arctan\frac{y}{x}$ and $\phi = \arctan\frac{z}{\sqrt{x^2+y^2}}$.

5. Superpositions of two infinity-harmonic functions, like
$3\sqrt{x_1^2 + x_2^2} - 27\arctan\frac{x_5}{\sqrt{x_3^2+x_4^2}} + 100.$

In this note, we use the ideas of ∞-harmonic morphisms (maps that preserve solutions of ∞-Laplace equations), ∞-harmonic coordinates, and the directional angles to construct many new families of smooth solutions of ∞-Laplace equations in space forms. We also determined all rotationally symmetric ∞-harmonic functions on such spaces and give explicit expressions of the ∞-Laplace operator on a Riemannian manifolds in local coordinates and with respect to local orthonormal frames.

2 ∞-harmonic morphisms and ∞-harmonic functions

Infinity-harmonic morphisms are maps between Riemannian manifolds that pull back locally defined ∞-harmonic functions to ∞-harmonic functions (see [33]). It was proved in [33] that these maps are precisely horizontally weakly conformal ∞-harmonic maps with dilations being constant along any horizontal curves. It was also proved in [33] that an ∞-harmonic morphism pulls back ∞-harmonic maps to ∞-harmonic maps. For the completeness of discussion, we give a proof of the following special case, which will be used in the rest of the paper.

Theorem 2.1 *If $\varphi : (M, g) \longrightarrow (N, h)$ is an ∞-harmonic morphism with dilation λ, then for any function $f : N \supseteq U \longrightarrow \mathbb{R}$ defined on an open subset with $\varphi^{-1}(U) \neq \phi$, we have $\Delta^M_\infty(f\circ\varphi) = \lambda^4(\Delta^N_\infty f)\circ\varphi$. In particular, an ∞-harmonic morphism preserves solutions of ∞-Laplace equations in the sense that if f is a solution of the ∞-Laplace equation in the target manifold, then the pull-back $f\circ\varphi$ is a solution of the ∞-Laplace equation in the domain manifold.*

Proof. Let φ be an ∞-harmonic morphism, i.e., a horizontally homothetic submersion with dilation λ. A straightforward computation gives $|\nabla(f\circ\varphi)|^2 = g^{ij}(f\circ\varphi)_i(f\circ\varphi)_j = g^{ij}f_\alpha\varphi_i^\alpha f_\beta\varphi_j^\beta = \lambda^2(h^{\alpha\beta}\circ\varphi)f_\alpha f_\beta = \lambda^2(|\nabla f|^2\circ\varphi)$, where the third equality was obtained by using the horizontal weak conformality equation $g^{ij}\varphi_i^\alpha\varphi_j^\beta = \lambda^2(h^{\alpha\beta}\circ\varphi)$. It follows that $\nabla(|\nabla(f\circ\varphi)|^2) = (\nabla\,\lambda^2)(|\nabla\,f|^2\circ\varphi) + \lambda^2\nabla\,(|\nabla\,f|^2\circ\varphi)$, and hence

$$
\begin{aligned}
\Delta_\infty^M(f\circ\varphi) &= \frac{1}{2}\,\langle\nabla\,(f\circ\varphi),\ \nabla\ |\nabla(f\circ\varphi)|^2\rangle \\
&= \frac{1}{2}\,\langle\nabla\,(f\circ\varphi),\ (\nabla\,\lambda^2)(|\nabla\,f|^2\circ\varphi)\rangle \qquad (2.1)\\
&+ \frac{1}{2}\,\langle\nabla\,(f\circ\varphi),\ \lambda^2\nabla\,(|\nabla\,f|^2\circ\varphi\rangle.
\end{aligned}
$$

A further computation shows that

$$
\frac{1}{2}\,\langle\nabla\,(f\circ\varphi),\ (\nabla\,\lambda^2)(|\nabla\,f|^2\circ\varphi)\rangle = \frac{1}{2}(|\nabla\,f|^2\circ\varphi)\mathrm{d}f\left(\mathrm{d}\varphi(\nabla\lambda^2)\right) = 0 \qquad (2.2)
$$

since $\nabla\lambda^2$ is vertical by the assumption that φ is a horizontally homothetic submersion.

On the other hand,

$$
\begin{aligned}
&\frac{1}{2}\,\langle\nabla\,(f\circ\varphi),\ \lambda^2\nabla\,(|\nabla\,f|^2\circ\varphi\rangle = \frac{1}{2}\,\lambda^2 g^{ij} f_\alpha\varphi_i^\alpha(|\nabla f|^2)_\beta\varphi_j^\beta \qquad (2.3)\\
&= \frac{1}{2}\,\lambda^4(h^{\alpha\beta}\circ\varphi)f_\alpha(|\nabla f|^2)_\beta = \frac{1}{2}\,\lambda^4\langle\nabla\,f,\ \nabla\ |\nabla f|^2\rangle_h\circ\varphi = \lambda^4(\Delta_\infty^N f)\circ\varphi.
\end{aligned}
$$

The theorem follows from (2.1), (2.2) and (2.3). □

Many known examples of ∞-harmonic morphisms come from Riemannian submersions or the projections of a warped product onto its first or second factor.

Example 2.1 The projection $\varphi : M^m \times_{\beta^2} N^n \longrightarrow (N^n, h),\ \varphi(x,y) = y$, of a warped product onto its second factor is a horizontally conformal submersion with dilation $\lambda = 1/\beta$ whose gradient is vertical, and hence φ is an ∞-harmonic morphism with nonconstant energy density provided that $\beta \neq$ constant. The projection onto the first factor is a Riemannian submersion and hence an ∞-harmonic morphism with constant energy density.

