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A new type of nilpotent BCI-algebras

Ardavan Najafi · Esfandiar Eslami ·
Arsham Borumand Saeid

Abstract In this paper we introduce a new notion of nilpotent BCI-algebras based on commutators rather than using nilpotent elements. To be specific and different from the old one (if we may, nilpotent algebras of type 1), call nilpotent BCI-algebras of type 2. We prove that every nilpotent BCI-algebra of type 2 is solvable. Also, we show that any finite BCI-algebra is solvable but is not nilpotent of type 2, generally. It is shown that every p-semisimple, associative and commutative BCI-algebras is solvable and nilpotent of type 2. Finally, the relationships between characteristic subalgebras, derived subalgebras and (K-)nil radical of BCI-algebras are investigated.

Keywords (K-)nil radical · commutators · characteristic subalgebra · nilpotent BCI-algebras of type 2

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1 Introduction

The notion of BCI-algebras was introduced by K. Iséki in 1966 as a generalization of BCK-algebras [9]. BCK and BCI-algebras are algebraic models of non-classical logics BCK and BCI-logics respectively. It is known that the class of associative BCI-algebras coincides with the class of Boolean groups [8]. The relation between p-semisimple BCI-algebras and abelian groups was presented by T. Lei and C. Xi in [11]. Nilpotent elements and order of elements in BCI-algebras are introduced by K. Iséki and Lin Dahua respectively [15].
Nil BCI-algebras are defined as BCI-algebras in which all elements are nilpotent [6]. There are some papers [10,15] in which such BCI-algebras are called nilpotent algebras. It is well-known that in algebraic structures including groups, rings and Lie algebras the notion of nilpotency is defined based on commutators [3,5] rather than on nilpotent elements. These motivate us to introduce a new notion of nilpotent BCI-algebras without using nilpotency of the elements. This concept is different from the nil (nilpotent)

BCI-algebras previously defined but it is consistent with the nilpotency of other mentioned algebras. To be specific we call such new BCI-algebras, nilpotent BCI-algebra of type 2.
We use the notion of nilpotent BCI-algebras of type 2 to develop other new concepts such as component series in these structures. We can also investigate the variety and some subvarieties of these specific type of BCI-algebras. Since nilpotency and solvability are two important notions, we extend these two notions to these BCI-algebras. The aim of the paper is to expound the relation between solvable and nilpotent BCI-algebras of type 2, and to discuss further properties of this concepts.

2 Preliminaries

Definition 2.1 *A BCI-algebra is a structure* $(X, \to, 1)$*, where* $\to$ *is a binary operation on* X *and* 1 *is an element of* X*, verifying, the axioms: for all* $x, y, z \in X$*,*
(I) $(x \to y) \to ((y \to z) \to (x \to z)) = 1,$
(II) $x \to ((x \to y) \to y) = 1,$
(III) $x \to x = 1,$
(IV) $y \to x = 1$ *and* $x \to y = 1$ *imply* $x = y$*.*

In any BCI-algebra $(X, \to, 1)$ the natural order can be defined by putting $x \leq y$ if and only if $x \to y = 1$, for all $x, y \in X$. If in a BCI-algebra $(X, \to, 1)$ the condition $x \to 1 = 1$, for all $x \in X$ holds, then it is a BCK-algebra. In BCI-algebra X for all $x, y, z \in X$ we have $((x \to y) \to y) \to y = x \to y$ and $x \to (y \to z) = y \to (x \to z)$ [4, 8]. A BCI-algebra X is called proper if it is not a BCK-algebra. A non-empty subset S of a BCI-algebra X is called a subalgebra of X, if $x \to y \in S$ whenever $x, y \in S$. Also a non-empty subset F of a BCI-algebra X is called a filter if: (i) $1 \in F$, (ii) $x \to y \in F$ and $x \in F$ imply $y \in F$, for all $x, y \in X$. An element x in a BCI-algebra X is called a positive element if it satisfies $x \to 1 = 1$. Let F be a filter of a BCI-algebra X. Then the relation θ_F defined by $(x, y) \in \theta_F$ if and only if $x \to y \in F$ and $y \to x \in F$ is a congruence relation on X. If C_x denote the class of $x \in X$, then $C_1 = F$. Assume that $X/F = \{C_x : x \in X\}$. Then $(X/F, \to, C_1)$ is a BCI-algebra, where $C_x \to C_y = C_{x \to y}$, for all $x, y \in X$. A mapping $f : X \longrightarrow Y$ from BCI-algebras is called a homomorphism, if for any $x, y \in X$, $f(x \to y) = f(x) \to f(y)$ holds. The BCI-algebra X is called commutative if $x \leq y$ implies $y = y \vee x$, where $x \vee y = (x \to y) \to y$. In any commutative BCI-algebra X for all $x, y \in X$ we have $(x \vee y) \to (y \vee x) = (y \to x) \to 1$. A BCI-algebra X is called p-semisimple if $(x \to 1) \to 1 = x$, for all $x \in X$. In any p-semisimple BCI-algebra X for all $x, y \in X$ we have $(x \to y) \to 1 = y \to x$ and $(x \to y) \to y = x$. A BCI-algebra X is called associative if $(x \to y) \to z = x \to (y \to z)$, for all $x, y, z \in X$ [4, 7, 8, 9].
From now on, in this paper, $(X, \to, 1)$ or simply X is a BCI-algebra, unless otherwise specified.

Definition 2.2 [6] *An element x of X is a nilpotent element if $x^n \to 1 = 1$ for some positive integer n, where $x^n \to y = \underbrace{x \to (... \to (x \to (x \to y))...)}_{n-times}$.*

A filter F of X is called a nil filter of X if every element of F is nilpotent. In particular, if every x in X is nilpotent, then X is called a nilpotent BCI-algebra [4] *or a nil BCI-algebra* [6].

For every positive integer k, we define

$$N_k(X) = \{x \in X : x^k \to 1 = 1\}$$
$$N(X) = \{x \in X : x \textit{ is a nilpotent element}\}$$

The set $N(X)$ is called the nil-radical of X, and $N_k(X)$ is called the k-nil radical set of X [4].

Definition 2.3 [14] *Let x_1, x_2 be elements of X. The element $((x_2 \to x_1) \to 1) \to ((x_1 \vee x_2) \to (x_2 \vee x_1))$ of X is called a pseudo-commutator of x_1 and x_2 and is denoted by $[x_1, x_2]$. i.e.,*

$$[x_1, x_2] =((x_2 \to x_1) \to 1) \to ((x_1 \vee x_2) \to (x_2 \vee x_1)).$$

Definition 2.4 [12, 13, 14] *Suppose that X_1, X_2 are non-empty subsets of X. Define a commutator of X_1 and X_2 to be*

$$[X_1, X_2] = \{\prod[x_1, x_2] : x_1 \in X_1, x_2 \in X_2\}.$$

Where $\prod[x_1, x_2] = [a_1, b_1] \to [a_2, b_2] \to ... \to [a_n, b_n]$, for $a_i \in X_1$ and $b_i \in X_2$. $[X, X]$ is called the commutator subalgebra or the derived subalgebra of X and is denoted by X'. Therefore
$X' = \{x_1 \to x_2 \to ... \to x_n : n \geq 1$, each x_i is a pseudo-commutator in $X\}$.

