

The center map of a centroaffine ruled surface

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Abstract We determine a nondegenerate centroaffine ruled surface such that the image of its center map lies on a plane containing the origin. A centroaffine minimal surface such that the image of the center map lies on a plane not containing the origin is also exhibited. Moreover, we study a nondegenerate centroaffine ruled surface such that the image of its center map lies on a curve, and illustrate a one parameter family of such surfaces having the same center map.

Keywords centroaffine surface · ruled surface · center map

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1 Introduction

It is an elementary fact in Euclidean differential geometry that the unit normals of a hypersphere in the Euclidean space meet the center. In equiaffine differential geometry, we consider the Blaschke normals for affine hypersurfaces in the affine space, instead of unit normals. In this setting, affine hyperspheres can be defined as Blaschke hypersurfaces whose affine shape operator is a scalar operator. An affine hypersphere with nonzero affine shape operator is called to be proper, and it is proved that the Blaschke normals of a proper affine hypersphere meet at one point, called the center. See [8, 12] for more about affine hyperspheres as well as affine differential geometry.

As a generalization of the center of an affine hypersurface, Furuhashi and Vrancken [4] introduced a notion of center map and studied affine hypersurfaces whose center map is centroaffinely congruent with the original hypersurface, called to be self congruent. In particular, they showed that a definite centroaffine surface in the three-dimensional vector space \mathbf{R}^3 which has a nondegenerate center map is self congruent if and only if the Tchebychev operator vanishes. See also [2, 5, 6, 13, 14] for more about related topics.

On the other hand, a class of ruled surfaces has importance in the surface theory in \mathbf{R}^3 . Yu, Yang and Liu [15] studied centroaffine ruled surfaces and obtained some classification results for linear Weingarten centroaffine ruled surfaces and proved that a nondegenerate centroaffine ruled surface is centroaffine minimal if and only if the curvature of the centroaffine metric is constant. In [5], Hu proved that a nondegenerate centroaffine ruled surface with nondegenerate centroaffine director curve whose center map is a point or a curve is a ruled surface with vanishing Pick function.

In this paper, we shall study the center map of a centroaffine ruled surface and characterize some centroaffine ruled surfaces by the center map geometrically as follows:

Theorem 1.1 *Let $f : M \rightarrow \mathbf{R}^3$ be a nondegenerate centroaffine ruled surface whose center map is a point or a curve. Then f is given by*

$$f(x, y) = A'(x) + yA(x), \quad (1.1)$$

where A is an \mathbf{R}^3 -valued function of x only such that

$$\det \begin{pmatrix} A \\ A' \\ A'' \end{pmatrix} \neq 0. \quad (1.2)$$

Theorem 1.2 *Let $f : M \rightarrow \mathbf{R}^3$ be a nondegenerate centroaffine ruled surface whose center map is a piece of a plane through the origin. Then, f is*

given by (1.1) such that $\det \begin{pmatrix} A \\ A' \\ A'' \end{pmatrix}$ is nonzero constant, or centroaffinely congruent with

$$f(u, v) = (e^{u+2v}, e^{u-v}, e^v). \quad (1.3)$$

2 The center map of an affine hypersurface

At first, we review a notion of center map of an affine hypersurface due to Furuhashi and Vrancken [4].

We express an n -dimensional hypersurface in the $(n + 1)$ -dimensional affine space \mathbf{R}^{n+1} by an immersion f from an n -dimensional manifold M into \mathbf{R}^{n+1} . If we choose a transversal vector field ξ along f , the Gauss-Weingarten formulas for f are given by

$$D_X f_* Y = f_* \nabla_X Y + h(X, Y)\xi, \quad D_X \xi = -f_* S X + \tau(X)\xi \quad (X, Y \in \mathfrak{X}(M)),$$

where D is the standard flat connection D on \mathbf{R}^{n+1} and $\mathfrak{X}(M)$ is the set of all vector fields on M . Then ∇ , h , S and τ define a torsion-free affine connection, a symmetric $(0, 2)$ -tensor field, a $(1, 1)$ -tensor field and a 1-form on M , respectively, called the induced connection, the affine fundamental form, the affine shape operator and the transversal connection form, respectively.