It is interesting to note that ∞-harmonic morphisms from nonnegatively curved manifolds are "basically" Riemannian submersions: it was proved in [31] that any ∞-harmonic morphism $\varphi : \mathbb{R}^{n+1} \longrightarrow (N^n, h)$ is a Riemannian submersion up to a homothety; if $\varphi : \mathbb{S}^{n+1} \longrightarrow (N^n, h)$ is an ∞-harmonic morphism, then $n = 2m$, (N^n, h) is isometric to $\mathbb{C}P^m$ and, up to a homothety, φ is a standard Hopf fibration $\mathbb{S}^{2m+1} \longrightarrow \mathbb{C}P^m$; the last statement was generalized in [32] as: any ∞-harmonic morphism from a compact manifold of nonnegative sectional curvature is a Riemannian submersion.

One application of Theorem 2.1 is to construct ∞-harmonic functions via pull-back or "lift" of ∞-harmonic functions on the base space by ∞-harmonic morphism including Riemannian submersions.

Corollary 2.2 *Any ∞-harmonic function (locally) defined on $\mathbb{R}^m$ is also an ∞-harmonic function on the hyperbolic space $(H^{m+1} = \mathbb{R}^m \times \mathbb{R}^+,\ x_{m+1}^{-2}\sum_{i=1}^{m+1}\mathrm{d}x_i^2)$. In particular, any affine function in the first m variables is a solution of the ∞-Laplace equation in hyperbolic space in upper-half space model.*

Proof. It is well known (see e.g., [4]) that the projection $\varphi : (H^{m+1} = \mathbb{R}^m \times \mathbb{R}^+, x_{m+1}^{-2}\sum_{i=1}^{m+1} \mathrm{d}x_i^2) \longrightarrow \mathbb{R}^m,\ \varphi(x_1, \ldots, x_m, x_{m+1}) = (x_1, \ldots, x_m)$ is a horizontally homothetic submersion and hence an ∞-harmonic morphism. By Theorem 2.1, any ∞-harmonic function on the base can be "lifted" or pulled back to be an ∞-harmonic function on the hyperbolic space. In particular, since any affine function $f(x_1, ..., x_m) = c_1x_1 + \ldots + c_mx_m + c_0$ is an ∞-harmonic function on $\mathbb{R}^m$ the pull-back $f \circ \varphi(x_1, ..., x_m, x_{m+1}) = f(x_1, ..., x_m) = c_1x_1 + \ldots + c_mx_m + c_0$ is an ∞-harmonic function on the hyperbolic space. □

Example 2.2 [There are many ∞-harmonic functions on the Heisenberg space] The Heisenberg space can be viewed as $\mathbb{H}_3 = (\mathbb{R}^3, g)$, endowed with a left invariant metric whose group of isometries has dimension 4. With respect to the standard coordinates (x, y, z) in $\mathbb{R}^3$, the metric can be written as $g = \mathrm{d}x^2 + \mathrm{d}y^2 + (\mathrm{d}z + \frac{y}{2}\mathrm{d}x - \frac{x}{2}\mathrm{d}y)^2$. It is easily checked that $\varphi : \mathbb{H}_3 \longrightarrow \mathbb{R}^2,\ \varphi(x, y, z) = (x, y)$ is a Riemannian submersion. It follows that any ∞-harmonic functions on $\mathbb{R}^2$, for instance, $f = ax + by + c,\ F = \arctan\frac{y}{x}$, are also ∞-harmonic functions on the Heisenberg space.

3 ∞-harmonic coordinates

In this section we show that in space forms and many other spaces we can always choose local coordinates so that all the coordinate functions are ∞-harmonic thus provides abundant smooth solutions of the ∞-Laplace equations.

It is well known that around any point of a Riemannian manifold (M, g) there is a local coordinate neighborhood $U \subseteq M$ whose coordinate functions are all harmonic functions (see, e.g., [34]). Such a local coordinate system is called a local harmonic coordinate system. In light of this, we call a local coordinate system $\{x_i\}$ in $U \subset M$ an *∞-harmonic coordinate system* if each of every coordinate functions $x_i(x_1, \ldots, x_m) = x_i,\ i = 1, \ldots, m$, is infinity-harmonic. Clearly, the standard Cartesian coordinates $(x_1, \ldots, x_m)$ in Euclidean space $\mathbb{R}^m$ are ∞-harmonic coordinates since any affine function in $\mathbb{R}^m$ is an ∞-harmonic function. The polar coordinates (r, θ) in $\mathbb{R}^2$ are also ∞-harmonic coordinates, this can be seen from the well-known fact that functions $r(x, y) = \sqrt{x^2 + y^2}$ and $\theta(x, y) = \arctan\frac{y}{x}$ are infinity-harmonic functions. Also, the cylindrical coordinates (r, θ, z) in $\mathbb{R}^3 \setminus \{\mathrm{x-axis}\}$ are infinity-harmonic coordinates because $r(x, y, z) = \sqrt{x^2 + y^2},\ \theta(x, y, z) = \arctan\frac{y}{x},\ z(x, y, z) = z$ are all ∞-harmonic functions. It would be interesting to know if there exists local infinity-harmonic coordinates around any point of a general Riemannian manifold. Though we do not know the answer to the question in general, we will show that in a space of constant sectional curvature (and in many other cases) it is always possible to have ∞-harmonic coordinates. Before going any further we need the following

Lemma 3.1 *Let (M^m, g) be a Riemannian manifold and $U \subset M$ be a local coordinate neighborhood with coordinates $\{x_i\}$. Then the k-th coordinate function $x_k(x_1, ..., x_m) = x_k$ is an ∞-harmonic function on (U, g_U) if and only if $g^{kj}\frac{\partial g^{kk}}{\partial x_j} = 0$, where g^{ij} denotes the contravariant components of the metric g with respect to the local coordinates $\{x_i\}$. In particular, an orthogonal coordinate system $\{x_i\}$ (meaning $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = 0$ for $i \neq j$) is an ∞-harmonic coordinate system if and only if $\frac{\partial g^{kk}}{\partial x_k} = 0,\ k = 1, \ldots, m$.*