Now, we give an explicit example of how obtain $[X_1, X_2]$ and X'.

Example 2.1 Let $X = \{0, a, b, c, d\}$. Define " $\to$ " on X by

$\to$	a	b	c	d	1
a	1	a	b	d	1
b	1	1	a	d	1
c	1	1	1	d	1
d	1	1	1	1	1
1	a	b	c	d	1

Then $(X, \to, 1)$ is a BCK-algebra and so a BCI-algebra. Put $X_1 = \{a, c\}$ and $X_2 = \{d\}$. Therefore $[X_1, X_2] = \{\prod[x_1, x_2] : x_1 \in X_1, x_2 \in X_2\} = \{[a, d] = a, [c, d] = c, [a, d] \to [c, d] = b, [c, d] \to [a, d] = 1, ([c, d] \to [a, d]) \to [c, d] = c, [c, d] \to ([a, d] \to [c, d]) = 1, ...\} = \{a, b, c, 1\}$. Also we obtain $X' = [X, X] = \{[1, a] = 1, [a, b] = 1, ..., [a, d] = a, [c, d] = c, [a, d] \to [a, d] = 1, [a, d] \to [c, d] = b, ([c, d] \to [a, d]) \to [c, d] = c, [c, d] \to ([a, d] \to [c, d]) = 1, ...\} = \{a, b, c, 1\}$.

Theorem 2.5 [12, 13, 14] *Suppose that F is a filter of X. Then*
i) X is commutative if and only if $X^{'} = \{1\}$,
ii) $X^{'}$ is a subalgebra of X,
iii) $[A/F, B/F] = [A, B]/F$ for two subsets A, B of X,
iv) if Y is a subalgebra of X, then $Y^{'}$ is a subalgebra of $X^{'}$.

Lemma 2.6 [13, 14] *Let f be a homomorphism from X to a BCI-algebra Y. Then $f([x, y]) = [f(x), f(y)]$, for all $x, y \in X$.*

Lemma 2.7 [4] *For any $x, y \in X$ and $n \in N$, $(x \to y)^n \to 1 = (x^n \to 1) \to (y^n \to 1)$.*

Although not all of the finite groups are necessarily nilpotent, based on the *Definition 2.2*, every finite BCI-algebra and every BCK-algebra is nilpotent [4]. This is the main difference between the *definitions* of nilpotent BCI-algebras and nilpotent groups. In addition, since p-semisimple BCI-algebras are convertible to abelian groups [1], it is expected that similar to abelian groups, any p-semisimple BCI-algebra be nilpotent, but this is not always the case.

Example 2.2 Let $C^* = C \backslash \{0\}$, where C is the set of all complex numbers. Then as it has mentioned in [6], $(C^*, \div, 1)$ is an infinite p-semisimple BCI-algebra. Now it is easy to check that $z \in C^*$ is a nilpotent element iff $z^n = 1$ for some $n \in N$. Therefore C^* is not a nilpotent BCI-algebra.

Notation 2.1 [14] We put $C^0(X) = X, C^1(X) = [X, X], \ldots, C^k(X) = [C^{k-1}(X), X]$ and $C_0(X) = X, C_1(X) = [X, X], \ldots, C_k(X) = [C_{k-1}(X), C_{k-1}(X)]$.

Definition 2.8 [14] *X is called solvable if there exists $n \in N$ such that $C_n(X) = \{1\}$. The smallest such n is called the derived length of X.*

Lemma 2.9 [2] *Let X be a BCK-algebras. Then $[x, y]$ is not a maximal element of X, for all $x, y \in X$.*

3 Nilpotent BCI-algebras of type 2

We have introduced and studied some properties of solvable BCI-algebras in [14]. In this section, we present some new results on solvable BCI-algebras. Also, we generalize the concept of nilpotency in groups theory into BCI-algebras and define the notion of nilpotent BCI-algebras of type 2 and discuss it properties.

Definition 3.1 *X is called nilpotent of type 2 if there exists $n \in N$ such that $C^n(X) = \{1\}$. The least such n is called the nilpotency class of X.*

$C^m(X)$ and $C_m(X)$ are non-empty subsets of X because $1 \in C_m(X)$ and $1 \in C^m(X)$, for all $m \in N$.

Example 3.1 [4] Let $X = \{a, b, c, d, 1\}$. We define " $\to$ " on X by

$\to$	a	b	c	d	1
a	1	b	b	b	1
b	b	1	a	a	b
c	b	1	1	a	b
d	b	1	1	1	b
1	a	b	c	d	1

$(X, \to, 1)$ is a BCI-algebra. The pseudo-commutators of elements of X are given by the following table

$[,]$	a	b	c	d	1
a	1	1	1	1	1
b	1	1	1	1	1
c	1	1	1	a	1
d	1	1	1	1	1
1	1	1	1	1	1

Then $C^0(X) = C_0(X) = X, C^1(X) = C_1(X) = \{a, 1\}$, $C^2(X) = C_2(X) = \{1\}$. Therefore X is solvable of length 2. Also X is nilpotent of type 2 and the nilpotency class is 2.

By Theorem 2.5, we have the following lemma.

Lemma 3.2 *X is commutative if and only if $C^1(X) = C_1(X) = \{1\}$.*

Proposition 3.3 *If $A \subseteq B \subseteq X$, then $[C, A] \subseteq [C, B]$ and $[A, C] \subseteq [B, C]$ for any $C \subseteq X$.*

Proof. Let $t \in [C, A]$. Then there exist $c_i \in C$ and $a_i \in A$, for $i \in I$ such that $t = \prod[c_i, a_i]$. Since $A \subseteq B$, $a_i \in B$ and $t = \prod[c_i, a_i] \in [C, B]$. Therefore $[C, A] \subseteq [C, B]$. The proof of the other statement is similar. □

In Example 3.1 if we put $A = \{a, 1\}$ and $B = \{b, 1\}$, then for every $C \subseteq X$ we have $[A, C] \subseteq [B, C]$ and $[C, A] \subseteq [C, B]$ but $A \nsubseteq B$. Therefore the converse of Proposition 3.3 is not true in general.
The above proposition leads to the following.

Proposition 3.4 *For any non-negative integer m, $C_m(X) \subseteq C^m(X)$.*

Proof. By induction on m. For $m = 0$, we have $C_0(X) = X = C^0(X)$. Now assume that $C_{m-1}(X) \subseteq C^{m-1}(X)$, for some $m > 0$. Then $C_m(X) = [C_{m-1}(X), C_{m-1}(X)] \subseteq [C_{m-1}(X), X] \subseteq [C^{m-1}(X), X] = C^m(X)$, hence the result holds for m. □

Now we describe the relation between nilpotent BCI-algebra of type 2 and solvable BCI-algebra.

