An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)
Tomul LXIV, 2018, f. 2

Exact algebraic and non-algebraic limit cycles for a class of integrable quintic and planar polynomial differential systems

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Abstract As an improvement of the work of J. Giné and M. Grau [8], we presente for the first time an integrable quintic planar differential system and prove by using the Poincaré return map that it admits a nested configuration formed by an algebraic and a non-algebraic limit cycles explicitely given was presented. Moreover, we show that this system do not possess other limit cycles.

Keywords Non-algebraic limit cycle · invariant curve · Poincaré return-map · first integral · Riccati equation

Mathematics Subject Classification (2010) 34C29 · 34C25

1 Introduction

It is a fact that most of research papers devoted to the study of limit cycles for planar and autonomous differential system of degree n, of the form

$$\begin{aligned} \dot{x} &= \frac{dx}{dt} = P_n(x,y), \\ \dot{y} &= \frac{dy}{dt} = Q_n(x,y), \end{aligned} \tag{1.1}$$

where $P_n(x,y)$ and $Q_n(x,y)$ are coprime polynomials of $\mathbb{R}[x,y]$ and $n = \max\{\deg P_n, \deg Q_n\}$, are concerned by their number, stability and location in the phase plane. We recall that a limit cycle of system (1.1) is an isolated periodic orbit in the set of its periodic orbits and it is said to be algebraic if it is contained in the zero set of an invariant algebraic curve of the system. We recall that an algebraic curve defined by $U(x,y) = 0$ is an invariant curve for (1.1) if there exists a polynomial $K(x,y)$ (called the cofactor) such that

$$P_n(x,y)\frac{\partial U}{\partial x}+Q_n(x,y)\frac{\partial U}{\partial x}=K(x,y)\,U(x,y)\,. \tag{1.2}$$

A natural problem is to express analytically the limit cycles. Until recently, the only limit cycles known in an explicit way were algebraic (see for instance [3], [4], [9], [10] and references therein). In 1998, M. Abdelkadder [1] presented for the first time an example of Liénard equations with an exact algebraic limit cycle. This example was recovered as a particular case by A. Bendjeddou and R. Cheurfa [3] when studying a more general class of planar systems.

Although, Limit cycles of planar polynomial differential systems are not in general algebraic. For instance, the limit cycle appearing in the van der Pol equation is non-algebraic as it is proved by Odani [11]. In the chronological order, the first examples were explicit non-algebraic limit cycles appeared are those of A. Gasull [7] and by Al-Dossary [2] for $n=5$ and by Benterki and Llibre [5] for $n=3$. The fist result for the coexistence of algebraic and non-algebraic limit cycles goes back to J. Giné and M. Grau [8] for $n=9$.

As we can see below, we extend in this paper this last result to the case $n=5$. As a main result, we shall prove the following result:

Theorem 1.1 *The quintic system*

$$\begin{aligned}\dot{x}&=P_5(x,y)\,,\\ \dot{y}&=Q_5(x,y)\,,\end{aligned} \tag{1.3}$$

where

$P_5(x,y)=x\left(x^2+y^2-1\right)^2+\left(-x+2y+xy^2+x^3\right)\left((a+b)\,x^2+(a-b)\,y^2\right)$
and
$Q_5(x,y)=y\left(x^2+y^2-1\right)^2+\left(-2x-y+x^2y+y^3\right)\left((a+b)\,x^2+(a-b)\,y^2\right)$
in which $a\in\mathbb{R}_+^*$ and $b\in\mathbb{R}^*$ possesses exactly two limit cycles: the circle $(\gamma_1): x^2+y^2-1=0$ surrounding a transcendental limit cycle (γ_*) explicitly given in polar coordinates $(r;\theta)$by the equation

$$r(\theta;r_*)=\sqrt{1+\frac{\exp(-\theta)}{1+\frac{1}{r_*^2-1}-\exp(-\theta)+f(\theta)}}, \tag{1.4}$$

with $f(\theta)=\int_0^\theta \frac{e^{-s}}{a+b\cos 2s}\,ds$ and $r_*=\sqrt{\frac{f(2\pi)}{1-e^{-2\pi}+f(2\pi)}}$,

when the following condition is assumed :

$$b^2-a^2<0. \tag{1.5}$$

Proof. Firstly, we have

$$xQ_5(x,y)-yP_5(x,y)=-2\left(x^2+y^2\right)\left((a+b)\,x^2+(a-b)\,y^2\right),$$

so from (1.5) the origin is the unique critical point at finite distance. Also, in accordance with (1.2) it is not difficult to see that the circle $(\gamma_1) : x^2 + y^2 - 1 = 0$ is an invariant curve, the associated cofactor being

$$K(x,y) = -4\left(x^2+y^2\right)\left((a+b+1)\,x^2 + (a-b+1)\,y^2 - 1\right). \tag{1.6}$$