Proof. A direct computation yields $|\nabla x_k|^2 = g^{ij}\frac{\partial x_k}{\partial x_i}\frac{\partial x_k}{\partial x_j} = g^{kk}$ and $\Delta_\infty x_k = \frac{1}{2}\langle \nabla x_k, \nabla |\nabla x_k|^2\rangle = \frac{1}{2}g^{ij}\frac{\partial x_k}{\partial x_i}\frac{\partial |\nabla x_k|^2}{\partial x_j} = g^{kj}\frac{\partial g^{kk}}{\partial x_j}$, from which the lemma follows. □

Example 3.1 Applying Lemma 3.1 we can easily see that the coordinates (t, x, y, z) in the generalized Kasner space-times $((0,\infty)\times\mathbb{R}^3, -\mathrm{d}t^2 + t^{2a}\mathrm{d}x^2 + t^{2b}\mathrm{d}y^2 + t^{2c}\mathrm{d}z^2)$ are ∞-harmonic coordinates. The standard coordinates in the Sol space, one model of Thurston's 3-dimensional geometries) are ∞-harmonic coordinates. The Sol space is ($\mathbb{R}^3$ provides with the Sol metric which, with respect to the standard coordinates (x, y, z), can be written as $g_{Sol} = e^{2z}\mathrm{d}x^2 + e^{-2z}\mathrm{d}y^2 + \mathrm{d}z^2$).

Corollary 3.2 *In $\mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$ with the standard Cartesian coordinates (x, y) we consider the twisted product metric $g = \mathrm{d}x^2 + \alpha^{-2}(x, y)\mathrm{d}y^2$. Then, the coordinates (x, y) are ∞-harmonic if and only $g = \mathrm{d}x^2 + \alpha^{-2}(x, y)\mathrm{d}y^2$ is a warped product, i.e., $\alpha^2(x, y)$ depends only on x.*

Proof. With respect to the coordinates (x, y) we have $g^{11} = 1,\ g^{22} = \alpha^2$. By Lemma 3.1, the coordinates are ∞-harmonic coordinates if and only if $\frac{\partial g^{22}}{\partial y} = \frac{\partial \alpha^2}{\partial y} = 0$, i.e., $\alpha(x, y)$ depends only on x, from which the lemma follows. □

Example 3.2 The coordinates (u, v) in the parametrized surface of revolution, $(u, v) \to (f(v)\cos u, f(v)\sin u, g(v)) \subseteq \mathbb{R}^3$, where $f'^2 + g'^2 = 1$, are ∞-harmonic coordinates. One can easily check that the induced metric on the surface of revolution with respect to the coordinates (u, v) is give by $g = f^2(v)\mathrm{d}u^2 + \mathrm{d}v^2$ which is a warped product metric. Therefore, by the above corollary, the coordinates (u, v) are ∞-harmonic . In particular, the coordinates (u, v) in the helicoid, where $(u, v) \longrightarrow (u\cos v, u\sin v, v) \subseteq \mathbb{R}^3$, are ∞-harmonic coordinates. One checks that the metric with respect to these coordinates is given by $g = \mathrm{d}u^2 + (1 + u^2)\mathrm{d}v^2$;

Example 3.3 The coordinates (θ, t) in the catenoid, where $(\theta,\ t) \to (\cosh t\cos\theta, \cosh t\sin\theta,\ t) \subseteq \mathbb{R}^3$, are not ∞-harmonic coordinates. One checks that the metric with respect to these coordinates is given by $g = \cosh^2 t(\mathrm{d}\theta^2 + \mathrm{d}t^2)$ and the second coordinate function $t(\theta, t) = t$ is not an ∞-harmonic function.

The following theorem guarantees the existence of an ∞-harmonic coordinate system around any point of a space of constant sectional curvature.

Theorem 3.3 (1) *The hyperspherical coordinates $(\rho, \phi_1, \ldots, \phi_m)$ are ∞-harmonic coordinates in a domain of Euclidean space $\mathbb{R}^{m+1}$;*

(2) *The angular coordinates $(\phi_1, \ldots, \phi_m)$ are ∞-harmonic coordinates in a domain of the sphere S^m;*

(3) *There is a globally defined ∞-harmonic coordinate system on the hyperbolic space H^m.*

Proof. Recall that the hyperspherical coordinates (see e.g., [11]) of $(m+1)$-dimensional Euclidean space are defined using a radial coordinate ρ , and angular coordinates

$\phi_1, \ldots, \phi_m$. If $(x_1, \ldots, x_{m+1})$ are the Cartesian coordinates, then we may define

$$\begin{cases} x_1 = \rho \cos\phi_1 \\ x_2 = \rho \sin\phi_1 \cos\phi_2 \\ x_3 = \rho \sin\phi_1 \sin\phi_2 \cos\phi_3 \\ \ldots \\ x_m = \rho \sin\phi_1 \sin\phi_2 \ldots \sin\phi_{m-1} \cos\phi_m \\ x_{m+1} = \rho \sin\phi_1 \sin\phi_2 \ldots \sin\phi_{m-1} \sin\phi_m, \end{cases} \tag{3.1}$$

where $0 \leq \phi_m \leq 2\pi,\ 0 \leq \phi_i \leq \pi, i = 1, 2, \ldots, m-1$. A straightforward computation using (3.1) and $g = \mathrm{d}x_1^2 + \ldots + \mathrm{d}x_{m+1}^2$ one obtains that, with respect to the hyperspherical coordinates $(\rho, \phi_1, \ldots, \phi_m)$, the covariant components g_{ij} and contravariant components g^{ij} of the metric tensor are given respectively by

$$\begin{cases} g_{11} = 1,\ g_{22} = \rho^2,\ g_{33} = \rho^2 \sin^2\phi_1, g_{44} = \rho^2 \sin^2\phi_1 \sin^2\phi_2, \ldots, \\ g_{m+1\, m+1} = \rho^2 \sin^2\phi_1 \ldots \sin^2\phi_{m-1},\ \text{all other } g_{ij} = 0, i \neq j; \\ g^{11}{=}1, g^{22}{=}\rho^{-2}, g^{33}{=}\rho^{-2}\sin^{-2}\phi_1, g^{44}{=}\rho^{-2}\sin^{-2}\phi_1 \sin^{-2}\phi_2, \ldots, \\ g^{m+1\, m+1} = \rho^{-2}\sin^{-2}\phi_1 \ldots \sin^{-2}\phi_{m-1},\ \text{all other } g^{ij} = 0, i \neq j; \end{cases} \tag{3.2}$$

From this and Lemma 3.1 we obtain statement (1).