Corollary 3.5 *Every nilpotent BCI-algebra of type 2 is solvable.*

Proof. Let X be nilpotent of type 2. Then there exists $m \in N$ such that $C^m(X) = \{1\}$. Since $C_m(X)$ is a non-empty set and $C_m(X) \subseteq C^m(X) = \{1\}$, $C_m(X) = \{1\}$. Therefore X is solvable. □

Theorem 3.6 *i) Every commutative BCI-algebra is nilpotent of type* 2 *and solvable.*
ii) Every p-semisimple BCI-algebra is nilpotent of type 2 *and solvable.*
iii) Every associative BCI-algebra is nilpotent of type 2 *and solvable.*

Proof. i) For any commutative BCI-algebra X, $C_1(X) = [X, X] = C^1(X) = \{1\}$. Hence X is nilpotent of type 2 and solvable.
ii) Let X be a p-semisimple BCI-algebra and $x, y \in X$. Then

$$\begin{aligned}[x,y] =&((y \to x) \to 1) \to ((x \vee y) \to (y \vee x))\\ =&((y \to x) \to 1) \to (x \to y)\\ =&(x \to y) \to (x \to y) = 1.\end{aligned}$$

Therefore $C_1(X) = [X, X] = C^1(X) = \{1\}$. Hence X is nilpotent of type 2 and solvable.
iii) Let X be an associative BCI-algebra and $x, y \in X$. Then

$$\begin{aligned}[x,y] =&((y \to x) \to 1) \to (((x \to y) \to y) \to ((y \to x) \to x))\\ =&(y \to (x \to 1)) \to ((x \to (y \to y)) \to (y \to (x \to x)))\\ =&(y \to (x \to 1)) \to ((x \to 1) \to (y \to 1))\\ =&((y \to (x \to 1)) \to (x \to 1)) \to (y \to 1)\\ =&(y \to ((x \to 1) \to (x \to 1))) \to (y \to 1)\\ =&(y \to 1) \to (y \to 1) = 1.\end{aligned}$$

Therefore $C_1(X) = C^1(X) = \{1\}$. Hence X is nilpotent of type 2 and solvable. □

Lemma 3.7 *If* $A \subseteq B \subseteq X$*, then* $C^m(A) \subseteq C^m(B)$ *and* $C_m(A) \subseteq C_m(B)$*, for all* $m \in N \cup \{0\}$*.*

Proof. We prove by induction on m. For $m = 0$, we have $C^0(A) = A \subseteq C^0(B) = B$ and $C_0(A) = A \subseteq C_0(B) = B$.
Now assume that $C^{m-1}(A) \subseteq C^{m-1}(B)$ and $C_{m-1}(A) \subseteq C_{m-1}(B)$, for some $m > 0$. Then $C^m(A) = [C^{m-1}(A), A] \subseteq [C^{m-1}(B), B] = C^m(B)$ and $C_m(A) = [C_{m-1}(A), C_{m-1}(A)] \subseteq [C_{m-1}(B), C_{m-1}(B)] = C_m(B)$. □

The following example shows that the converses of statements in Corollary 3.5, Theorem 3.6, as well as the equality of Proposition 3.4 may not hold.

Example 3.2 i) [4] Let $X = \{a, b, c, d, 1\}$. We define the operation " $\to$ " on X by

$\to$	a	b	c	d	1
a	1	b	c	c	1
b	1	1	c	c	1
c	c	c	1	a	c
d	c	c	1	1	c
1	a	b	c	d	1

Then $(X, \to, 1)$ is a BCI-algebra. It is easy to check that $C^0(X) = C_0(X) = X, C^1(X) = C_1(X) = \{a, 1\}$ and $C_2(X) = \{1\}$. Also $C^2(X) = C^3(X) = C^4(X) = ... = \{a, 1\}$. Therefore X is solvable of length 2 but X is not nilpotent of type 2.
ii) The BCI-algebra X in Example 3.1 is nilpotent of type 2 and solvable. It is not commutative because $d \le c$ but $(c \to d) \to d = a \to d = b \ne c$. Since $(a \to 1) \to 1 = 1 \ne a$, X is not p-semisimple. Also $b = a \to (1 \to d) \ne (a \to 1) \to d = d$. Then X is not an associative BCI-algebra.

Open problem 1. *Under what suitable conditions is the converse of the Corollary* 3.5 *and Theorem* 3.6 *true*?

Theorem 3.8 *i)* $C_n(X)$ *is a subalgebra of* $C_{n-1}(X)$*, for any* $n \in N$.
ii) $C^n(X)$ *is a subalgebra of* $C^{n-1}(X)$*, for any* $n \in N$.

Proof. i) We prove by induction on n. For $n = 1$, we have $C_1(X) = X'$ is a subalgebra of $C_0(X) = X$ by Theorem 2.5 (ii). Now assume that $C_n(X)$ is a subalgebra of $C_{n-1}(X)$, for some $n \ge 1$. Then $C_{n+1}(X) = (C_n(X))'$, which is a subalgebra of $(C_{n-1}(X))' = C_n(X)$ by Theorem 2.5 (iv).
ii) For $n = 1$, we have $C^1(X) = X'$ is a subalgebra of $C^0(X) = X$. Now assume that $C^n(X)$ is a subalgebra of $C^{n-1}(X)$, for some $n \ge 1$. Then $C^{n+1}(X) = [C^n(X), X] \subseteq [C^{n-1}(X), X] = C^n(X)$, so $C^{n+1}(X) \subseteq C^n(X)$. Let $x, y \in C^{n+1}(X)$, then there exist $a_i, b_i \in C^n(X)$ and $c_i, d_i \in X$ such that $x = [a_1, c_1] \to [a_2, c_2] \to ... \to [a_n, c_n], y = [b_1, d_1] \to [b_2, d_2] \to ... \to [b_m, d_m]$. Therefore $x \to y = ([a_1, c_1] \to [a_2, c_2] \to ... \to [a_n, c_n]) \to ([b_1, d_1] \to [b_2, d_2] \to ... \to [b_m, d_m])$. So $x \to y \in C^{n+1}(X)$. Whence $C^{n+1}(X)$ is a subalgebra of $C^n(X)$. Hence the result holds for n in the both case. □

Theorem 3.9 *Let* X, Y *be two BCI-algebras. Then for subsets* A, C *of* X *and* B, D *of* Y *we have* $[A \times B, C \times D] = [A, C] \times [B, D]$.

