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Some generalizations of Prešić type mappings and applications

Satish Shukla · Stojan Radenović

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Abstract In this paper, we prove some common fixed point theorems for the mappings satisfying Prešić type contractive conditions in metric spaces. Our results generalize and extend the result of Prešić for some new type of contractive conditions. The common fixed point of mappings is approximated by a k-step iterative sequence. Some examples are provided to illustrate the results. An application of Prešić type mappings to second order difference equations is also given.

Keywords Prešić type mapping · common fixed point · difference equation · equilibrium point

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1 Introduction and preliminaries

Let (X, d) be any metric space and $f\colon X \to X$ be any mapping, then f is said to be a contraction on X if there exists $\lambda \in [0, 1)$ such that $d(fx, fy) \leq \lambda d(x, y)$, for all $x, y \in X$. A point $x \in X$ is called fixed point of f if $fx = x$. Banach [1] proved that every contraction on a complete metric space has a unique fixed point and this result is known as the Banach contraction principle. There are several generalizations of this famous principle. One such generalization is given by Prešić [15,16]. When studying the convergence of some particular sequences, Prešić [15,16] proved the following theorem.

Satish Shukla
Department of Applied Mathematics,
Shri Vaishnav Institute of Technology & Science,
Gram Baroli, Sanwer Road, Indore (M.P.) 453331, India
E-mail: satishmathematics@yahoo.co.in

Stojan Radenović
Faculty of Mechanical Engineering,
University of Belgrade,
Kraljice Marije 16, 11120 Beograd, Serbia
E-mail: radens@beotel.net

Theorem 1.1 *Let (X, d) be a complete metric space, k a positive integer and $f\colon X^k \to X$ a mapping satisfying the following contractive type condition:*

$$d(f(x_1, x_2, \ldots, x_k), f(x_2, x_3, \ldots, x_{k+1})) \leq \sum_{i=1}^{k} q_i d(x_i, x_{i+1}), \tag{1.1}$$

for every $x_1, x_2, \ldots, x_{k+1} \in X$, where $q_1, q_2, \ldots, q_k$ are nonnegative constants such that $q_1+q_2+\cdots+q_k < 1$. Then there exists a unique point $x \in X$ such that $f(x, x, \ldots, x) = x$. Moreover, if $x_1, x_2, \ldots, x_k$ are arbitrary points in X and for $n \in \mathbb{N}$,

$$x_{n+k} = f(x_n, x_{n+1}, \ldots, x_{n+k-1}), \tag{1.2}$$

then the sequence $\{x_n\}$ is convergent and $\lim x_n = f(\lim x_n, \lim x_n, ..., \lim x_n)$.

Note that, the k-step iterative sequence given by (1.2) represent a nonlinear difference equation. In view of Prešić's theorem it is obvious that if this sequence is convergent (which is ensured by the Prešić's theorem) then the limit of sequence is a fixed point of f. The result of Prešić is generalized by several authors and some generalizations and applications of Prešić's theorem can be seen in [3–5, 7, 10–13, 17–27].

In this paper, we prove some common fixed point theorems for mappings satisfying Prešić type contractive conditions in complete metric spaces. The common fixed point of such mappings is approximated by a k-step iterative sequence. Results of this paper extend and generalize the results of Prešić, Păcurar [14] and the recent result of Khan et al. [7]. Some examples are given to illustrate the results. An application of Prešić type mappings to nonlinear difference equations is given.

For the following definition we refer to [5] and the references therein.

Definition 1.2 Let X be a nonempty set, k a positive integer and $f : X^k \to X$ be a mapping. If $f(x, \ldots, x) = x$, then $x \in X$ is called a fixed point of f.

Definition 1.3 Let X be a nonempty set, k a positive integer, $f : X^k \to X$ and $g : X \to X$ be mappings.

(a) An element $x \in X$ said to be a coincidence point of f and g if $gx = f(x, \ldots, x)$.
(b) If $w = gx = f(x, \ldots, x)$, then w is called a point of coincidence of f and g.
(c) If $x = gx = f(x, \ldots, x)$, then x is called a common fixed point of f and g.
(d) Mappings f and g are said to be commuting if $g(f(x, ..., x)) = f(gx, ..., gx)$, for all $x \in X$.
(e) Mappings f and g are said to be weakly compatible if they commute at their coincidence points.

Lemma 1.4 ([11]) *Let X be a nonempty set, k a positive integer and $f : X^k \to X$, $g : X \to X$ two weakly compatible mappings. If f and g have a unique point of coincidence $y = f(x, x, \ldots, x) = g(x)$, then y is the unique common fixed point of f and g.*

Remark 1.1 The function $F\colon X \to X$ defined by $F(x) = f(x, x, \ldots, x)$, for all $x \in X$ is called the associate operator of f. Note that, $x \in X$ is a fixed point of f if and only if it is a fixed point of F. Similarly, $x \in X$ is a common fixed point of f and g if and only if it is a common fixed point of F and g.