For statement (2), notice that the standard metric on the unit sphere S^m is obtained by restricting the Euclidean metric (3.2) on S^m via the standard embedding. It follows that, we can substitute $\rho = 1$ into the Euclidean metric

$$g = \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\phi_1^2 + \rho^2 \sin^2\phi_1 \mathrm{d}\phi_2^2 + \ldots + \rho^2 \sin^2\phi_1 \ldots \sin^2\phi_{m-1} \mathrm{d}\phi_m^2 \tag{3.3}$$

to get the standard metric h on the unit sphere which, with respect to the angular coordinates $(\phi_1, \ldots, \phi_m)$ on S^m, has the following components

$$\begin{cases} h_{11} = 1,\ h_{22} = \sin^2\phi_1,\ h_{33} = \sin^2\phi_1 \sin^2\phi_2, \ldots, \\ h_{mm} = \sin^2\phi_1 \ldots \sin^2\phi_{m-1}, \quad \text{all other } h_{ij} = 0, \qquad i \neq j; \\ h^{11} = 1,\ h^{22} = \sin^{-2}\phi_1,\ h^{33} = \sin^{-2}\phi_1 \sin^{-2}\phi_2, \ldots, \\ h^{mm} = \sin^{-2}\phi_1 \ldots \sin^{-2}\phi_{m-1}, \quad \text{all other } h^{ij} = 0, \qquad i \neq j. \end{cases} \tag{3.4}$$

This and Lemma 3.1 give statement (2).

To prove statement (3), we use upper-half space model for hyperbolic space ($H^m = \mathbb{R}^{m-1} \times \mathbb{R}^+, x_m^{-2} \sum_{i=1}^{m} \mathrm{d}x_i^2$). It is easy to check that the standard coordinates $(x_1, \ldots, x_m)$ are not ∞-harmonic because the last coordinate function $x_m(x_1, \ldots, x_m) = x_m$ is not infinity-harmonic by Lemma 3.1 and the fact that $g^{mm} = x_m^2$. However, we can change the coordinates $(x_1, \ldots, x_m)$ into $(x_1, \ldots, x_{m-1}, t)$ via an isometry

$$\varphi : \left(H^m{=}\mathbb{R}^{m-1} \times \mathbb{R}^+, x_m^{-2} \sum_{i=1}^{m} \mathrm{d}x_i^2 \right) \to (\mathbb{R}^{m-1} \times \mathbb{R}, e^{-2t} \sum_{i=1}^{m-1} \mathrm{d}x_i^2 + \mathrm{d}t^2) \tag{3.5}$$

given by

$$\begin{cases}\varphi^i(x_1,\ldots,x_m)=x_i, \quad i=1,\ldots,(m-1)\\ \varphi^m(x_1,\ldots,x_m)=\ln x_m.\end{cases}$$

It follows that with respect to the coordinates $\{x_1,\ldots,x_{m-1},t)$ we have

$$\begin{cases}g^{ii}=e^{2t},\ i=1,\ldots,(m-1);\\ g^{mm}=1,\\ g^{ij}=0, \quad i\neq j.\end{cases} \tag{3.6}$$

From this and Lemma 3.1 we conclude that the coordinates $(x_1,\ldots,x_{m-1},t)$ are infinity-harmonic coordinates in the hyperbolic space $(H^m=\mathbb{R}^{m-1}\times\mathbb{R}, e^{-2t}\sum_{i=1}^{m-1}\mathrm{d}x_i^2+\mathrm{d}t^2)$. □

4 Smooth solutions of the ∞-Laplace equation in space forms

In this section we will use the ideas of ∞-harmonic morphisms, ∞-harmonic coordinates, and the directional angles to show that there many smooth families of solutions of ∞-Laplace equations given by elementary functions.

The following proposition is a consequence of Theorem 2.1

Proposition 4.1 *For any horizontally homothetic submersion (in particular, any Riemannian submersion) $\varphi:(M^m,g)\longrightarrow(N^n,h)$. With respect to any local ∞-harmonic coordinates $\{y^\alpha\}$ in a neighborhood $V\subseteq N$ and any local coordinates $\{x^i\})$ in a neighborhood $U=\varphi^{-1}(V)\subseteq M$, all the component functions $\varphi^\alpha(x^1,\ldots,x^m)$, $\alpha=1,\ldots,n$ of φ are ∞-harmonic harmonic functions on $U=\varphi^{-1}(V)\subseteq M$.*

Proof. This follows from Theorem 2.1 and the fact that a horizontally homothetic submersion (in particular, any Riemannian submersion) is an ∞-harmonic morphism and that the component functions are nothing but the pull-backs of the coordinate functions, $\varphi^\alpha(x_1,\ldots,x_m)=\varphi^*(y^\alpha)(x_1,\ldots,x_m)=y^\alpha(\varphi(x_1,\ldots,x_m))$. □

Example 4.1 [The standard coordinates in both Nil and Heisenberg spaces are ∞-harmonic coordinates] Nil space can be viewed as $(\mathbb{R}^3,g_{Nil})$, where the metric with respect to the standard coordinates (x,y,z) in $\mathbb{R}^3$ takes the form $g_{Nil}=\mathrm{d}x^2+\mathrm{d}y^2+(\mathrm{d}z-x\mathrm{d}y)^2$. An easy computation gives the following components of Nil metric:

$$g_{11}=1,\ g_{12}=g_{13}=0,\ g_{22}=1+x^2,\ g_{23}=-x,\ g_{33}=1;$$
$$g^{11}=1,\ g^{12}=g^{13}=0,\ g^{22}=1,\ g^{23}=x,\ g^{33}=1+x^2.$$

Heisenberg Space is a 3-dimensional homogeneous space, denoted by $\mathbb{H}_3=(\mathbb{R}^3,g)$, endowed with a left invariant metric whose group of isometries has dimension 4.