We fix a volume form ω on \mathbf{R}^{n+1} which is parallel with respect to D and define a volume form θ on M by

$$\theta(X_1, \dots, X_n) = \omega(f_*X_1, \dots, f_*X_n, \xi) \quad (X_1, \dots, X_n \in \mathfrak{X}(M)),$$

called the volume form induced by f and ξ . Then we have

$$\nabla_X \theta = \tau(X)\theta \quad (X \in \mathfrak{X}(M)).$$

We call f to be nondegenerate, definite or indefinite if h is nondegenerate, definite or indefinite, respectively. Note that the nondegeneracy of f is independent of the choice of ξ as well as the definiteness or the indefiniteness. If f is nondegenerate, we can find a transversal vector field ξ such that $\tau = 0$, i.e., $\nabla \theta = 0$. We call f with such ξ an equiaffine hypersurface. Moreover, we can find a unique ξ up to sign such that $\tau = 0$ and θ is equal to the volume form with respect to h . We call f with such ξ a Blaschke hypersurface, and ξ the Blaschke normal vector field of f . A Blaschke hypersurface is called an affine hypersphere if S is a scalar operator. An affine hypersphere is called to be proper or improper if S is nonzero or zero, respectively.

Let $f : M \rightarrow \mathbf{R}^{n+1}$ be an affine hypersurface. Using the Blaschke normal vector field ξ of f , we decompose f as

$$f = Z + r\xi, \tag{2.1}$$

where Z is an \mathbf{R}^{n+1} -valued function on M tangent to $f(M)$, and r is a function on M , called the equiaffine support function from the origin. Then we have the following:

Proposition 2.1 (cf. [4, Proposition 2.2]) *A Blaschke hypersurface f is a proper affine hypersphere if and only if Z is constant.*

Moreover, if f is a proper affine hypersphere, it is shown that Z is a point where the Blaschke normals meet, called the center. After these facts, Furuhashi and Vrancken called Z in (2.1) the center map of f .

3 Centroaffine hypersurfaces

A centroaffine hypersurface is an affine hypersurface given by an immersion $f : M \rightarrow \mathbf{R}^{n+1}$ with a transversal vector field ξ defined as the restriction of the radial vector field to f :

$$\xi = - \left(\sum_{i=1}^{n+1} x_i \frac{\partial}{\partial x_i} \right) \Big|_f,$$

where (x_1, \dots, x_{n+1}) are affine coordinates. Then the Gauss-Weingarten formulas for f are given by

$$D_X f_* Y = f_* \nabla_X Y + h(X, Y)\xi, \quad D_X \xi = -f_* X \quad (X, Y \in \mathfrak{X}(M)).$$

If f is nondegenerate, we call the affine fundamental form h the centroaffine metric, and denote the Levi-Civita connection by $\hat{\nabla}$. Then we have a $(1, 2)$ -tensor field C , a function J , a vector field T and a $(1, 1)$ -tensor field \mathcal{T} on M defined by

$$C = \nabla - \hat{\nabla}, \quad J = \frac{1}{n(n-1)} \|C\|_h^2, \quad T = \frac{1}{n} \operatorname{tr}_h C, \quad \mathcal{T} = \hat{\nabla} T,$$

called the difference tensor field, the Pick function, the Tchebychev vector field and the Tchebychev operator, respectively. We denote the traceless part of C by \tilde{C} , which is defined by

$$\tilde{C}(X, Y) = C(X, Y) - \frac{n}{n+2} (h(T, X)Y + h(T, Y)X + h(X, Y)T)$$

for $X, Y \in \mathfrak{X}(M)$. Then the generalized Pick function \tilde{J} is defined by

$$\tilde{J} = \frac{1}{n(n-1)} \|\tilde{C}\|_h^2.$$

Moreover, f is called to be centroaffine minimal if it is an extremal for the volume integral of h , which is equivalent to the condition that $\operatorname{tr} \mathcal{T} = 0$ ([16]).