In order to prove our main results we shall need the following lemmas which can be seen in [2,15].

Lemma 1.5 *Let k be a positive integer and $\alpha_1, \alpha_2, \ldots, \alpha_k \geq 0$ are such that $\sum_{i=1}^{k} \alpha_i < 1$. If $\{\triangle_n\}$ is a sequence of positive numbers satisfying $\triangle_{n+k} \leq \alpha_1\triangle_n + \alpha_2\triangle_{n+1} + \cdots + \alpha_k\triangle_{n+k-1}$, $n \geq 1$, then there exists $L \geq 0$ and $\tau \in (0,1)$ such that $\triangle_n \leq L\tau^n$, for all $n \geq 1$.*

Lemma 1.6 *Let $\{a_n\}, \{b_n\}$ be two sequences of positive real numbers and $q \in [0,1)$ such that $a_{n+1} \leq qa_n + b_n, n \geq 0$ and $b_n \to 0$ as $n \to \infty$. Then $a_n \to 0$ as $n \to \infty$.*

Let Θ be the set of functions $\theta\colon [0,\infty)^4 \to [0,\infty)$ satisfying the following conditions:

(i) θ is continuous;
(ii) for all $a, b, c, d \in [0,\infty)$, $\theta(a,b,c,d) = 0$ if and only if $abcd = 0$.

For examples of function θ we refer to [7].

Now we can state our main results.

2 Main results

Following theorem extends Theorem 2.1 of [7].

Theorem 2.1 *Let (X,d) be a complete metric space, k a positive integer and $f\colon X^k \to X$, $g\colon X \to X$ be two mappings such that $f(X^k) \subset g(X)$ and $g(X)$ is closed. Suppose following condition is satisfied:*

$$d(f(x_0,x_1,\ldots,x_{k-1}), f(x_1,x_2,\ldots,x_k)) \leq \sum_{i=1}^{k} \alpha_i d(gx_{i-1}, gx_i) + \beta \sum_{i=0}^{k}\sum_{j=0}^{k} d(gx_i, Fx_j) + \theta(d(gx_0,Fx_0), d(gx_k,Fx_k), d(gx_0,Fx_k), d(gx_k,Fx_0)), \quad \text{(A)}$$

for all $x_0, x_1, \ldots, x_k \in X$, where α_i, β are nonnegative constants such that $\sum_{i=1}^{k} \alpha_i + \beta k(k+1)^2 < 1$, F is the associate operator of f and $\theta \in \Theta$. Then f and g have a unique point of coincidence. In addition, if f and g are weakly compatible then f and g have a unique common fixed point. Moreover, for any $x_0 \in X$ the sequence $\{y_n\}$ defined by $y_n = gx_n = f(x_{n-1}\ldots, x_{n-1})$, for all $n \in \mathbb{N}$, converges to the common fixed point of f and g.

Proof. Let $x_0 \in X$ be arbitrary. As, $f(X^k) \subset g(X)$, we define a sequence $\{y_n\}$ by $y_n = gx_n = f(x_{n-1}, \ldots, x_{n-1}) = Fx_{n-1}$, for all $n \in \mathbb{N}$, where F is the associate operator for f. We shall show that F will satisfy a particular contractive condition.

Note that, for all $x, y \in X$,

$$\begin{aligned} d(Fx,Fy) &= d(f(x,\ldots,x), f(y,\ldots,y)) \\ &\leq d(f(x,\ldots,x), f(x,\ldots,x,y)) + d(f(x,\ldots,x,y), f(x,\ldots,x,y,y)) \\ &\quad + \cdots + d(f(x,y,\ldots,y), f(y,\ldots,y)). \end{aligned}$$

Using (A) we obtain

$$\begin{aligned} d(Fx,Fy) \le {} & \{\alpha_k d(gx,gy) + k^2\beta d(gx,Fx) + k\beta d(gx,Fy) + k\beta d(gy,Fx) \\ & + \beta d(gy,Fy)\} + \{\alpha_{k-1} d(gx,gy) + (k-1)^2\beta d(gx,Fx) \\ & + 2(k-1)\beta d(gx,Fy) + 2(k-1)\beta d(gy,Fx) + 2^2\beta d(gy,Fy)\} \\ & + \cdots + \{\alpha_1 d(gx,gy) + \beta d(gx,Fx) + (k-1)\beta d(gx,Fy) \\ & + (k-1)\beta d(gy,Fx) + k^2\beta d(gy,Fy)\} \\ & + k\theta(d(gx,Fx), d(gy,Fy), d(gx,Fy), d(gy,Fx)) \end{aligned}$$