With respect to the standard coordinates (x, y, z) in $\mathbb{R}^3$, the metric can be written as $g = \mathrm{d}x^2 + \mathrm{d}y^2 + (\mathrm{d}z + \frac{y}{2}\mathrm{d}x - \frac{x}{2}\mathrm{d}y)^2$ with components given by:

$$g_{11} = 1 + \frac{y^2}{4}, g_{12} = -\frac{xy}{4}, g_{13} = \frac{y}{2}, g_{22} = 1 + \frac{x^2}{4}, g_{23} = -\frac{x}{2},\ g_{33} = 1; \tag{4.1}$$

$$g^{11} = 1, g^{12} = 0, g^{13} = -\frac{y}{2}, g^{22} = 1,\ g^{23} = \frac{x}{2}, g^{33} = 1 + \frac{x^2 + y^2}{4}. \tag{4.2}$$

It follows from Lemma 3.1 that the standard coordinates (x, y, z) in each of these model spaces are ∞-harmonic coordinates.

Note that there is an isometry between Nil and Heisenberg Spaces. We can check that $\sigma : (\mathbb{H}_3, g) \longrightarrow (\mathbb{R}^3, g_{Nil})$ with $\sigma(X, Y, Z) = (X, Y, Z + XY/2)$ is an isometry from Heisenberg space onto Nil space. By Proposition 4.1, all the component functions of this isometry and its inverse are ∞-harmonic functions.

Example 4.2 We know (see e.g., [4]) that $F : (R^4 \setminus \{0\} \longrightarrow R^3$ given by $F(x_1, x_2, x_3, x_4) = (x_1^2 + x_2^2 - x_3^2 - x_4^2,\ 2x_1x_3 - 2x_2x_4,\ 2x_1x_4 + 2x_2x_3)$ is a horizontally conformal submersion with dilation $\lambda = 2|X|$. We can perform a conformal change on the domain metric turning the map $F : (R^4 \setminus \{0\}, 4\,|X|^2\,\delta_{ij}) \longrightarrow R^3$ into a Riemannian submersion. It follows from Proposition 4.1 that all the component functions of F are ∞-harmonic functions in conformally flat space $(R^4 \setminus \{0\}, 4\,|X|^2\,\delta_{ij})$.

Theorem 4.2 (1) *All component functions*

$$\begin{cases} x_1(\rho, \phi_1, \ldots, \phi_{m-1}) = \rho \cos \phi_1 \\ x_2(\rho, \phi_1, \ldots, \phi_{m-1}) = \rho \sin \phi_1 \cos \phi_2 \\ x_3(\rho, \phi_1, \ldots, \phi_{m-1}) = \rho \sin \phi_1 \sin \phi_2 \cos \phi_3 \\ \ldots \\ x_{m-1}(\rho, \phi_1, \ldots, \phi_{m-1}) = \rho \sin \phi_1 \sin \phi_2 \ldots \sin \phi_{m-2} \cos \phi_{m-1} \\ x_m(\rho, \phi_1, \ldots, \phi_{m-1}) = \rho \sin \phi_1 \sin \phi_2 \ldots \sin \phi_{m-2} \sin \phi_{m-1} \end{cases} \tag{4.3}$$

of the transformation from hyperspherical into Cartesian coordinates are solutions of the ∞-Laplace equation in Euclidean space $\mathbb{R}^m$;

(2) *All component functions*

$$\begin{cases} \rho(x_1, \ldots, x_m) = \sqrt{x_m^2 + x_{m-1}^2 + \ldots + x_1^2}, \\ \phi_1(x_1, \ldots, x_m) = \arctan \frac{\sqrt{x_m^2 + x_{m-1}^2 + \ldots + x_2^2}}{x_1}, \\ \phi_2(x_1, \ldots, x_m) = \arctan \frac{\sqrt{x_m^2 + x_{m-1}^2 + \ldots + x_3^2}}{x_2} \\ \ldots \\ \phi_{m-2}(x_1, \ldots, x_m) = \arctan \frac{\sqrt{x_m^2 + x_{m-1}^2}}{x_{m-2}} \\ \phi_{m-1}(x_1, \ldots, x_m) = \arctan \frac{x_m}{x_{m-1}} \end{cases} \tag{4.4}$$

of the transformation from Cartesian into hyperspherical coordinates are solutions of the ∞-Laplace equation in Euclidean space $\mathbb{R}^m$;

(3) *Any linear combination of functions from the family* (4.3) *is a solution of the ∞-Laplace equation in Euclidean space $\mathbb{R}^m$.*

(4) *Any function from the family* (4.4) *is also a solution of the* ∞*-harmonic equation in hyperbolic space* $(H^{m+1} = \mathbb{R}^m \times \mathbb{R}^+, x_{m+1}^{-2} \sum_{i=1}^{m+1} \mathrm{d}x_i^2)$.

Proof. Since the transformation and the inverse transformation are isometries and hence they are ∞-harmonic morphisms. As we have proved in Theorem 3.3 that the hyperspherical coordinates and the Cartesian coordinates are both ∞-harmonic coordinates we obtain statements (1) and (2) by using Proposition 4.1. Statement (3) follows from Theorem 2.1 and the fact that any linear function $f(x_1, \ldots, x_m) = c_1x_1 + \ldots + c_m x_m$ in $\mathbb{R}^m$ is an ∞-harmonic function and the pull-back of f by φ is precisely a linear combination of the functions from family (4.3). Finally, we use statement (2) and Corollary 2.2 to obtain statement (4). $\square$

Remark 4.1 1. We would like to point out that in the above argument, or in Proposition 4.1 in general, the coordinates in the target manifold being ∞-harmonic is essential. One can easily find examples to show that even for an isometry, if the coordinates in the target space are not ∞-harmonic then the component functions need not be infinity-harmonic. The inverse transformation of the isometry given in (3.5) is one of such examples.