Proof. Let $t \in [A \times B, C \times D]$. Therefore $t = [(a, b), (c, d)]$, where $(a, b) \in A \times B$ and $(c, d) \in C \times D$. Then

$$\begin{aligned} t = [(a,b),(c,d)] &= (((c,d) \to (a,b)) \to (1,1)) \\ &\to ((((a,b) \to (c,d)) \to (c,d)) \to (((c,d) \to (a,b)) \to (a,b))) \\ &= (((c \to a) \to 1), ((d \to b) \to 1)) \to ((((a \to c) \to c) \end{aligned}$$

$$
\begin{aligned}
&,((b \to d) \to d)) \to (((c \to a) \to a), ((d \to b) \to b))) \\
=&(((c \to a) \to 1), ((d \to b) \to 1)) \to ((((a \to c) \to c) \\
\to&((c \to a) \to a)), (((b \to d) \to d) \to ((d \to b) \to b))) \\
=&((((c \to a) \to 1) \to (((a \to c) \to c) \to ((c \to a) \to a))) \\
&,(((d \to b) \to 1) \to (((b \to d) \to d) \to ((d \to b) \to b)))) \\
=&([a, c], [b, d]) \in [A, C] \times [B, D].
\end{aligned}
$$

Therefore $[A \times B, C \times D] \subseteq [A, C] \times [B, D]$.
Now, let $t \in [A, C] \times [B, D]$. Then $t = (x, y)$ for some $x \in [A, C]$ and $y \in [B, D]$. Then there exist $a \in A, b \in B, c \in C$ and $d \in D$ such that $x = [a, c]$ and $y = [b, d]$. So $t = (x, y) = ([a, c], [b, d]) = [(a, b), (c, d)] \in [A \times B, C \times D]$. Thus $[A, C] \times [B, D] \subseteq [A \times B, C \times D]$. Hence $[A \times B, C \times D] = [A, C] \times [B, D]$. □

Theorem 3.10 *Let X and Y be two BCI-algebras. Then for all non-negative integer n, we have*
i) $C_n(X \times Y) = C_n(X) \times C_n(Y)$,
ii) $C^n(X \times Y) = C^n(X) \times C^n(Y)$.

Proof. i) We prove by induction on n, for $n = 0$, $C_0(X \times Y) = X \times Y = C_0(X) \times C_0(Y)$. Now assume that $C_{n-1}(X \times Y) = C_{n-1}(X) \times C_{n-1}(Y)$, for some positive integer n. Then

$$
\begin{aligned}
C_n(X \times Y) =&[C_{n-1}(X \times Y), C_{n-1}(X \times Y)] \\
=&[C_{n-1}(X) \times C_{n-1}(Y), C_{n-1}(X) \times C_{n-1}(Y)] \\
=&[C_{n-1}(X), C_{n-1}(X)] \times [C_{n-1}(Y), C_{n-1}(Y)] \\
=&C_n(X) \times C_n(Y).
\end{aligned}
$$

ii) We proceed by induction on n. For $n = 0$, $C^0(X \times Y) = X \times Y = C^0(X) \times C^0(Y)$. Now assume that $C^{n-1}(X \times Y) = C^{n-1}(X) \times C^{n-1}(Y)$, for some positive integer n. Then

$$
\begin{aligned}
C^n(X \times Y) =&[C^{n-1}(X \times Y), X \times Y] \\
=&[C^{n-1}(X) \times C^{n-1}(Y), X \times Y] \\
=&[C^{n-1}(X), X] \times [C^{n-1}(Y), Y] \\
=&C^n(X) \times C^n(Y).
\end{aligned}
$$

Hence the result holds for n. □

Corollary 3.11 *The product of nilpotent BCI-algebras of type* 2 *(solvable) is a nilpotent BCI-algebra of type* 2 *(solvable).*

Remark 3.1 We consider the BCI-algebra $X = \{a, b, c, d, 1\}$ from Example 3.2 (i). X is nilpotent but is not nilpotent of type 2. Thus every nilpotent need not be a nilpotent of type 2 in general. Also the BCI-algebra C^* in Example 2.2 is nilpotent of type 2 but is not nilpotent, since $1 \div x^n \neq 1$ for any positive integer n and any real number $x \neq \pm 1$.

Lemma 3.12 *$[x, y]$ is a positive element of X, for all $x, y \in X$.*

Proof.

$$
\begin{aligned}
[x,y] \to 1 = &(((y \to x) \to 1) \to (((x \to y) \to y) \to ((y \to x) \to x))) \to 1 \\
=&(((y \to x) \to 1) \to 1) \to ((((x \to y) \to y) \to 1) \to \\
\to&(((y \to x) \to x) \to 1)) \\
=&(((y \to 1) \to (x \to 1)) \to 1) \\
\to&((((x \to y) \to 1) \to (y \to 1)) \to (((y \to x) \to 1) \to (x \to 1))) \\
=&(((y \to 1) \to (x \to 1)) \to 1) \to ((((x \to 1) \to (y \to 1)) \\
\to&(y \to 1)) \to (((y \to 1) \to (x \to 1)) \to (x \to 1))) \\
=&(((y \to 1) \to 1) \to ((x \to 1) \to 1)) \to ((((x \to 1) \\
\to&(y \to 1)) \to (y \to 1)) \to (((y \to 1) \to (x \to 1)) \to (x \to 1))) \\
=&(((x \to 1) \to (y \to 1)) \to (y \to 1)) \to ((((y \to 1) \to 1) \\
\to&((x \to 1) \to 1)) \to (((y \to 1) \to (x \to 1)) \to (x \to 1))) \\
=&(((x \to 1) \to (y \to 1)) \to (y \to 1)) \to (((y \to 1) \to (x \to 1)) \\
\to&((((y \to 1) \to 1) \to ((x \to 1) \to 1)) \to (x \to 1))) \\
=&(((x \to 1) \to (y \to 1)) \to (y \to 1)) \to (((y \to 1) \to (x \to 1)) \\
\to&(x \to (((y \to 1) \to 1) \to 1) \to (((x \to 1) \to 1) \to 1)) \\
=&(((x \to 1) \to (y \to 1)) \to (x \to 1)) \to (x \to (((y \to 1) \to (x \to 1)) \\
\to&((((y \to 1) \to 1) \to 1) \to ((x \to 1) \to 1) \to 1))) \\
\to&(((x \to 1) \to (y \to 1)) \to (x \to 1)) \\
=&(((x \to 1) \to (y \to 1)) \to (y \to 1)) \\
\to&(x \to (((y \to 1) \to (x \to 1)) \to ((y \to 1) \to (x \to 1)))) \\
=&(((x \to 1) \to (y \to 1)) \to (y \to 1)) \to (x \to 1) \\
=&x \to ((((x \to 1) \to (y \to 1)) \to (y \to 1)) \to 1) \\
=&x \to (y \to (((x \to 1) \to (y \to 1)) \to 1) \to 1) \\
=&x \to ((y \to 1) \to (((x \to 1) \to 1) \to 1) \to (((y \to 1) \to 1) \to 1)) \\
=&x \to ((y \to 1) \to ((x \to 1) \to (y \to 1))) \\
=&x \to ((x \to 1) \to ((y \to 1) \to (y \to 1))) \\
=&x \to ((x \to 1) \to 1) \\
=&1.
\end{aligned}
$$

Then $[x, y]$ is a positive element of X. $\square$

Lemma 3.13 $[x, y] \to 1 = [x \to 1, y \to 1] = 1$ *, for all $x, y \in X$.*

Proof. By *Definition 2.3* and *Lemma 2.7* we obtain

$$\begin{aligned}[x \to 1, y \to 1] =& (((y \to 1) \to (x \to 1)) \to 1) \to ((((x \to 1) \to (y \to 1)) \\ &\to (y \to 1)) \to (((y \to 1) \to (x \to 1)) \to (x \to 1))) \\ =& (((y \to x) \to 1) \to (((x \to y) \to y) \to ((y \to x) \to x))) \to 1 \\ =& [x, y] \to 1 = 1. \quad \square\end{aligned}$$

Lemma 3.14 *For all positive integers m, n, we have*
i) $C_n(C_m(X)) = C_{n+m}(X)$,
ii) $C^n(C_m(X)) = C_{n+m}(X)$.