A centroaffine hypersurface whose center map is centroaffinely congruent with the original hypersurface is called to be self congruent. Concerning the self congruency for centroaffine hypersurfaces, we have the following:

Theorem 3.1 ([4, Theorem 3.1]) *Let $f : M \rightarrow \mathbf{R}^{n+1}$ be a nondegenerate centroaffine hypersurface. If f is self congruent, then the Tchebychev operator of f vanishes identically. In particular, f is centroaffine minimal.*

In the case of $n = 2$ and f is definite, the converse of Theorem 3.1 is true:

Theorem 3.2 ([4, Theorem 2.10]) *Let $f : M \rightarrow \mathbf{R}^3$ be a definite centroaffine surface such that the center map is an immersion. If the Tchebychev operator of f vanishes identically, then f is self congruent.*

Remark 3.1 It is shown in [2, Theorem A.5] that the converse of Theorem 3.1 is not true in the case of $n = 2$ and f is indefinite. See also Example 5.3.

4 Centroaffine surfaces

In order to prove our main theorems, we have only to consider an indefinite centroaffine surface $f : M \rightarrow \mathbf{R}^3$ since any nondegenerate centroaffine ruled

surface is indefinite. Then we can take asymptotic line coordinates (u, v) and the Gauss formula becomes as follows (cf. [10, Theorem 1]):

$$\begin{cases} f_{uu} = \left(\frac{\varphi_u}{\varphi} + \rho_u\right) f_u + \frac{a}{\varphi} f_v, \\ f_{vv} = \left(\frac{\varphi_v}{\varphi} + \rho_v\right) f_v + \frac{b}{\varphi} f_u, \\ f_{uv} = -\varphi f + \rho_v f_u + \rho_u f_v \end{cases} \quad (4.1)$$

with the integrability conditions:

$$(\log |\varphi|)_{uv} = -\varphi - \frac{ab}{\varphi^2} + \rho_u \rho_v, \quad a_v + \rho_u \varphi_u = \rho_{uu} \varphi, \quad b_u + \rho_v \varphi_v = \rho_{vv} \varphi. \quad (4.2)$$

We remark that $h = 2\varphi du dv$ is the centroaffine metric, $\pm e^\rho$ is the equiaffine support function from the origin, and

$$a = \varphi \det \begin{pmatrix} f \\ f_u \\ f_{uu} \end{pmatrix} / \det \begin{pmatrix} f \\ f_u \\ f_v \end{pmatrix}, \quad b = \varphi \det \begin{pmatrix} f \\ f_v \\ f_{vv} \end{pmatrix} / \det \begin{pmatrix} f \\ f_u \\ f_v \end{pmatrix}. \quad (4.3)$$

Moreover, the difference tensor field C , the Pick function J , the Tchebychev vector field T , the Tchebychev operator \mathcal{T} , the traceless part of the difference tensor field \tilde{C} and the generalized Pick function \tilde{J} are computed as follows (cf. [3]):

$$C(\partial_u, \partial_u) = \rho_u \partial_u + \frac{a}{\varphi} \partial_v, \quad C(\partial_v, \partial_v) = \frac{b}{\varphi} \partial_u + \rho_v \partial_v, \quad C(\partial_u, \partial_v) = \rho_v \partial_u + \rho_u \partial_v,$$

$$J = \frac{3\rho_u \rho_v}{\varphi} + \frac{ab}{\varphi^3}, \quad T = \frac{\rho_v}{\varphi} \partial_u + \frac{\rho_u}{\varphi} \partial_v, \quad (4.4)$$

$$\mathcal{T}(\partial_u) = \frac{\rho_{uv}}{\varphi} \partial_u + \frac{a_v}{\varphi^2} \partial_v, \quad \mathcal{T}(\partial_v) = \frac{b_u}{\varphi^2} \partial_u + \frac{\rho_{uv}}{\varphi} \partial_v, \quad (4.5)$$

$$\tilde{C}(\partial_u, \partial_u) = \frac{a}{\varphi} \partial_v, \quad \tilde{C}(\partial_v, \partial_v) = \frac{b}{\varphi} \partial_u, \quad \tilde{C}(\partial_u, \partial_v) = 0,$$