2. The statement 3 was a little surprised to us as the ∞-Laplace equation is nonlinear so a linear combinations of solutions need not be a solution again. As an easy example, we know that the polar coordinate function r, θ are ∞-harmonic functions, however, their sum $u(r, \theta) = r + \theta$ is not infinity-harmonic.

In the next theorem we will use the notion of the directional angles to construct new families of smooth solutions of the ∞-Laplace equation in Euclidean space.

Theorem 4.3 *The* m *families of two parameters of smooth functions* $\phi^i(x_1, \ldots, x_m) = C_1 \arccos \frac{x_i}{\sqrt{x_1^2+\ldots+x_m^2}} + C_2 : \mathbb{R}^m \setminus \{0\} \longrightarrow \mathbb{R}$, *where* $\arccos \frac{x_i}{\sqrt{x_1^2+\ldots+x_m^2}}$, $i = 1, 2 \ldots, m$ *is the directional angle of the position vector* x *in* $\mathbb{R}^m$, *are all* ∞*-harmonic functions on the whole Euclidean space with only one point deleted.*

Proof. We will show that the family of functions $\phi(x_1, \ldots, x_m) = C_1 \arccos \frac{x_m}{\sqrt{x_1^2+\ldots+x_m^2}} + C_2$ are ∞-harmonic and the proof for the other families follows in a similar fashion. By the nature of the infinity-Laplace equation it is enough to check that the function $\phi(x_1, \ldots, x_m) = \arccos \frac{x_m}{\sqrt{x_1^2+\ldots+x_m^2}}$ is ∞-harmonic. To this end, we compute $\nabla\phi = (\frac{\partial\phi}{\partial x_1}, \ldots, \frac{\partial\phi}{\partial x_m})$, where $\frac{\partial\phi}{\partial x_i} = \frac{x_i\, x_m}{\rho^2\sqrt{\Sigma_{j\neq m} x_j^2}}, i = 1, 2, \ldots, (m-1)$ and $\frac{\partial\phi}{\partial x_m} = -\frac{\Sigma_{j\neq m} x_j^2}{\rho^2\sqrt{\Sigma_{j\neq m} x_j^2}}$ with $\rho^2 = \Sigma_{j=1}^m\, x_j^2$.

A further computation yields $|\nabla\phi|^2 = \frac{1}{\rho^2}$, and

$$\nabla|\nabla\phi|^2 = \left(\frac{-2x_1}{\rho^4}, \ldots, \frac{-2x_{m-1}}{\rho^4}, \frac{-2x_m}{\rho^4}\right).$$

It follows that $\Delta_\infty\phi = \frac{1}{2}\langle\nabla\phi, \nabla|\nabla\phi|^2\rangle = \frac{1}{2}\, \frac{-2x_m \Sigma_{j\neq m}\, x_j^2 + 2x_m \Sigma_{j\neq m}\, x_j^2}{\rho^6\sqrt{\Sigma_{j\neq m}\, x_j^2}} = 0$, which shows that ϕ is indeed an ∞-harmonic function. $\square$

Remark 4.2 One can also check that the families $\phi^i(x_1, \ldots, x_m) = C_1 \arcsin \frac{x_i}{\sqrt{x_1^2+\ldots+x_m^2}} + C_2 : \mathbb{R}^m \setminus \{0\} \longrightarrow \mathbb{R}$ are also ∞-harmonic functions. However, it follows from an exercise of Calculus that $\arcsin x = \frac{\pi}{2} + \arccos x$ and hence they have been included in the families in Theorem 4.3.

By using Corollary 2.2 we have

Corollary 4.4 *The m families of two parameters of smooth functions $\phi^i(x_1, \ldots, x_m) = C_1 \arccos \frac{x_i}{\sqrt{x_1^2+\ldots+x_m^2}} + C_2 : \mathbb{R}^m \setminus \{0\} \longrightarrow \mathbb{R}$, where $\arccos \frac{x_i}{\sqrt{x_1^2+\ldots+x_m^2}}$, $i = 1, 2\ldots, m$ is the directional angle of the position vector x in $\mathbb{R}^m$, are all ∞-harmonic functions on the hyperbolic space $(H^{m+1} = \mathbb{R}^m \times \mathbb{R}^+,\ x_{m+1}^{-2}\sum_{i=1}^{m+1} \mathrm{d}x_i^2)$.*

In what follows we shall determine all rotationally symmetric solutions of the ∞-Laplace equation in space forms.

Theorem 4.5 *Let $u : \mathbb{R}^m \setminus \{0\} \longrightarrow \mathbb{R}$ be a rotationally symmetric function with $u = u(r)$, where $r^2 = x_1^2 + x_2^2 + \ldots + x_m^2$. Then*

$$\Delta_\infty u = \frac{\partial^2 u}{\partial r^2}(\frac{\partial u}{\partial r})^2, \tag{4.5}$$

and hence $u(r)$ is an ∞-harmonic function if and only if $u(r) = c_1 r + c_2, or, u(x) = c_1\sqrt{x_1^2 + \ldots + x_m^2} + c_2$, where c_1 and c_2 are constants.