that is

$$\begin{aligned} d(Fx,Fy) &\le \sum_{i=1}^{k} \alpha_i d(gx,gy) + \beta[1 + 2^2 + \cdots + k^2][d(gx,Fx) + d(gy,Fy)] \\ &\quad + \beta[k + 2(k-1) + \cdots + 2(k-1) + k][d(gx,Fy) + d(gy,Fx)] \\ &\quad + k\theta(d(gx,Fx), d(gy,Fy), d(gx,Fy), d(gy,Fx)) \\ &= \sum_{i=1}^{k} \alpha_i d(gx,gy) + \beta\frac{k(k+1)(2k+1)}{6}[d(gx,Fx) + d(gy,Fy)] \\ &\quad + \beta\frac{k(k+1)(k+2)}{6}[d(gx,Fy) + d(gy,Fx)] \\ &\quad + k\theta(d(gx,Fx), d(gy,Fy), d(gx,Fy), d(gy,Fx)) \end{aligned}$$

or

$$\begin{aligned} d(Fx,Fy) \le {} & Ad(gx,gy) + B[d(gx,Fx) + d(gy,Fy)] \\ & + C[d(gx,Fy) + d(gy,Fx)] + M(x,y), \end{aligned} \tag{2.1}$$

where $A = \sum_{i=1}^{k} \alpha_i$, $B = \beta\frac{k(k+1)(2k+1)}{6}$ and $C = \beta\frac{k(k+1)(k+2)}{6}$ and $M(x,y) = k\theta(d(gx,Fx), d(gy,Fy), d(gx,Fy), d(gy,Fx))$.

Note that

$$\begin{aligned} A + 2B + 2C &= \sum_{i=1}^{k} \alpha_i + 2\beta\frac{k(k+1)(2k+1)}{6} + 2\beta\frac{k(k+1)(k+2)}{6} \\ &= \sum_{i=1}^{k} \alpha_i + 2\beta\frac{k(k+1)(3k+3)}{6} = \sum_{i=1}^{k} \alpha_i + \beta k(k+1)^2 < 1. \end{aligned}$$

Now, as $y_n = gx_n = f(x_{n-1}\ldots, x_{n-1})$, for all $n \in \mathbb{N}$, therefore for any $n \in \mathbb{N}$, it follows from (2.1) that

$$\begin{aligned} d(y_{n+1}, y_n) = d(Fx_n, Fx_{n-1}) &\le Ad(gx_n, gx_{n-1}) \\ &\quad + B[d(gx_n, Fx_n) + d(gx_{n-1}, Fx_{n-1})] \\ &\quad + C[d(gx_n, Fx_{n-1}) + d(gx_{n-1}, Fx_n)] + M(x_n, x_{n-1}) \\ &= Ad(y_n, y_{n-1}) + B[d(y_n, y_{n+1}) + d(y_{n-1}, y_n)] + C[d(y_n, y_n) \\ &\quad + d(y_{n-1}, y_{n+1})] + M(x_n, x_{n-1}) \\ &\le Ad(y_n, y_{n-1}) + B[d(y_n, y_{n+1}) + d(y_{n-1}, y_n)] + C[d(y_{n-1}, y_n) \end{aligned}$$

$$+ d(y_n, y_{n+1})] + M(x_n, x_{n-1}),$$

where $M(x_n, x_{n-1}) = \theta(d(gx_n, Fx_n), d(gx_{n-1}, Fx_{n-1}), d(gx_n, Fx_{n-1}), d(gx_{n-1}, Fx_n)) = 0$. Therefore

$$d(y_{n+1}, y_n) \leq \frac{A+B+C}{1-B-C} d(y_n, y_{n-1}),$$
$$d(y_{n+1}, y_n) \leq \lambda d(y_n, y_{n-1}) \leq \lambda^n d(y_1, y_0),$$

where $\lambda = \frac{A+B+C}{1-B-C} \in [0,1)$ (as $0 \leq A + 2B + 2C < 1$). Therefore, for $m, n \in \mathbb{N}$ with $m > n$ we obtain

$$\begin{aligned} d(y_n, y_m) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{m-1}, y_m) \\ &\leq \lambda^n d(y_1, y_0) + \lambda^{n+1} d(y_1, y_0) + \cdots + \lambda^{m-1} d(y_1, y_0) \\ &\leq \frac{\lambda^n}{1-\lambda} d(y_1, y_0). \end{aligned}$$

As, $\lambda \in [0,1)$, we obtain $\{y_n\} = \{gx_n\}$ is a Cauchy sequence, and as $g(X)$ is closed, there exists $u, v \in X$ such that $v = gu$ and

$$\lim_{n\to\infty} d(y_n, v) = \lim_{n\to\infty} d(gx_n, gu) = 0. \tag{2.2}$$

We shall prove that $Fu = gu$. It follows from (2.1) that

$$\begin{aligned} d(gu, Fu) &\leq d(gu, y_{n+1}) + d(y_{n+1}, Fu) = d(y_{n+1}, v) + d(Fx_n, Fu) \\ &\leq d(y_{n+1}, v) + Ad(gx_n, gu) + B[d(gx_n, Fx_n) + d(gu, Fu)] \\ &+ C[d(gx_n, Fu) + d(gu, Fx_n)] + M(x_n, u) \\ &= d(y_{n+1}, v) + Ad(y_n, v) + B[d(y_n, y_{n+1}) + d(v, Fu)] \\ &+ C[d(y_n, v) + d(v, Fu) + d(v, y_{n+1})] + M(x_n, u). \end{aligned} \tag{2.3}$$