$$\tilde{J} = \frac{ab}{\varphi^3}. \quad (4.6)$$

The center map Z is given by

$$Z = \frac{\rho_v}{\varphi} f_u + \frac{\rho_u}{\varphi} f_v \quad (4.7)$$

(cf. [2, Proposition 3.1]). We also note that the centroaffine curvature κ , the curvature of h , is given by

$$\kappa = -\frac{(\log |\varphi|)_{uv}}{\varphi}. \quad (4.8)$$

5 Examples

Before we prove our main theorems, we illustrate examples of centroaffine ruled surfaces and their center maps. First, we treat surfaces in Theorem 1.1, which are studied by Fujioka [2] and Hu [5]. We recall the following:

Example 5.1 ([2, Remark 5.3]) Let f be a nondegenerate centroaffine ruled surface in (1.1) constructed from an \mathbf{R}^3 -valued function A of one variable. Note that f is characterized as a centroaffine minimal surface such that the Pick function $J = 0$ and the centroaffine curvature $\kappa = 1$.

Set

$$\alpha_A = \det \begin{pmatrix} A \\ A' \\ A'' \end{pmatrix} (\neq 0). \quad (5.1)$$

Then the center map of f is given by

$$Z = -\frac{1}{2} \frac{\alpha_A'}{\alpha_A} A. \quad (5.2)$$

If α_A is nonzero constant, f is a proper affine sphere centered at the origin (cf. [9] and [11, Theorem 4.3.1]).

Proposition 5.1 ([5, Corollary 4.4]) *Let B be an \mathbf{R}^3 -valued function of one variable such that $\alpha_B \neq 0$ as in (5.1) and λ a solution of the ordinary differential equation:*

$$3\lambda' + \frac{\alpha_B'}{\alpha_B} \lambda + 2 = 0. \quad (5.3)$$

Then (1.1) for $A = \lambda B$ gives a nondegenerate centroaffine ruled surface whose center map coincides with B .

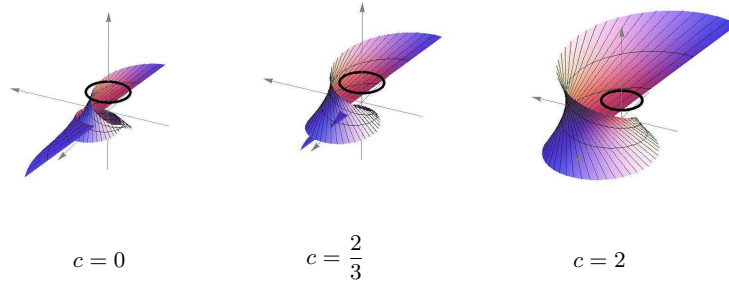
Proposition 5.2 *Let f be a ruled surface in (1.1) with center map Z . If Z is an affine map, then f is a proper affine sphere centered at the origin.*

Proof. We have

$$\begin{aligned} \left(\frac{\alpha_A'}{\alpha_A} A \right)' &= \left(\frac{\alpha_A'}{\alpha_A} \right)' A + \frac{\alpha_A'}{\alpha_A} A', \\ \left(\frac{\alpha_A'}{\alpha_A} A \right)'' &= \left(\frac{\alpha_A'}{\alpha_A} \right)'' A + 2 \left(\frac{\alpha_A'}{\alpha_A} \right)' A' + \frac{\alpha_A'}{\alpha_A} A'', \end{aligned}$$

which means that the condition $Z'' = 0$ implies $\alpha_A'/\alpha_A = 0$. \square

Example 5.2 For a space curve $Z(u) = (\cos u, \sin u, 1)$, we have $\alpha_Z' = 0$. The solution of (5.3) is given by $\lambda(u) = -\frac{2}{3}u + c$ for some constant c , and then we obtain a one parameter family of centroaffine ruled surfaces f_c as in (1.1) whose center map is the same Z .