Proof. A straightforward computation gives

$$\frac{\partial u}{\partial x_i} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x_i} = \frac{\partial u}{\partial r}\frac{x_i}{r}, \tag{4.6}$$

and

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\partial u}{\partial r}\frac{x_i}{r}\right) = \frac{\partial^2 u}{\partial r^2}\frac{x_i x_j}{r^2} + \left(\frac{1}{r}\delta_{ij} - \frac{x_i x_j}{r^3}\right)\frac{\partial u}{\partial r}. \tag{4.7}$$

Using equations (4.6) and (4.7) we have

$$\begin{aligned}
\Delta_\infty u &= \sum_{i,j=1}^m \frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} \\
&= \sum_{i,j=1}^m \left(\frac{\partial^2 u}{\partial r^2}\frac{x_i x_j}{r^2} + \frac{1}{r}\delta_{ij}\frac{\partial u}{\partial r} - \frac{x_i x_j}{r^3}\frac{\partial u}{\partial r}\right)\left(\frac{\partial u}{\partial r}\right)^2 \frac{x_i x_j}{r^2} \qquad (4.8) \\
&= \frac{\partial^2 u}{\partial r^2}\left(\frac{\partial u}{\partial r}\right)^2 \frac{\sum_{i,j=1}^m x_i^2 x_j^2}{r^4} + \left(\frac{\partial u}{\partial r}\right)^3 \frac{\sum_{i,j=1}^m \delta_{ij} x_i x_j}{r^3} \\
&- \left(\frac{\partial u}{\partial r}\right)^3 \frac{\sum_{i,j=1}^m x_i^2 x_j^2}{r^5} = \frac{\partial^2 u}{\partial r^2}\left(\frac{\partial u}{\partial r}\right)^2,
\end{aligned}$$

where in obtaining the last equality we have used the fact that $\sum_{i,j=1}^m \delta_{ij} x_i x_j = r^2$ and $\sum_{i,j=1}^m x_i^2 x_j^2 = r^4$. The proposition follows from equation (4.8). □

Theorem 4.6 (Rotationally symmetric ∞-harmonic functions on spheres) *Let (S^m, g_{can}) be the m-dimensional sphere with the standard metric. Identify $(S^m \setminus \{N\}, g_{can})$ with $(\mathbb{R}^m, F^{-2}\delta_{ij})$, where $F = 2^{-1}(1+|x|^2)$. Then, all rotationally symmetric ∞-functions on $S^m \setminus \{\pm N\} \equiv \mathbb{R}^m \setminus \{0\}$ are given by $u(x_1, ..., x_m) = c_1 \arctan\left(\sqrt{x_1^2 + \ldots + x_m^2}\,\right) + c_2$, where c_1 and c_2 are constants.*

Proof. It follows from [33] that the ∞-Laplace equation on sphere S^m can be written as

$$\Delta_\infty^{\mathbb{R}^m} u + \frac{2\,|\nabla u|^2}{1+|x|^2}\langle x, \nabla u\rangle = 0, \quad x \in \mathbb{R}^m, \tag{4.9}$$

where $|,|$ and ∇ denote the norm and the gradient defined by the standard metric $\langle,\rangle$ on Euclidean space $\mathbb{R}^m$. When $u = u(r)$ is rotationally symmetric we have

$$|\nabla u|^2 = (\frac{\partial u}{\partial r})^2, \tag{4.10}$$

$$\langle x, \nabla u\rangle = \sum_{i=1}^m x_i \frac{\partial u}{\partial x_i} = \frac{\partial u}{\partial r}\frac{\sum_{i=1}^m x_i^2}{r} = r\frac{\partial u}{\partial r}. \tag{4.11}$$

Substituting equations (4.5), (4.10) and (4.11) into equation (4.9) we have $\frac{\partial^2 u}{\partial r^2}(\frac{\partial u}{\partial r})^2 + \frac{2r}{1+r^2}(\frac{\partial u}{\partial r})^3 = 0$, which is equivalent to

$$\frac{\partial^2 u}{\partial r^2} + \frac{2r}{1+r^2}\frac{\partial u}{\partial r} = 0. \tag{4.12}$$

Solving this equation we have $u(r) = c_1 \arctan r + c_2$, from which the theorem follows. $\square$

Theorem 4.7 (Rotationally symmetric ∞-harmonic functions on hyperbolic spaces) *Let (B^m, g^H) be the m-dimensional hyperbolic space with open-ball model, where $B^m = \{x \in \mathbb{R}^m : |x| < 1\}$ and $g^H = F^{-2}\delta_{ij}$ with $F = 2^{-1}(1 - |x|^2)$. Then, all rotationally symmetric ∞-functions on $B^m \setminus \{0\}$ are given by $u(x_1, \ldots, x_m) = c_1 \ln \frac{1+\sqrt{x_1^2+\ldots+x_m^2}}{1-\sqrt{x_1^2+\ldots+x_m^2}} + c_2$, where c_1 and c_2 are constants.*

Proof. It follows from [33] that the ∞-Laplace equation on hyperbolic space B^m can be written as

$$\Delta_\infty^{\mathbb{R}^m} u - \frac{2\,|\nabla u|^2}{1-|x|^2}\langle x, \nabla u\rangle = 0, \quad x \in B^m. \tag{4.13}$$

where $|,|$ and ∇ denote the norm and the gradient defined by the Euclidean metric $\langle,\rangle$ on $B^m \subset \mathbb{R}^m$. When $u = u(r)$ is rotationally symmetric, we substitute equations (4.5), (4.10) and (4.11) into equation (4.13) to have $\frac{\partial^2 u}{\partial r^2}\left(\frac{\partial u}{\partial r}\right)^2 - \frac{2r}{1-r^2}\left(\frac{\partial u}{\partial r}\right)^3 = 0$, which is equivalent to

$$\frac{\partial^2 u}{\partial r^2} - \frac{2r}{1-r^2}\frac{\partial u}{\partial r} = 0. \tag{4.14}$$

Solving this equation we have $u(r) = c_1 \ln\frac{1+r}{1-r} + c_2$, from which the theorem follows. $\square$

5 More on ∞-Laplace equations on Riemannian manifolds

In this final section we derive the ∞-Laplace equation in a Riemannian manifold with respect to local coordinates and with respect to a local orthonormal frame. We also give a "unified" form of the ∞-Laplace equation on a surface with respect to a geodesic polar coordinates.