Now

$$\begin{aligned} M(x_n, u) &= \theta(d(gx_n, Fx_n), d(gu, Fu), d(gx_n, Fu), d(gu, Fx_n)) \\ &= \theta(d(y_n, y_{n+1}), d(v, Fu), d(y_n, Fu), d(v, y_{n+1})). \end{aligned}$$

Using (2.2) and properties of function θ we obtain $M(x_n, u) = 0$.

Therefore, it follows from (2.2) and (2.3) that

$$d(v, Fu) \leq (B + C) d(v, Fu).$$

As, $B + C < 1$, we obtain $Fu = f(u, \ldots, u) = gu = v$. Thus, v is point of coincidence of f and g. If, v' is another point of coincidence of f and g then there exists $u' \in X$ such that $Fu' = f(u', \ldots, u') = gu' = v'$. By (2.1) we obtain

$$\begin{aligned} d(v, v') = d(Fu, Fu') &\leq Ad(gu, gu') + B[d(gu, Fu) + d(gu', Fu')] \\ &+ C[d(gu, Fu') + d(gu', Fu)] + M(u, u') \\ &= (A+2C)d(v, v') + \theta(d(gu, Fu), d(gu'Fu'), d(gu, Fu'), d(gu', Fu)) \\ &+ \theta(d(gu, Fu), d(gu', Fu'), d(gu, Fu'), d(gu', Fu)) \\ &= (A + 2C) d(v, v'). \end{aligned}$$

As, $A + 2C < 1$ we obtain from above inequality that $d(v, v') = 0$, that is $v = v'$. Thus, point of coincidence v is unique. Further, if f and g are weakly compatible then by Lemma 1.4, v is the unique common fixed point of f and g. □

Following corollaries are immediate consequences of above theorem and are new Prešić type fixed point results.

Corollary 2.2 *Let (X, d) be a complete metric space, k a positive integer and $f : X^k \to X$ be a mapping. Suppose following condition is satisfied:*

$$d(f(x_0, x_1, \ldots, x_{k-1}), f(x_1, x_2, \ldots, x_k)) \leq \sum_{i=1}^{k} \alpha_i d(x_{i-1}, x_i) + \beta \sum_{i=0}^{k} \sum_{j=0}^{k} d(x_i, Fx_j) + \theta(d(x_0, Fx_0), d(x_k, Fx_k), d(x_0, Fx_k), d(x_k, Fx_0)),$$

for all $x_0, x_1, \ldots, x_k \in X$, where α_i, β are nonnegative constants such that $\sum_{i=1}^{k} \alpha_i + \beta k(k+1)^2 < 1$, F is the associate operator of f and $\theta \in \Theta$. Then f has a unique fixed point in X. Moreover, for any $x_0 \in X$ the sequence $\{x_n\}$ defined by $x_n = f(x_{n-1}, \ldots, x_{n-1})$, for all $n \in \mathbb{N}$, converges to the unique fixed point of f.

Corollary 2.3 *Let (X, d) be a complete metric space, k a positive integer and $f : X^k \to X$, $g\colon X \to X$ be two mappings such that $f(X^k) \subset g(X)$ and $g(X)$ is closed. Suppose following condition is satisfied:*

$$d(f(x_0, x_1, \ldots, x_{k-1}), f(x_1, x_2, \ldots, x_k)) \leq \sum_{i=1}^{k} \alpha_i d(gx_{i-1}, gx_i) + \beta \sum_{i=0}^{k} \sum_{j=0}^{k} d(gx_i, Fx_j),$$

for all $x_0, x_1, \ldots, x_k \in X$, where α_i, β are nonnegative constants such that $\sum_{i=1}^{k} \alpha_i + \beta k(k+1)^2 < 1$ and F the is associate operator of f. Then f and g have a unique point of coincidence. In addition, if f and g are weakly compatible then f and g have a unique common fixed point. Moreover, for any $x_0 \in X$ the sequence $\{y_n\}$ defined by $y_n = gx_n = f(x_{n-1}, \ldots, x_{n-1})$, for all $n \in \mathbb{N}$, converges to the common fixed point of f and g.

Following theorem is a generalization of results of Prešić [12] and [15] and its proof is similar as the proof of Theorem 2.1, therefore we omit the proof.