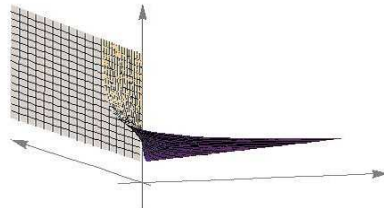


Example 5.2

Example 5.3 The surface f in (1.3) is a flat centroaffine surface with vanishing Tchebychev operator. In particular, f is centroaffine minimal. The center map is given by

$$Z(u, v) = (3e^{u+2v}, 0, e^v). \tag{5.4}$$

Accordingly, f is an indefinite centroaffine surface with vanishing Tchebychev operator which is not self congruent.



Example 5.3 ((1.3) and (5.4))

Example 5.4 Using the notation in §4, Fujioka ([1, Theorem 2.4]) classified centroaffine minimal surfaces such that $a = b$ and κ is constant, and obtained an explicit example of indefinite centroaffine minimal surfaces with $\mathcal{T} \neq 0$:

$$f(u, v) = \left(\frac{e^{-(u+v)}}{u+v} \cos(u-v), \frac{e^{-(u+v)}}{u+v} \sin(u-v), 1 - \frac{1}{u+v} \right).$$

The center map is given by

$$Z(u, v) = (-(u+v+1)e^{-(u+v)} \cos(u-v), -(u+v+1)e^{-(u+v)} \sin(u-v), 1).$$

In particular, Z is a piece of a plane not containing the origin.

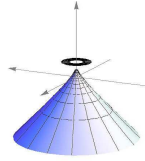
Example 5.5 Similar to Example 5.4, using a holomorphic coordinate z , we have an explicit example of definite centroaffine minimal surfaces:

$$f(x, y) = \left(\frac{e^{-(z+\bar{z})+\sqrt{-1}(z-\bar{z})}}{z+\bar{z}}, \frac{e^{-(z+\bar{z})-\sqrt{-1}(z-\bar{z})}}{z+\bar{z}}, 1 - \frac{1}{z+\bar{z}} \right),$$

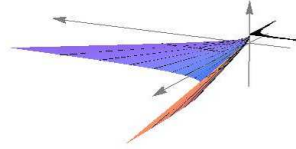
where $z = x + \sqrt{-1}y$ ([2, Theorem 3.5]). Then the center map is given by

$$Z(x, y) = (-(z+\bar{z}+1)e^{-(z+\bar{z})+\sqrt{-1}(z-\bar{z})}, -(z+\bar{z}+1)e^{-(z+\bar{z})-\sqrt{-1}(z-\bar{z})}, 1),$$

which is a piece of a plane not containing the origin.



Example 5.4



Example 5.5

6 Centroaffine ruled surfaces

A surface in \mathbf{R}^3 is said to be ruled if each point of the surface lies on a line segment in the surface. We note that the following fundamental fact holds:

Proposition 6.1 *Let $f : M \rightarrow \mathbf{R}^3$ be an indefinite centroaffine surface with generalized Pick function \tilde{J} . Then f is ruled if and only if $\tilde{J} = 0$.*

Proof. It is known in equiaffine differential geometry that an indefinite equiaffine surface is ruled if and only if the equiaffine Pick function J^e vanishes ([8, II Theorems 11.3, 11.4]). We remark that J^e vanishes if and only if so does \tilde{J} , since the traceless part of the difference tensor field for any affine hypersurface is independent of the choice of the relative normalization ([12, Corollary 5.1.4]). \square

Note that Liu and Jung [7] studied indefinite centroaffine surfaces with vanishing \tilde{J} . In this paper, we take another method to prove our main theorems.

Let $f : M \rightarrow \mathbf{R}^3$ be a nondegenerate centroaffine ruled surface. Since f is indefinite, we can take asymptotic line coordinates (u, v) and use the same

notation in §4. From Proposition 6.1, the first equation of (4.4) and (4.6), we have

$$ab = 0, \quad J = \frac{3\rho_u\rho_v}{\varphi}. \quad (6.1)$$

In the case of $J = 0$, from the first equation of (4.2), (4.8) and (6.1), we have $\kappa = 1$. On the other hand, from the second equation of (6.1), we have $\rho_{uv} = 0$, which implies that f is centroaffine minimal from (4.5). Then by [3, Theorem 1.2], f is given by (1.1) with (1.2).