Proof. i) Induction on n. For $n = 1$, $C_1(C_m(X)) = [C_m(X), C_m(X)] = C_{m+1}(X) = C_{1+m}(X)$ by *definition*. Assume the result is true for all m, and for a specific n. Want $C_{n+1}(C_m(X)) = C_{n+m+1}(X)$. But

$$\begin{aligned}C_{n+1}(C_m(X)) =& [C_n(C_m(X)), C_n(C_m(X))] \\ =& [C_{n+m}(X), C_{n+m}(X)] \\ =& C_{n+m+1}(X).\end{aligned}$$

ii) We prove by induction on n, for $n = 1$, $C^1(C_m(X)) = C_1(C_m(X)) = [C_m(X), C_m(X)] = C_{m+1}(X) = C_{1+m}(X)$.
Let $C^{n-1}(C_m(X)) = C_{n-1+m}(X)$, for all m and for a specific n. Then

$$\begin{aligned}C^n(C_m(X)) =& [C^{n-1}(C_m(X)), C_m(X)] \\ =& [C_{n-1+m}(X), C_m(X)] \\ =& [C_{n-1}(C_m(X)), C_m(X)] \\ =& C_n(C_m(X)) \\ =& C_{n+m}(X). \quad \square\end{aligned}$$

Remark 3.2 It is possible, for some positive integers m, n, to have
i) $C_{n+m}(X) \neq C_n(C^m(X)) \neq C^{n+m}(X)$,
ii) $C^n(C_m(X)) \neq C^{n+m}(X)$,
iii) $C_{n+m}(X) \neq C^n(C^m(X)) \neq C^{n+m}(X)$.
For example, if we consider the BCI-algebra $X = \{a, b, c, d, 1\}$ from Example 3.2 (i), we have $C_1(C^4(X)) = [C^4(X), C^4(X)] = [\{a, 1\}, \{a, 1\}] = \{1\} \neq \{a, 1\} = C^5(X)$. Also $C^2(C_2(X)) = \{1\} \neq C^4(X)$.

Any finite BCK-algebra is solvable [2]. In the following theorem we show that this statement holds for finite BCI-algebras.

Theorem 3.15 *Any finite BCI-algebra is solvable.*

Proof. Let X be a finite BCI-algebra and B be the set of all it positive elements. Suppose that $|X| = n \geq 2$.
Case I: Let X be a BCK-algebra and M be the set of maximal elements of

X. Then $C_1(X)$ is a subalgebra contained in $X \setminus M$ by Theorem 2.5 (ii) and Lemma 2.9. Hence $|X| > |C_1(X)|$. By induction $|X| > |C_1(X)| > |C_2(X)| > ... > |C_k(X)| = 1$ for some $k \leq n$ (for more details see [2]).
Case II: If X is not a BCK-algebra, then there exists at least an element $x \in X$ such that $x \to 1 \neq 1$. Since $[x, y]$ is a positive element for every $x, y \in X$, $C_1(X)$ is a subalgebra contained in B. Since B itself is a BCK-algebra, by case I, B is solvable. Therefore $C_1(X)$ is solvable. Hence there exists $n \in N$ such that $C_n(C_1(X)) = \{1\}$. So $C_{n+1}(X) = \{1\}$. Therefore X is solvable. □

The following example shows that there are finite BCK/BCI-algebras that are not nilpotent of type 2, in general.

Example 3.3 [4] Let $X = \{a, b, c, d, 1\}$. Define " $\to$ " on X by

$\to$	a	b	c	d	1
a	1	a	b	d	1
b	1	1	a	d	1
c	1	1	1	d	1
d	1	1	1	1	1
1	a	b	c	d	1

Then $(X, \to, 1)$ is a BCK-algebra. We obtain $C^n(X) = \{a, b, c, 1\}$, for every $n \geq 1$. Therefore X is not nilpotent of type 2.

Corollary 3.16 *There are proper inclusions of classes:*
$\{commutative\ BCI-algebras\} \subseteq \{nilpotent\ BCI-algebras\ of\ type\ 2\} \subseteq \{solvable\ BCI-algebras\}$

Proof. By Corollary 3.5, Theorem 3.6 and Example 3.2 is obvious. □

Corollary 3.17 *Suppose that X is nilpotent of type 2. Then the nilpotency class of X is less or equal n iff $[[x_1, x_2, ..., x_{n-1}], x_n] = 1$, for all $x_i \in X$.*

Proof. X is nilpotent of type 2 of class less or equal n if and only if $C^n(X) = \{1\}$ if and only if $[C^{n-1}(X), X] = \{1\}$ if and only if $[[x_1, x_2, ..., x_{n-1}], x_n] = 1$, for all $x_i \in X$. □

4 Some properties of $C_n(X), C^n(X)$

In this section, we investigate more properties of $C_n(X), C^n(X)$.

Theorem 4.1 *Let A and B are two subsets of X and f be a homomorphism of X to a BCI-algebra Y. Then for every non-negative integer n, we have*
i) $f([A, B]) = [f(A), f(B)]$,
ii) $f(C^n(A)) = C^n(f(A))$,
iii) $f(C_n(A)) = C_n(f(A))$.

Proof. *i*) Let $y \in f([A,B])$. Then there exists $x \in [A,B]$ such that $y = f(x)$. So $x = [a,b]$ for some $a \in A$ and $b \in B$ and hence $y = f(x) = f([a,b]) = [f(a), f(b)] \in [f(A), f(B)]$. Therefore $f([A,B]) \subseteq [f(A), f(B)]$.
Conversely, let $y \in [f(A), f(B)]$. Therefore there exist $y_1 \in f(A)$ and $y_2 \in f(B)$ such that $y = [y_1, y_2]$. Thus, $y_1 = f(a)$ and $y_2 = f(b)$ for some $a \in A$ and $b \in B$. Hence $y = [y_1, y_2] = [f(a), f(b)] = f([a,b]) \in f([A,B])$. Whence $[f(A), f(B)] \subseteq f([A,B])$. Therefore $[f(A), f(B)] = f([A,B])$.
ii) We prove by induction on n. If $n = 0$, then $f(C^0(A)) = f(A) = C^0(f(A))$. Now assume that $f(C^{n-1}(A)) = C^{n-1}(f(A))$, for some $n \in N$. Then

$$\begin{aligned} f(C^n(A)) =& f([C^{n-1}(A), A]) \\ =& [f(C^{n-1}(A)), f(A)] \\ =& [C^{n-1}(f(A)), f(A)] \\ =& C^n(f(A)). \end{aligned}$$

iii) We prove by induction on n. If $n = 0$, then $f(C_0(A)) = f(A) = C_0(f(A))$. Now assume that $f(C_{n-1}(A)) = C_{n-1}(f(A))$, for some $n \in N$. Then

$$\begin{aligned} f(C_n(A)) =& f([C_{n-1}(A), C_{n-1}(A)]) \\ =& [f(C_{n-1}(A)), f(C_{n-1}(A))] \\ =& [C_{n-1}(f(A)), C_{n-1}(f(A))] \\ =& C_n(f(A)). \quad \square \end{aligned}$$