Theorem 2.4 *Let (X, d) be a complete metric space, k a positive integer and $f\colon X^k \to X$, $g\colon X \to X$ be two mappings such that $f(X^k) \subset g(X)$ and $g(X)$ is closed. Suppose following condition is satisfied:*

$$d(f(x_0, x_1, \ldots, x_{k-1}), f(x_1, x_2, \ldots, x_k)) \leq \sum_{i=1}^{k} \alpha_i d(gx_{i-1}, gx_i) + \beta \sum_{i=0}^{k} d(gx_i, Fx_i),$$

for all $x_0, x_1, \ldots, x_k \in X$, where α_i, β are nonnegative constants such that $\sum_{i=1}^{k} \alpha_i + \beta k(k+1) < 1$ and F is the associate operator of f. Then f and g have a unique point of coincidence. In addition, if f and g are weakly compatible then f and g have a unique common fixed point. Moreover, for any $x_0 \in X$ the sequence $\{y_n\}$ defined by $y_n = gx_n = f(x_{n-1}. \ldots, x_{n-1})$, for all $n \in \mathbb{N}$, converges to the common fixed point of f and g.

In the next theorem, we approximate the point of coincidence of f and g by k-step iterative sequence.

Theorem 2.5 *Let (X, d) be a complete metric space, k a positive integer and $f\colon X^k \to X$, $g\colon X \to X$ be two mappings such that $f(X^k) \subset g(X)$ and $g(X)$ is closed. Suppose following condition is satisfied:*

$$\begin{aligned} d(f(x_0, x_1, \ldots, x_{k-1}), f(x_1, x_2, \ldots, x_k)) &\le \sum_{i=1}^{k} \alpha_i d(gx_{i-1}, gx_i) \\ &+ L \min\{\theta(d(gx_0, Fx_0), d(gx_k, Fx_k), d(gx_0, Fx_k), d(gx_k, Fx_0)), \\ &\qquad d(gx_k, f(x_0, x_1, \ldots, x_{k-1}))\}, \quad \text{(B)} \end{aligned}$$

for all $x_0, x_1, \ldots, x_k \in X$, where α_i, L are nonnegative constants such that $\sum_{i=1}^{k} \alpha_i < 1$, F is the associate operator of f and $\theta \in \Theta$. Then:

1) *f and g have a unique point of coincidence, say $v \in X$;*
2) *the sequence $\{y_n\}$ defined by $y_n = gx_n = f(x_{n-1}, \ldots, x_{n-1})$, for all $n \in \mathbb{N}$, converges to v for any starting point $x_0 \in X$;*
3) *the k-step iterative sequence $\{z_n\}$ defined by $x_0, x_1, \ldots, x_{k-1} \in X$ and $z_n = gx_n, 0 \le n \le k-1$ and $z_{n+k} = gx_{n+k} = f(x_n, \ldots, x_{n+k-1}), n \ge 0$, also converges to v.*

In addition, if f and g are weakly compatible then f and g have a unique common fixed point.

Proof. Note that $\min\{a, b\} \le a$ for all real a, b therefore, condition (B) implies (A) and so 1) and 2) follows from Theorem 2.1, that is, there exists $u, v \in X$ such that $v = gu = Fu = f(u, \ldots, u)$.

For 3), let $\{z_n\}$ be sequence defined by $z_n{=}gx_n, 0{\le}n{\le}k-1$ and $z_{n+k} = gx_{n+k} = f(x_n, \ldots, x_{n+k-1}), n \ge 0$, where $x_0, x_1, \ldots, x_{k-1} \in X$ are arbitrary. For any $n \in \mathbb{N}$ we have

$$\begin{aligned} d(z_{n+k}, v) &= d(f(x_n, \ldots, x_{n+k-1}), f(u, \ldots, u)) \\ &\le d(f(x_n, \ldots, x_{n+k-1}), f(x_{n+1}, \ldots, x_{n+k-1}, u)) \\ &+ d(f(x_{n+1}, \ldots, x_{n+k-1}, u), f(x_{n+2}, \ldots, x_{n+k-1}, u, u)) \qquad (2.4) \\ &+ \cdots + d(f(x_{n+k-1}, u, \ldots, u), f(u, \ldots, u)). \end{aligned}$$