Remark 6.1 In [15], a centroaffine minimal ruled surface with $J = 0$ is given in the form of

$$f(x, y) = \alpha(x) + y\beta(x),$$

where α and β are \mathbf{R}^3 -valued functions of x only such that $\det \begin{pmatrix} \alpha \\ \beta \\ \beta' \end{pmatrix} = 0$.

In the case of $J \neq 0$, we have

$$\rho_u \neq 0, \quad \rho_v \neq 0. \quad (6.2)$$

From the first equation of (6.1), exchanging u with v , if necessary, we may assume that

$$a = 0. \quad (6.3)$$

Then the second equation of (4.2) can be integrated as

$$\varphi = C(v)\rho_u,$$

where C is a nonzero function of v only. Changing the coordinate v , if necessary, we may assume that

$$\varphi = \rho_u. \quad (6.4)$$

Then the first and third equations of (4.2) become

$$(\log |\rho_u|)_{uv} = -\rho_u + \rho_u\rho_v, \quad b_u + \rho_v\rho_{uv} = \rho_{vv}\rho_u. \quad (6.5)$$

From (6.3) and (6.4), the first equation of (4.1) becomes

$$f_{uu} = \left(\frac{\rho_{uu}}{\rho_u} + \rho_u \right) f_u,$$

which can be integrated as

$$f = C_1(v)e^{\rho} + C_2(v), \quad (6.6)$$

where C_1 and C_2 are \mathbf{R}^3 -valued functions of v only. Note that $C_1 \neq 0$ since f is an immersion. Hence if we substitute (6.4) and (6.6) into the third equation of (4.1), then C_1 and C_2 satisfy a linear ordinary differential equation:

$$C_2' - C_2 = p(v)C_1, \quad (6.7)$$

where p is a function of v only defined by

$$p = \left(\frac{\rho_{uv}}{\rho_u} - \rho_v + 1 \right) e^\rho. \quad (6.8)$$

Substituting (6.4) and (6.6) into the second equation of (4.1) and using (6.7) and (6.8), we have

$$qC_2' = C_1'' + C_1' + rC_1,$$

where

$$q = \left(\frac{\rho_{uv}}{\rho_u} + \rho_v - 1 \right) e^{-\rho}, \quad r = \left(\frac{\rho_{uv}}{\rho_u} \right)_v - \rho_v^2 + \rho_v - b. \quad (6.9)$$

From the first equation of (6.5), we have $q_u = 0$. On the other hand, from (6.5), we have $r_u = 0$. Hence q and r are functions of v only. In particular, C_1 and C_2 satisfy a linear ordinary differential equation:

$$q(v)C_2' = C_1'' + C_1' + r(v)C_1. \quad (6.10)$$

Note that from (6.6) and (6.7), we have

$$f_u = C_1\rho_u e^\rho, \quad f_v = C_1(\rho_v e^\rho + p) + C_1' e^\rho + C_2. \quad (6.11)$$

Since f is a centroaffine surface, from (6.6) and (6.11), the vectors C_1, C_1', C_2 are linearly independent for each v .

Therefore we have the following:

Lemma 6.2 *Let $f : M \rightarrow \mathbf{R}^3$ be a nondegenerate centroaffine ruled surface whose Pick function does not vanish. Then f is given by (6.6), where C_1 and C_2 are \mathbf{R}^3 -valued functions of v only satisfying (6.7) and (6.10) such that C_1, C_1', C_2 are linearly independent for each v , and ρ and b satisfy (6.5).*

7 Proofs of the main theorems

Now we shall prove Theorems 1.1 and 1.2. Let f be an indefinite centroaffine ruled surface, and use the same notation in §6. Each proof is divided into the two cases: $J = 0$ and $J \neq 0$.

Proof of Theorem 1.1. In the case of $J = 0$, as mentioned before Remark 6.1, f is given by (1.1) and so the center map is given by (5.2), which is a point or a curve.