Lemma 5.1 (1) *In local coordinate $\{x^i\}$ in (M, g), we have*

$$\Delta_\infty u := g^{ij} g^{kl} u_{kj} u_i u_l - g^{ij} g^{kl} \Gamma^t_{jk} u_t u_i u_l. \tag{5.1}$$

(2) *With respect to a local orthonormal base $\{e_i\}$ in (M, g) we have*

$$\Delta_\infty u := \sum_{i,k=1}^{m} (e_i e_k u)(e_i u)(e_k u). \tag{5.2}$$

Proof. For statement (1), a straightforward computation yields

$$\begin{aligned}
\Delta_\infty u :&= \frac{1}{2} g(\operatorname{grad} u, \operatorname{grad} |\operatorname{grad} u|^2) = \frac{1}{2} g^{ij} \nabla_i u \nabla_j (g^{kl} \nabla_k u \nabla_l u) \\
&= \frac{1}{2} g^{ij} g^{kl} \nabla_i u \nabla_j (\nabla_k u \nabla_l u) = \frac{1}{2} g^{ij} g^{kl} \nabla_i u (2 \nabla j \nabla_k u \nabla_l u) \\
&= g^{ij} g^{kl} (\nabla_j \nabla_k u) \nabla_i u \nabla_l u,
\end{aligned} \tag{5.3}$$

where $\nabla_i u = \nabla_{\frac{\partial}{\partial x^i}} u = u_i$. Substituting $\nabla_j \nabla_k u = u_{kj} - \Gamma^t_{kj} u_t$ into (5.3) we get (5.1).

For statement (2), we note that with respect to an orthonormal frame $\{e_i\}$ in (M, g) we have

$$\begin{aligned}
\Delta_\infty u :&= \frac{1}{2} g(\operatorname{grad} u, \operatorname{grad} |\operatorname{grad} u|^2) = \frac{1}{2} g\left(\sum_i (e_i |\nabla u|^2) e_i, \ \sum_j (e_j u) e_j \right) \\
&= \frac{1}{2} \sum_i (e_i |\nabla u|^2)(e_i u).
\end{aligned} \tag{5.4}$$

Substituting $e_i |\nabla u|^2 = e_i \sum_k (e_k u)^2 = 2 \sum_k (e_k u) e_i e_k u$ into (5.4) we get (5.2). $\square$

The ∞-Laplacian on surfaces. Let Σ be a surface with geodesic polar coordinates (r, θ). Then, the metric takes the form $g = \mathrm{d}r^2 + f(r)^2 \mathrm{d}\theta^2$. We can chose $e_1 = \frac{\partial}{\partial r}$, $e_2 = \frac{1}{f} \frac{\partial}{\partial \theta}$ as an orthonormal frame. Then we have

$$e_1 u = u_r, \ e_2 u = \frac{1}{f} u_\theta, \ e_1 e_1 u = u_{rr}, \ e_2 e_1 u = \frac{1}{f} u_{r\theta}, \tag{5.5}$$

$$e_1 e_2 u = -\frac{f'}{f^2} u_\theta + \frac{1}{f} u_{r\theta}, \ e_2 e_2 u = \frac{1}{f^2} u_{\theta\theta}, \tag{5.6}$$

from which we obtain the ∞-Laplace operator in $\mathbb{R}^2$ in polar coordinates

$$\Delta_\infty u := u_{rr} u_r^2 + \frac{2}{f^2} u_{r\theta} u_r u_\theta + \frac{1}{f^4} u_{\theta\theta} u_\theta^2 - \frac{f'}{f^3} u_r u_\theta^2. \tag{5.7}$$

It follows that the only rotationally symmetric solutions $u(r,\theta)=u(r)$ of ∞-Laplace equations in surfaces are of the form $u(r)=C_1r+C_0$.

For 2-dimensional space forms, the ∞-Laplace operators take the following forms with respect to the geodesic polar coordinates.

1. In Euclidean space $\mathbb{R}^2$, $g=\mathrm{d}r^2+r^2\mathrm{d}\theta^2$, we apply equation (5.7) with $f(r)=r$ to have

$$\Delta_\infty u := u_{rr}u_r^2+\frac{2}{r^2}u_{r\theta}u_ru_\theta+\frac{1}{r^4}u_{\theta\theta}u_\theta^2-\frac{1}{r^3}u_ru_\theta^2. \tag{5.8}$$

2. In the sphere S^2, $g=\mathrm{d}r^2+\sin^2 r\,\mathrm{d}\theta^2$, we apply equation (5.7) with $f(r)=\sin r$ to have

$$\Delta_\infty u := u_{rr}u_r^2+2\csc^2 r\,u_{r\theta}u_ru_\theta+\csc^4 r\,u_{\theta\theta}u_\theta^2-\cot r\csc^2 r\,u_ru_\theta^2. \tag{5.9}$$

3. In hyperbolic space H^2, $g=\mathrm{d}r^2+\sinh^2 r\,\mathrm{d}\theta^2$, we apply equation (5.7) with $f(r)=\sinh r$ to have

$$\Delta_\infty u := u_{rr}u_r^2+2\mathrm{csch}^2 r\,u_{r\theta}u_ru_\theta+\mathrm{csch}^4 r\,u_{\theta\theta}u_\theta^2-\coth r\,\mathrm{csch}^2 r\,u_ru_\theta^2. \tag{5.10}$$

A straightforward checking shows that both coordinates transformation functions $x=r\cos\theta,\ y=r\sin\theta$ satisfy equation (5.7) and hence they are ∞-harmonic functions.

Corollary 5.2 *Let (r,θ) be the polar coordinates in $\mathbb{R}^2$, then $u(r,\theta)=r\alpha(\theta)$ is an ∞-harmonic function if and only if $\alpha=C$ or $\alpha(\theta)=a\cos\theta+b\sin\theta$.*

Proof. A straightforward computation shows that if $u(r,\theta)=r\alpha(\theta)$, then $\Delta_\infty u=(\alpha+\alpha'')\alpha'^2$. Solving the equation $(\alpha+\alpha'')\alpha'^2=0$ we obtain the corollary. □

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