Using (B) for each term of the sum on the right hand side of above inequality we obtain

$$\begin{aligned} &d(f(x_n, \ldots, x_{n+k-1}), f(x_{n+1}, \ldots, x_{n+k-1}, u)) \\ &\le \alpha_1 d(gx_n, gx_{n+1}) + \cdots + \alpha_{k-1} d(gx_{n+k-2}, gx_{n+k-1}) + \alpha_k d(gx_{n+k-1}, gu) \\ &+ L \min\{\theta(d(gx_n, Fx_n), d(gu, Fu), d(gx_n, Fu), d(gu, Fx_n)), \end{aligned}$$

$$d(gu, f(x_n, x_{n+1}, \ldots, x_{n+k-1}))\},$$
$$d(f(x_{n+1}, \ldots, x_{n+k-1}, u), f(x_{n+2}, \ldots, x_{n+k-1}, u, u))$$
$$\leq \alpha_1 d(gx_{n+1}, gx_{n+2}) + \cdots + \alpha_{k-2} d(gx_{n+k-2}, gx_{n+k-1}) + \alpha_{k-1} d(gx_{n+k-1}, gu)$$
$$+ L \min\{\theta(d(gx_{n+1}, Fx_{n+1}), d(gu, Fu), d(gx_{n+1}, Fu), d(gu, Fx_{n+1})),$$
$$d(gu, f(x_{n+1}, x_{n+2}, \ldots, x_{n+k-1}, u))\},$$

and

$$d(f(x_{n+k-1}, u, \ldots, u), f(u, \ldots, u)) \leq \alpha_1 d(gx_{n+k-1}, gu)$$
$$+L \min\{\theta(d(gx_{n+k-1}, Fx_{n+k-1}), d(gu, Fu), d(gx_{n+k-1}, Fu), d(gu, Fx_{n+k-1})),$$
$$d(gu, f(x_{n+k-1}, u, \ldots, u))\}.$$

As, $d(gu, Fu) = 0$ and $\theta \in \Theta$, (2.4) leads to

$$d(z_{n+k}, v) \leq \alpha_1 d(gx_n, gx_{n+1}) + (\alpha_1 + \alpha_2) d(gx_{n+1}, gx_{n+2})$$
$$+ \cdots + (\alpha_1 + \cdots + \alpha_{k-1}) d(gx_{n+k-2}, gx_{n+k-1}) + [\sum_{i=1}^{k} \alpha_i] d(gx_{n+k-1}, gu),$$

that is,

$$d(z_{n+k}, v) \leq E_n + [\sum_{i=1}^{k} \alpha_i] d(z_{n+k-1}, v). \tag{2.5}$$

Where $E_n = \alpha_1 d(gx_n, gx_{n+1}) + \cdots + (\alpha_1 + \cdots + \alpha_{k-1}) d(gx_{n+k-2}, gx_{n+k-1})$.

We shall show that $\lim_{n\to\infty} E_n = 0$. Note that, for any $n \in \mathbb{N}$ we have

$$\begin{aligned} d(gx_{n+k}, gx_{n+k+1}) &= d(f(x_n, x_{n+1}, \ldots, x_{n+k-1}), f(x_{n+1}, x_{n+2}, \ldots, x_{n+k})) \\ &\leq \alpha_1 d(gx_n, gx_{n+1}) + \cdots + \alpha_k d(gx_{n+k-1}, gx_{n+k}) \\ &\quad + L \min\{\theta(d(gx_n, Fx_n), d(gx_{n+k}, Fx_{n+k}), d(gx_n, Fx_{n+k}), \\ &\quad d(gx_{n+k}, Fx_n)), d(gx_{n+k}, f(x_n, \ldots, x_{n+k-1}))\} \end{aligned}$$

As, $gx_{n+k}=f(x_n, \ldots, x_{n+k-1})$ we have $d(gx_{n+k}, gx_{n+k+1})=\alpha_1 d(gx_n, gx_{n+1})$ $+\alpha_2 d(gx_{n+1},\ gx_{n+2}) + ... + \alpha_k d(gx_{n+k-1},\ gx_{n+k})$. Using Lemma 1.5 in above inequality we have, there exist $N>0$ and $\lambda \in (0,1)$ such that $d(gx_n, gx_{n+1}) \leq N\lambda^n$, for all $n \in \mathbb{N}$. Therefore, we obtain $\lim_{n\to\infty} E_n = 0$, and so, it follows from (2.5) and Lemma 1.6 that $\lim_{n\to\infty} d(z_n, v) = 0$. If f and g are weakly compatible, then by Lemma 1.4, v is the unique common fixed point of f and g. □

Remark 2.1 Let (X, d) be a metric space, then $f\colon X^k \to X$ is called almost Prešić contraction [14] if there exists nonnegative constants α_i, L such that

$$\begin{aligned} d(f(x_0, x_1, \ldots, x_{k-1}), f(x_1, x_2, \ldots, x_k)) &\leq \sum_{i=1}^{k} \alpha_i d(x_{i-1}, x_i) \\ &\quad + L \min\{d(x_0, Fx_0), d(x_k, Fx_k), d(x_0, Fx_k), d(x_k, Fx_0), \\ &\qquad d(x_k, f(x_0, x_1, \ldots, x_{k-1}))\}, \end{aligned}$$

for all $x_0, x_1, \ldots, x_k \in X$, where F is the associate operator of f.

For $\beta = 0$, $\theta(a, b, c, d) = L \min\{a, b, c, d\}$ and $g = I_X$ in Theorem 2.5, we get the fixed point result of [14] for almost Prešić contraction.