Theorem 4.2 *Let f be an isomorphism from X to a BCI-algebra Y. Then X is nilpotent of type 2 if and only if Y is nilpotent of type 2.*

Proof. Since $f(X) = Y$, $C^k(Y) = C^k(f(X)) = f(C^k(X))$, for any $k \in N$. But X is nilpotent of type 2, then there exists $n \in N$ such that $C^n(X) = \{1\}$. Therefore, $\{1\} = f(\{1\}) = f(C^n(X)) = C^n(Y)$. Hence, Y is a nilpotent of type 2.
Conversely, let Y be a nilpotent BCI-algebra of type 2. Then $f(C^n(X)) = C^n(f(X)) = C^n(Y) = \{1\}$, for some $n \in N$. Therefore $f(C^n(X)) = \{1\} = f(\{1\})$. Hence $C^n(X) = \{1\}$, that is, X is nilpotent of type 2. $\square$

Theorem 4.3 *Let F be a filter of X. Then $C^n(X/F) = C^n(X)/F$ and $C_n(X/F) = C_n(X)/F$, for any positive integer n.*

Proof. For $n = 1$, we have $C^1(X/F) = [X/F, X/F] = [X,X]/F = C^1(X)/F$. Also $C_1(X/F) = [X/F, X/F] = [X,X]/F = C_1(X)/F$.
Now inductively assume that $C^{n-1}(X/F) = C^{n-1}(X)/F$ and $C_{n-1}(X/F) = C_{n-1}(X)/F$, for some $n \geq 1$. Then

$$\begin{aligned} C^n(X/F) =& [C^{n-1}(X/F), X/F] \\ =& [C^{n-1}(X)/F, X/F] \\ =& [C^{n-1}(X), X]/F \\ =& C^n(X)/F. \end{aligned}$$

and

$$\begin{aligned}C_n(X/F) =&[C_{n-1}(X/F), C_{n-1}(X/F)]\\ =&[C_{n-1}(X)/F, C_{n-1}(X)/F]\\ =&[C_{n-1}(X), C_{n-1}(X)]/F\\ =&C_n(X)/F. \quad \square\end{aligned}$$

Lemma 4.4 *The quotient of a nilpotent BCI-algebra of type 2 (solvable) is a nilpotent BCI-algebra of type 2 (solvable).*

Proof. Let F be a filter of a nilpotent BCI-algebra X of type 2. Then for some $n \in N$, $C^n(X) = \{1\}$. Hence $C^n(X/F) = C^n(X)/F = \{1\}/F = \{F\}$. Therefore X/F is a nilpotent BCI-algebra of type 2.
Also, if X is a solvable, then for some $n \in N$, $C_n(X) = \{1\}$. Hence $C_n(X/F) = C_n(X)/F = \{1\}/F = \{F\}$. Therefore X/F is a solvable BCI-algebra. □

Theorem 4.5 *Subalgebras and homomorphic images of nilpotent BCI-algebra of type 2 are nilpotent of type 2.*

Proof. Let Y be a subalgebra of X. Then $Y \subseteq X$ and so $C^n(Y)$ is a subset of $C^n(X)$, for all $n \in N$. Since X is a nilpotent of type 2, there exists $m \in N$ such that $C^m(X) = \{1\}$ and so $C^m(Y) = \{1\}$. Thus, Y is a nilpotent BCI-algebra of type 2. Now, let $(X, \to, 1)$ and $(Y, \to', 1')$ be two BCI-algebras and X is nilpotent of type 2, $f : X \longrightarrow Y$ be an epimorphism from X onto Y. Then for any non-negative integer n, we have $f(C^n(X)) = C^n(Y)$. Hence $\{1\} = f(\{1\}) = f(C^n(X)) = C^n(Y)$. So, $f(C^n(X)) = C^n(Y) = \{1\}$. Thus Y is a nilpotent BCI-algebra of type 2. □

Lemma 4.6 *Let F be a filter of X. Then F and X/F are solvable iff X is a solvable BCI-algebra.*

Proof. Let f be the natural homomorphism from X onto X/F. Since X/F is solvable, for some $n \in N$, $f(C_n(X)) = C_n(f(X)) = C_n(X/F) = F$. Hence $C_n(X)$ is a subalgebra of $Ker(f) = F$. Since F is a solvable, there exists a positive integer m such that $C_m(F) = \{1\}$. Therefore, $C_m(C_n(X)) = C_{m+n}(X) \subseteq C_m(F) = \{1\}$. Hence $C_{m+n}(X) = \{1\}$. i.e., X is solvable.
Conversely, let X be solvable. Therefore, for some $n \in N$, $C_n(X) = \{1\}$. Since $F \subseteq X$, $C_n(F) \subseteq C_n(X) = \{1\}$. Then $C_n(F) = \{1\}$. Hence F is solvable, also X/F is a solvable by Lemma 4.4. □

Remark 4.1 In Example 3.2 (i), $F = \{a, 1\}$ is a filter of X which is nilpotent of type 2. The quotient BCI-algebra induced by this filter is $X/F = \{C_1, C_b, C_c, C_d\}$. We obtain $C^1(X/F) = C^1(X)/F = \{C_1\}$. Therefore F and X/F are nilpotent of type 2, but X is not a nilpotent BCI-algebra of type 2.

Theorem 4.7 *Let $(X_i, \to_i, 1_i)$ be an indexed family of BCI-algebras. Then*
i) $\prod_{i\in I} X_i$ is nilpotent of type 2 if and only if X_i is nilpotent of type 2, for all $i \in I$.
ii) $\prod_{i\in I} X_i$ is solvable if and only if X_i is solvable, for all $i \in I$.

Proof. Let $\prod_{i\in I} X_i$ be nilpotent of type 2, then there exists $n \in N$ such that $C^n(\prod_{i\in I} X_i) = \{1\}$. The project mapping $p : \prod_{i\in I} X_i \to X_i$ is an epimorphism. Therefore $\{1\} = p(\{1\}) = p(C^n(\prod_{i\in I} X_i)) = C^n(p(\prod_{i\in I} X_i)) = C^n(X_i)$. Hence X_i is nilpotent of type 2.
Conversely, let X_i be nilpotent of type 2. If $C^1(X_i) = \{1\}$, then we show that $C^1(\prod_{i\in I} X_i) = \{1\}$. Suppose that $t \in C^1(\prod_{i\in I} X_i)$, then there exist sequences $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$ in $\prod_{i\in I} X_i$ such that

$$\begin{aligned} t =&[(a_i)_{i\in I}, (b_i)_{i\in I}] \\ =&([a_1, b_1], [a_2, b_2], ...) \\ =&(1_1, 1_2, ...) \\ =&1. \end{aligned}$$