In the case of $J \neq 0$, we will prove that the center map is an immersion by using Lemma 6.2. Substituting (6.4) and (6.11) into (4.7), we have

$$Z = C_1(2\rho_v e^\rho + p) + C_1' e^\rho + C_2. \quad (7.1)$$

Then we have

$$Z_u = 2C_1(\rho_{uv} + \rho_u \rho_v) e^\rho + C_1' \rho_u e^\rho. \quad (7.2)$$

From (6.7), (6.10) and (7.1), we have

$$\begin{aligned} Z_v &= C_1\{(2\rho_{vv} + 2\rho_v^2 + pq - r)e^\rho + p + p'\} \\ &\quad + C_1'\{(3\rho_v - 1)e^\rho + p\} + C_2(qe^\rho + 1). \end{aligned} \tag{7.3}$$

Note that from (6.2) the coefficient of C_1' in the right hand side of (7.2) does not vanish. Moreover, since q is a function of v only, from (6.2) the coefficient of C_2 in the right hand side of (7.3) does not vanish. Since C_1, C_1', C_2 are linearly independent for each v , the center map Z is an immersion. \square

Remark 7.1 In [5], Hu proved that a nondegenerate centroaffine ruled surface with nondegenerate centroaffine director curve has vanishing Pick function if the center map is a point or a curve. The director curve A of the ruled surface given by (1.1) with (1.2) defines a nondegenerate centroaffine space curve if and only if $\det \begin{pmatrix} A' \\ A'' \\ A''' \end{pmatrix} \neq 0$.

Proof of Theorem 1.2. If $J = 0$, then f is given by (1.1) and the center map is given by (5.2). Since the center map is a piece of a plane through the origin, we have

$$0 = \det \begin{pmatrix} Z \\ Z' \\ Z'' \end{pmatrix} = -\frac{(\alpha_A')^3}{\alpha_A^2},$$

which implies that $\alpha_A' = 0$, that is, $Z = 0$.

In the following, we consider the case $J \neq 0$. From (7.2) we have

$$Z_{uu} = C_1\{2(\rho_{uv} + \rho_u\rho_v)e^\rho\}_u + C_1'(\rho_u e^\rho)_u. \tag{7.4}$$

By the assumption, we have

$$0 = \det \begin{pmatrix} Z \\ Z_u \\ Z_{uu} \end{pmatrix} = 2 \begin{pmatrix} \rho_{uv} \\ \rho_u \end{pmatrix}_u + \rho_v \det \begin{pmatrix} C_1 \\ C_1' \\ C_2 \end{pmatrix},$$

which is equivalent to that the function $\rho_{uv}/\rho_u + \rho_v$ depends on v only since C_1, C_1', C_2 are linearly independent for each v . From (6.2) and the first equation of (6.9), we have

$$\frac{\rho_{uv}}{\rho_u} + \rho_v = 1, \tag{7.5}$$

which is equivalent to

$$q = 0. \tag{7.6}$$

From (6.8), (6.10), (7.1), (7.5) and (7.6), we have

$$Z = 2C_1e^\rho + C_1'e^\rho + C_2, \quad Z_u = 2C_1\rho_u e^\rho + C_1'\rho_u e^\rho, \tag{7.7}$$

$$Z_{uv} = (2 - r)C_1\rho_u e^\rho + 2C_1'\rho_u e^\rho.$$

By the assumption, we have

$$0 = \det \begin{pmatrix} Z \\ Z_u \\ Z_{uv} \end{pmatrix} = (r + 2)\rho_u^2 e^{2\rho} \det \begin{pmatrix} C_1 \\ C_1' \\ C_2 \end{pmatrix},$$

which is equivalent to

$$r = -2, \quad (7.8)$$

since C_1, C_1', C_2 are linearly independent for each v .

From (6.7), (6.10), (7.6), (7.7) and (7.8), we have

$$0 = \det \begin{pmatrix} Z \\ Z_u \\ Z_v \end{pmatrix} = -p\rho_u e^\rho \det \begin{pmatrix} C_1 \\ C_1' \\ C_2 \end{pmatrix},$$

which is equivalent to

$$p = 0, \quad (7.9)$$

since C_1, C_1', C_2 are linearly independent for each v and (6.2) holds.