Example 2.1 Let $X = \mathbb{R}$ and d is usual metric on X. Then (X, d) is a complete metric space. For $k = 2$, define $f\colon X^2 \to X$ and $g\colon X \to X$ by $f(x, y) = \frac{3x-2y}{4}$, for all $x, y \in X$ and $gx = 2x$, for all $x \in X$. Then it is easy to see that f is not an almost Prešić contraction. Indeed, for $x = y = 0, z = 5$ we have $d(f(x, y), f(y, z)) = d(f(0, 0), f(0, 5)) = d(0, -5/2) = 5/2$, and $\alpha_1 d(x, y) + \alpha_2 d(y, z) + L \min\{d(x, Fx), d(z, Fz), d(x, Fz), d(z, Fx)\} = 5\alpha_2 + L \min\{d(0, 0), d(5, F5), d(0, F5), d(5, F0)\} = 5\alpha_2$, therefore for f to be an almost Prešić contraction we must have $\alpha_2 \geq 1/2$. Similarly, for $x = 5, y = z = 0$ we must have $\alpha_1 \geq 3/4$, but then $\alpha_1 + \alpha_2 \geq 5/4 > 1$. Thus f is not an almost Prešić contraction.

While f and g satisfies all the conditions of Theorem 2.5, with $\alpha_1 = \frac{3}{8}, \alpha_2 = \frac{1}{4}$ and with arbitrary value of L. Note that f and g gave a unique common fixed point $0 = g0 = f(0, 0)$.

3 Applications to difference equation

The study of nonlinear difference equations, which plays an important role in modelling different problems appearing in economics, biology, ecology, genetics, psychology, sociology, probability theory and others (see introduction from [14] and the references therein) is a topic of great interest, in particular the difference equations of order greater than one is studied by many authors. First, we give some definitions.

Definition 3.1 Let I be an interval of real numbers. A point $v \in I$ is called an equilibrium point of k-th order nonlinear difference equation

$$x_{n+k} = f(x_n, x_{n+1}, \ldots, x_{n+k-1}), \quad n = 1, 2, \ldots, \tag{3.1}$$

if $v = f(v, v, \ldots, v)$, i.e., if v is a fixed point of $f\colon I^k \to I$.

Definition 3.2 Let v be an equilibrium point of Equation (3.1), then v is called

(1) locally stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, $x_0, x_1, \ldots, x_{k-1} \in I$ and $|x_0 - v| + |x_1 - v| + \cdots + |x_{k-1} - v| < \delta$ implies $|x_n - v| < \varepsilon$, for all $n \geq 0$;
(2) locally asymptotically stable if it is locally stable and there exists $\delta > 0$ such that, $x_0, x_1, \ldots, x_{k-1} \in I$ and $|x_0 - v| + |x_1 - v| + \cdots + |x_{k-1} - v| < \delta$ implies $\lim_{n\to\infty} x_n = v$;
(3) global attractor if for every $x_0, x_1, \ldots, x_{k-1} \in I$ we have $\lim_{n\to\infty} x_n = v$;
(4) globally asymptotically stable if it is asymptotically stable and a global attractor;
(5) unstable if v is not locally stable.

Theorem 3.3 *If $f\colon I^k \to I$ be a Prešić operator then for every set of initial conditions $\{x_0, x_1, \ldots, x_{k-1}\} \subset I$, the difference equation*

$$x_{n+k} = f(x_n, x_{n+1}, \ldots, x_{n+k-1}), \quad n = 0, 1, 2, \ldots, \tag{3.2}$$

has a unique equilibrium point in I and the sequence $\{x_n\}$ defined by $x_{n+k} = f(x_n, x_{n+1}, \ldots, x_{n+k-1})$, $n = 0, 1, 2, \ldots$, converges to the equilibrium point. Furthermore, the equilibrium point is globally asymptotically stable (here I is considered to be complete with the usual metric).

Proof. We note that, I is complete metric space with usual metric. Suppose, f is a Prešić operator then for every $x_0, x_1, \ldots, x_{k-1}, x_k \in I$ we have

$$|f(x_0, x_1, \ldots, x_{k-1}) - f(x_1, x_2, \ldots, x_k)| \leq \sum_{i=1}^{k} \alpha_i |x_{i-1} - x_i|, \tag{3.3}$$

where $0 \leq \sum_{i=1}^{k} \alpha_1 < 1$.