Thus, $t = 1$ and hence $C^1(\prod_{i\in I} X_i) = \{1\}$. By induction we can show that if $C^n(X_i) = \{1\}$, then $C^n(\prod_{i\in I} X_i) = \{1\}$. So $\prod_{i\in I} X_i$ is nilpotent of type 2. The proof (ii) is similar. □

Lemma 4.8 *For every $k \in N$, $N_1(X) \subseteq N_k(X) \subseteq N(X)$ and $N(X) = \bigcup_{k\in N} N_k(X)$.*

Proof. Let $x \in N_1(X)$. Then $x \to 1 = 1$. Since $x^k \to 1 = \underbrace{x \to (... \to (x \to (x \to 1))...)}_{k-times} = ... = x \to 1 = 1$, $x \in N_k(X)$. Therefore $N_1(X) \subseteq N_k(X)$. If $x \in N_k(X)$ for $k \in N$, then $x^k \to 1 = 1$. Therefore x is a nilpotent element of X. i.e., $x \in N(X)$. So, $N_k(X) \subseteq N(X)$ for any positive integer k.
Obviously, $\bigcup_{k\in N} N_k(X) \subseteq N(X)$. Let $x \in N(X)$. Then there exists $k \in N$ such that $x^k \to 1 = 1$. Then $x \in N_k(X)$, hence $x \in \bigcup_{k\in N} N_k(X)$. Therefore $N(X) \subseteq \bigcup_{k\in N} N_k(X)$. □

Lemma 4.9 *Let $f : X \to Y$ be a monomorphism from X to a BCI-algebra Y. Then $f(N_k(X)) = N_k(f(X))$ for all $k \in N$ and $f(N(X)) = N(f(X))$.*

Proof. Let $y \in f(N_k(X))$. Then there exists $x \in N_k(X)$ such that $y = f(x)$. Since $x \in N_k(X)$, $x^k \to 1 = 1$. Hence $f(1) = f(x^k \to 1) = f(x^k) \to f(1) = (f(x))^k \to f(1)$. Therefore $y = f(x) \in N_k(f(X))$. So, $f(N_k(X)) \subseteq N_k(f(X))$.
Conversely, let $y = f(x) \in N_k(f(X))$. Then $(f(x))^k \to f(1) = f(1) = f(x^k \to 1)$. Since f is one to one, $x^k \to 1 = 1$. Whence $x \in N_k(X)$, so $f(x) \in f(N_k(X))$. Thus $N_k(f(X)) \subseteq f(N_k(X))$. Then $f(N_k(X)) = N_k(f(X))$.Since, $N(X) = \bigcup_{k\in N} N_k(X)$, $f(N(X)) = f(\bigcup_{k\in N} N_k(X)) = \bigcup_{k\in N} f(N_k(X))$
$= \bigcup_{k\in N} N_k(f(X)) = N(f(X))$. □

Theorem 4.10 *Let $\{A_i : i \in I\}$ be a family of BCI-algebras. Then* $N(\prod_{i\in I} A_i) = \prod_{i\in I} N(A_i)$.

Proof. Suppose that $x \in \prod_{i\in I} N(A_i)$. Then $x = (x_1, ..., x_n)$ where x_i is nilpotent in A_i. Thus, for each $1 \leq i \leq n$, there exists $m_i \in N$ such that $x_i^{m_i} \to 1_i = 1_i$. Put $m = m_1 m_2 ... m_n$. Then $x^m \to 1 = 1$, that is $x \in N(\prod_{i\in I} A_i)$. Therefore $\prod_{i\in I} N(A_i) \subseteq N(\prod_{i\in I} A_i)$.
Conversely, let $x \in N(\prod_{i\in I} A_i)$. Then $x^m \to 1 = 1$ for some $m \in N$. But $x^m \to 1 = (x_1, x_2, ..., x_n)^m \to (1_1, 1_2, ..., 1_n) = (1_1, 1_2, ..., 1_n)$. Hence $x_i^m \to 1_i = 1_i$, for $i = 1, 2, ..., n$. Therefore $x_i \in N(A_i)$ and consequence $x = (x_1, x_2, ..., x_n) \in \prod_{i\in I} N(A_i)$. □

5 Characteristic subalgebras

In this section, we introduce the notion of characteristic subalgebra of BCI-algebras and investigate the relation between this concept and the derived subalgebras, nil radicals and k-nil radicals.

Definition 5.1 *A subalgebra Y is called characteristic subalgebra of X if every automorphism of X maps Y to itself. That is, if $f \in Aut(X)$, then $f(Y) \subseteq Y$.*

Obviously, $X, \{1\}$ are characteristic subalgebras of X.

Example 5.1 Let $X = \{a, b, c, 1\}$. Define $\to$ on X as follows:

$\to$	a	b	c	1
a	1	b	b	1
b	a	1	a	1
c	1	1	1	1
1	a	b	c	1

Then $(X, \to, 1)$ is a BCK-algebra and so a BCI-algebra. The functions $f_1(x) = x$ and f_2 defined by

$$f_2(x) = \begin{cases} b \text{ if } x = a \\ a \text{ if } x = b \\ x \text{ otherwise} \end{cases}$$

are all automorphisms of X. Then $Y = \{a, b, 1\}$ is a subalgebra of X. These automorphisms map Y to itself. Therefore, Y is a characteristic subalgebra of X. Also $Z = \{a, 1\}$ is a subalgebra of X. f_2 maps Z to $\{b, 1\}$. Therefore Z is not a characteristic subalgebra of X.

Theorem 5.2 *The derived subalgebra of X is a characteristic subalgebra.*

Proof. Obviously, X' is a subalgebra of X. Let f be an automorphism of X. Then $f([X, X]) = [f(X), f(X)]$, thus $f(X') = (f(X))' = X'$. It follows that $[X, X]$ is a characteristic subalgebra of X. □

Theorem 5.3 *For any $k \in N$, $N_k(X)$ is a characteristic subalgebra of X.*

Proof. Clearly, $1 \in N_k(X)$, then $N_k(X)$ is non-empty. Put $x, y \in N_k(X)$. Then $x^k \to 1 = 1 = y^k \to 1$. Hence, $(x \to y)^k \to 1 = (x^k \to 1) \to (y^k \to 1) = 1 \to 1 = 1$. Therefore $x \to y \in N_k(X)$ and $N_k(X)$ is a subalgebra of X. Now, suppose that f is an automorphism of X and $x \in N_k(X)$, then $x^k \to 1 = 1$. Hence $1 = f(1) = f(x^k \to 1) = f(x^k) \to f(1) = (f(x))^k \to 1$. Therefore $f(x) \in N_k(X)$. It follows that $N_k(X)$ is a characteristic subalgebra of X. □