From (7.9) and (7.6), we have $\rho_v = 1$, which can be integrated as

$$\rho(u, v) = v + \gamma(u), \quad (7.10)$$

where γ is a function of u only. From the second equation of (6.9), (7.8) and (7.10), we have $b = 2$. Changing the coordinate u , if necessary, we may assume that $\gamma(u) = u$, i.e.,

$$\rho(u, v) = u + v. \quad (7.11)$$

Then from (6.4) we have $\varphi = 1$.

Moreover, from (7.6), (7.8) and (7.9), the differential equations (6.7) and (6.10) become

$$C_2' = C_2, \quad C_1'' + C_1' - 2C_1 = 0,$$

which can be integrated as

$$C_1 = Pe^v + Qe^{-2v}, \quad C_2 = Re^v \quad (P, Q, R \in \mathbf{R}^3). \quad (7.12)$$

Since C_1, C_1', C_2 are linearly independent for each v , the vectors P, Q, R are also linearly independent. From (6.6), the first equation of (7.7), (7.11) and (7.12), we have

$$f(u, v) = Pe^{u+2v} + Qe^{u-v} + Re^v$$

whose center map is given by

$$Z(u, v) = 3Pe^{u+2v} + Re^v,$$

which is a piece of a plane through the origin. \square

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References

1. FUJIOKA, A. – *Centroaffine minimal surfaces with constant curvature metric*, Kyungpook Math. J., **46** (2006), no. 2, 297-305.
2. FUJIOKA, A. – *Centroaffine minimal surfaces with non-semisimple centroaffine Tchebychev operator*, Results Math., **56**, (2009), no. 1-4, 177-195.
3. FUJIOKA, A. – *Centroaffine minimal surfaces whose centroaffine curvature and Pick function are constants*, J. Math. Anal. Appl., **365** (2010), no. 2, 694-700.
4. FURUHATA, H.; VRANCKEN, L. – *The center map of an affine immersion*, Results Math., **49** (2006), no. 3-4, 201-217.
5. HU, N. – *Centroaffine surfaces with degenerate center map*, J. Geom. Phys., **65** (2013), 45-54.
6. KATOU, M. – *Center maps of affine minimal ruled hypersurfaces*, Interdiscip. Inform. Sci., **12** (2006), no. 1, 53-56.
7. LIU, H.; JUNG, S.D. – *Indefinite centroaffine surfaces with vanishing generalized Pick function*, J. Math. Anal. Appl., **329** (2007), no. 1, 712-720.
8. NOMIZU, K.; SASAKI, T. – *Affine differential geometry. Geometry of affine immersions*, Cambridge Tracts in Mathematics, 111, Cambridge University Press, Cambridge (1994), xiv+263 pp.
9. RADON, J. – *Zur Affingeometrie der Regelflächen* (German), Leipziger Berichte, **70** (1918), 147-155.
10. SCHIEF, W.K. – *Hyperbolic surfaces in centro-affine geometry. Integrability and discretization, Integrability and chaos in discrete systems* (Brussels, 1997), Chaos Solitons Fractals **11** (2000), no. 1-3, 97-106.
11. SIMON, U. – *Local classification of two-dimensional affine spheres with constant curvature metric*, Differential Geom. Appl., **1** (1991), no. 2, 123-132.
12. SIMON, U.; SCHWENK-SCHELLSCHMIDT, A.; VIESEL, H. – *Introduction to the affine differential geometry of hypersurfaces*, Lecture Notes of the Science University of Tokyo, Science University of Tokyo, Tokyo (1991), viii+161 pp.
13. TRABELSI, H. – *Generic affine hypersurfaces with self congruent center map*, Results Math., **51** (2007), no. 1-2, 127-140.
14. TRABELSI, H. – *Improper affine hyperspheres with self-congruent center map*, Monatsh. Math., **152** (2007), no. 1, 73-81.
15. YU, Y.; YANG, Y.; LIU, H. – *Centroaffine ruled surfaces in \mathbb{R}^3* , J. Math. Anal. Appl., **365** (2010), no. 2, 683-693.
16. WANG, C.P. – *Centroaffine minimal hypersurfaces in \mathbb{R}^{n+1}* , Geometriae Dedicata, **51** (1994), no. 1, 63-74.

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