By Prešić's theorem, f has a unique fixed point v and $\lim_{n\to\infty} x_n = v$, for all $x_0, x_1, \ldots, x_{k-1} \in I$ with $x_{n+k} = f(x_n, x_{n+1}, \ldots, x_{n+k-1})$. Therefore, v is the equilibrium point and it is globally asymptotically stable. □

Many authors studied the global behavior of the recursive sequence

$$x_{n+2} = \frac{a + bx_{n+1}}{c + dx_n}, \quad n = 0, 1, 2, \ldots, \tag{3.4}$$

where a, b, c, d are nonnegative real numbers (see, [6,8,9]). Note that, if $b, d \neq 0$, then replacing x_n by $(\frac{b}{d})x_n$, equation (3.4) can be written as

$$x_{n+2} = \frac{\alpha + x_{n+1}}{\beta + x_n}, \quad n = 0, 1, 2, \ldots, \tag{3.5}$$

where α, β are nonnegative real numbers. As an example, we consider the second order difference equation (3.5) and prove following corollary of Theorem 3.3.

Corollary 3.4 *If $\alpha \leq \beta$ and $\beta > 2$, then difference equation* (3.5) *has a unique globally asymptotically stable equilibrium point in the interval $I = [0, \infty)$.*

Proof. Note that, I is complete metric space with usual metric. Here the corresponding operator $f : I^2 \to I$ for equation (3.5) is given by $f(x, y) = \frac{\alpha+y}{\beta+x}$, for all $x, y \in I$. Now, for any $x, y, z \in X$ we have

$$\begin{aligned}
|f(x, y) - f(y, z)| &= |\frac{\alpha + y}{\beta + x} - \frac{\alpha + z}{\beta + y}| \\
&= \frac{|\alpha(y - x) + \beta(y - z) + y^2 - xz|}{(\beta + x)(\beta + y)} \\
&= \frac{|(\alpha + y)(y - x) + (\beta + x)(y - z)|}{(\beta + x)(\beta + y)} \\
&\leq \frac{\alpha + y}{(\beta + x)(\beta + y)}|y - x| + \frac{1}{\beta + y}|y - z| \\
&\leq \frac{1}{\beta}[|y - x| + |y - z|] \quad (\text{as } \alpha \leq \beta \text{ and } x, y \in [0, \infty)).
\end{aligned}$$

As, $\beta > 2$ therefore, f is a Prešić operator with $\alpha_1 = \alpha_2 = \frac{1}{\beta}$. Thus by Theorem 3.3, equation (3.5) has a unique globally asymptotically stable equilibrium point in the interval $I = [0, \infty)$. □

Remark 3.1 We note that, if f is a Prešić operator then f satisfies the condition (B) of Theorem 2.5 (with $g = I_X$), therefore the equilibrium point of equation (3.5) can be obtained by the single-step iterative sequence defined by $x_{n+1} = f(x_n, x_n) = F(x_n), n = 0, 1, 2, \ldots$ with arbitrary initial value $x_0 \in I$, instead by the two-step iterative sequence $x_{n+2} = f(x_n, x_{n+1}), n = 0, 1, 2, \ldots$ with two arbitrary initial values $x_0, x_1 \in I$.

We consider a particular case of difference equation (3.5) as an example.

Example 3.1 Let us consider the difference equation (3.5) with $\alpha = 1, \beta = 4$ and $I = [0, \infty)$. Note that $\alpha < \beta$ and $\beta > 2$, therefore by Corollary 3.4, equation has a unique globally asymptotically stable equilibrium point in the interval $I = [0, \infty)$. Note that, the single-step iterative sequence $\{x_n\} = \{f(x_{n-1}, x_{n-1})\}$, $n = 1, 2, \ldots$ and two-step iterative sequence $\{y_n\} = \{f(y_{n-2}, y_{n-1})\}$, $n = 2, 3, \ldots$ (say) both converge to the equilibrium point $v = \frac{\sqrt{13}-3}{2} \approx 0.302775 \in I$. The comparison between the convergence of these two sequences is given in the following table and one can see that with some particular initial values the single-step iterative sequence converges faster than the two-step iterative sequence.

n	$y_{n+1} = \dfrac{1+y_n}{4+y_{n-1}}, y_0 = 0, y_1 = 1$	$x_n = \dfrac{1+x_{n-1}}{4+x_{n-1}}, x_0 = 1$
1	0.5	0.4
2	0.3	0.318182
3	0.288889	0.305263
4	0.299741	0.303178
5	0.303048	0.302841
6	0.303053	0.302786
7	0.302821	0.302777
8	0.302767	0.302775
9	0.302774	0.302775
10	0.302775	0.302775

In a similar way, we can prove the following corollary, which ensure a unique globally asymptotically stable equilibrium point of the difference equation $x_{n+2} = \frac{a+bx_{n+1}}{c+dx_n^2}$ or if $b, d \neq 0$, then

$$x_{n+2} = \frac{\alpha + x_{n+1}}{\beta + x_n^2}. \tag{3.6}$$

Corollary 3.5 *If $\alpha^2 \leq \beta$ and $\beta > 5$, then difference equation (3.6) has a unique globally asymptotically stable equilibrium point in the interval $I = [0, \sqrt{\beta}]$.*

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