Theorem 5.4 *The nil-radical $N(X)$ is a characteristic subalgebra of X.*

Proof. Since 1 is nilpotent, $N(X)$ is non-empty. Let $x, y \in N(X)$. Then there exist $m, n \in N$ such that $x^m \to 1 = 1$ and $y^n \to 1 = 1$. Then $x^{mn} \to 1 = (x^m \to 1) \to (x^{m(n-1)} \to 1) = x^{m(n-1)} \to 1 = ... = x^m \to 1 = 1$. Likewise, $y^{mn} \to 1 = 1$. By Lemma 2.7, we get that $(x \to y)^{mn} \to 1 = (x^{mn} \to 1) \to (y^{mn} \to 1) = 1 \to 1 = 1$. Hence $x \to y \in N(X)$. Thus $N(X)$ is a subalgebra of X. Suppose that f is an automorphism of X and $x \in N(X)$, then $x^k \to 1 = 1$ for some $k \in N$. Hence $1 = f(1) = f(x^k \to 1) = f(x^k) \to f(1) = (f(x))^k \to 1$. Therefore $f(x) \in N(X)$. It follows that $N(X)$ is a characteristic subalgebra of X. □

Example 5.2 i) In Example 5.1, $N_k(X) = X$, for any $k \geq 1$ and $N(X) = X$. Also $C^n(X) = C_n(X) = \{1\}$, for any $n \geq 1$. Thus $C^n(X) \neq N_k(X)$, $C_n(X) \neq N_k(X)$ and $C^n(X) \neq N(X)$, $C_n(X) \neq N(X)$ for every $k, n \in N$.
ii) Let $X = [0, 1]$. Define " $\to$ " on X by

$$x \to y = \begin{cases} 1 \text{ if } x \leq y \\ y \text{ otherwise} \end{cases}$$

Then $(X, \to, 1)$ is a BCI-algebra. $C^1(X) = X = [0, 1], C^2(X) = [X, X] = [0, 1), ..., C^k(X) = [X, C^{k-1}(X)] = [0, 1)$ for any $k \in N$ and $C_1(X) = X, C_2(X) = [X, X] = ... = C_k(X) = [C_{k-1}(X), C_{k-1}(X)] = [0, 1)$.
Since, $x \to 1 = 1$ for any $x \in X$, X is nilpotent. But X is not nilpotent of type 2, because $C^k(X) = [0, 1) \neq \{1\}$ for any $k \in N$.

Theorem 5.5 *$C^n(X)$ and $C_n(X)$ are characteristic subalgebras of X.*

Proof. We prove by induction on n. If $n = 0$, then $C_0(X) = X$ that is a subalgebra of X. Now assume that $C_k(X)$ for any $k < n$ is a subalgebra of X. Therefore, $C_n(X) = C_1(C_{n-1}(X))$, which is a subalgebra of $C_1(X)$. Since $C_1(X)$ is a subalgebra of X, $C_n(X)$ is a subalgebra of X.
Clearly, $0 \in C^n(X)$, then $C^n(X)$ is non-empty. Let $x, y \in C^n(X)$. Then there exist $a_i, b_i, c_i, d_i \in C^{n-1}(X)$ such that $x = \prod[a_i, b_i] = x_1 \to ... \to x_m$ and $y = \prod[c_i, d_i] = y_1 \to ... \to y_l$. Hence, $x \to y = \prod[a_i, b_i] \to \prod[c_i, d_i] = (x_1 \to ... \to x_m) \to (y_1 \to ... \to y_l)$. Therefore $x \to y \in C^n(X)$. Whence $C^n(X)$ is a subalgebra of X.
Now, suppose that f is an automorphism of X. Since $f(C^n(X)) = C^n(f(X))$

$= C^n(X)$ and $f(C_n(X)) = C_n(f(X)) = C_n(X)$, for every $n \in N$, then f maps $C^n(X)$ to $C^n(X)$ also f maps $C_n(X)$ to itself. Hence $C^n(X)$ and $C_n(X)$ are characteristic subalgebras of X. $\square$

The following example shows that the converse of Theorems 5.2, 5.3, 5.4 and 5.5 may not always hold.

Example 5.3 i) In Example 5.1, Y is a characteristic subalgebra of X, but it is not a commutator subalgebra of X, since $C^n(X) = C_n(X) = \{1\}$, for every positive integer n.
ii) Consider the BCI-algebra C^* in Example 2.2. It is clear that C^* is a characteristic subalgebra. We obtain that $N_1(C^*) = \{1\}$, $N_2(C^*) = \{-1, 1\}$,...,$N_n(C^*) = \{z \in C^* : z^n = 1\}$. Thus $C^* \neq N_k(C^*)$ for any $k \in N$, also $C^* \neq N(C^*)$.
iii) In Example 5.1 consider $Y = \{a, b, 1\}$. Y is a characteristic subalgebra of X, but $\{1\} = C^n(X) = C_n(X) \neq Y$ for all $n \in N$.

Theorem 5.6 *If A is a characteristic subalgebra of X, then $C^n(A), C_n(A)$ $N_k(A)$ and $N(A)$ are characteristic subalgebras of X.*

Proof. Let A be a characteristic subalgebra of X. Then $C^n(A), C_n(A)$ $N_k(A)$ and $N(A)$ are subalgebras of X. Suppose that $f \in Aut(X)$, then $f(A) \subseteq A$. Therefore $f(C^n(A)) = C^n(f(A)) \subseteq C^n(A)$. Also $f(C_n(A)) = C_n(f(A)) \subseteq C_n(A)$, for any $n \in N$. Hence $C^n(A)$ and $C_n(A)$ are characteristic subalgebras of X. Since $f(N_k(A)) \subseteq N_k(f(A)) \subseteq N_k(A)$ and $f(N(A)) \subseteq N(f(A)) \subseteq N(A)$, $N_k(A)$ and $N(A)$ are characteristic subalgebras of X. $\square$

6 Conclusion

The notion of nilpotent BCI-algebras was formulated first by W. Huang. In this paper, we characterized the notion of nilpotent BCI-algebras of type 2 and we studied relationships between nilpotent BCI-algebras of type 2 and nilpotent BCI-algebras. The results of this paper show that:
(1) Every nilpotent BCI-algebras of type 2 is solvable.
(2) Any finite BCI-algebra is solvable but is not nilpotent of type 2 in general.
(3) commutative BCI algebras $\subseteq$ nilpotent BCI-algebras of type 2 $\subseteq$ solvable BCI-algebras.
(4) $C^n(X)$, $C_n(X)$, $N_k(X)$ and $N(X)$ are characteristic subalgebras of X.
Some important issues for future work are:
i) Considering concept nilpotent BCI-algebras of type 2 in simple BCI-algebras and other specific BCI-algebras.
ii) Making Engle BCI-algebras similar with Engle groups by using nilpotent BCI-algebras of type 2.
iii) Considering filters and subalgebras in nilpotent BCI-algebras of type 2.

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Authors

Ardavan Najafi,
Department of Pure Mathematics,
Faculty of Mathematics and Computer,
Shahid Bahonar University of Kerman,
Kerman, Iran

Esfandiar Eslami,
Department of Pure Mathematics,
Faculty of Mathematics and Computer,
Shahid Bahonar University of Kerman,
Kerman, Iran

Arsham Borumand Saeid (Corresponding author),
Department of Pure Mathematics,
Faculty of Mathematics and Computer,
Shahid Bahonar University of Kerman,
Kerman, Iran
E-mail: arsham@uk.ac.ir