Of course (γ_1) defines a periodic solution of system (1.3), since it do not pass through the origin. To see whether or not (γ_1) is in fact a limit cycle, we can proceed as follow: If T denotes the period of (γ_1), we consider the integral $I(\gamma_1)$, where

$$I(\gamma_1) = \int_0^T \operatorname{div}(x,y)\,dt. \tag{1.7}$$

We know from [9] that this integral can be computed via

$$I(\gamma_1) = \int_0^T K(x(t), y(t))\,dt. \tag{1.8}$$

From (1.5) again, the curve $K(x,y) = 0$ do not cross (γ_1) with $K(x,y) > 0$ inside $(\gamma_1) / \{(0,0)\}$, so $I(\gamma_1) < 0$. Consequently (γ_1) defines a stable algebraic limit cycle for system (1.3).
The search for the non-algebraic limit cycle, requires the integration of our system. In polar coordinates, this system becomes

$$\begin{aligned} \dot{r} &= (a+b\cos 2\theta+1)\,r^5 - (a+b\cos 2\theta+2)\,r^3 + r, \\ \dot{\theta} &= -2\,(a+b\cos 2\theta)\,r^2. \end{aligned} \tag{1.9}$$

Taking θ as an independent variable, we obtain the equation

$$(a+b\cos 2\theta)\frac{dr}{d\theta} = -\frac{1}{2}(a+b\cos 2\theta+1)\,r^3 + \frac{1}{2}(a+b\cos 2\theta+2)\,r - \frac{1}{2}. \tag{1.10}$$

Via the change of variables $\rho = r^2$, this equation is transformed into the Riccati equation

$$(a+b\cos 2\theta)\frac{d\rho}{d\theta} = -(a+b\cos 2\theta+1)\,\rho^2 + (a+b\cos 2\theta+2)\,\rho - 1. \tag{1.11}$$

Fortunately, this equation is integrable, since it possesses the particular solution $\rho = 1$ corresponding of course to the limit cycle (γ_1). The general solution of this equation is

$$\rho(\theta) = 1 + \frac{\exp(-\theta)}{k - \exp(-\theta) + \int_0^\theta \frac{\exp(-s)}{a+b\cos 2s}ds},$$

Consequently, for suitable values of the constant k, the general solution of eq. (1.10) is

$$r(\theta;k)=\sqrt{1+\frac{\exp(-\theta)}{k-\exp(-\theta)+\int_0^{\theta}\frac{\exp(-s)}{a+b\cos 2s}ds}}. \tag{1.12}$$

By passing to Cartesian coordinates, we deduce the first integral

$$F(x,y)=\frac{x^2+y^2}{x^2+y^2-1}e^{-\arctan\frac{y}{x}}-\int_0^{\arctan\frac{y}{x}}\frac{e^{-s}}{a+b\cos 2s}\,ds. \tag{1.13}$$

The trajectories of system (1.3) are the level curves $F(x,y)=k$, $k\in\mathbb{R}$. We show that all this curves are non-algebraic (if we exclude of course the curve (γ_1) corresponding to $k\to+\infty$). To see that, we must prove for instance that there is no value of the integer n for which $\frac{\partial^n F}{\partial y^n}=0$. For this purpose, let us compute $\frac{\partial F}{\partial y}(x,y)$ and $\frac{\partial^2 F}{\partial y^2}(x,y)$. We find that they can be put on the form

$$\frac{\partial F}{\partial y}(x,y)=-\frac{e^{-\arctan\frac{y}{x}}}{\left(a+b\cos\left(2\arctan\frac{y}{x}\right)\right)(x^2+y^2)(x^2+y^2-1)^2}R_5(x,y),$$

and

$$\frac{\partial^2 F}{\partial y^2}(x,y)=\frac{e^{-\arctan\frac{y}{x}}}{\left(a+b\cos\left(2\arctan\frac{y}{x}\right)\right)^2(x^2+y^2)^3(x^2+y^2-1)^3}R_{10}(x,y),$$

where R_5 and R_{10} are two non-vanishing polynomials of degrees 5 and 10 respectively, their expressions do not matter here. The transcendental term $\frac{e^{-\arctan\frac{y}{x}}}{a+b\cos\left(2\arctan\frac{y}{x}\right)}$ which appears in $\frac{\partial^2 F}{\partial y^2}(x,y)$ appears already in $\frac{\partial F}{\partial y}(x,y)$, hence it can not be cancelled when higher derivatives are performed, so $\frac{\partial^n F}{\partial y^n}$do not vanishes identically as claimed. Thus any other limit cycle, if exists, should also be non-algebraic.To go a steep further, we remark that the solution such as $r(0;r_0)=r_0>0$, corresponds to the value $k=\frac{r_0^2}{r_0^2-1}$ provided a rewriting of the general solution of eq.(1.10) as

$$r(\theta;r_0)=\sqrt{1+\frac{\exp(-\theta)}{1+\frac{1}{r_0^2-1}-\exp(-\theta)+\int_0^{\theta}\frac{\exp(-s)}{a+b\cos 2s}ds}}, \tag{1.14}$$

Any periodic solution of system (1.3) must satisfy the condition

$$r(2\pi;r_0)=r_0, \tag{1.15}$$

provided two distinct positive values of r_0: $r_1=1$ corresponding obviously to the algebraic limit cycle (γ_1) and thanks to (1.6), a well defined second value

$$r_*=\sqrt{\frac{f(2\pi)}{1-e^{-2\pi}+f(2\pi)}}. \tag{1.16}$$

By injecting the second value r_* of r_0 in eq. (1.14), we obtain the second candidate solution given by the statement of the theorem through eq. (1.4). In the sequel, the notations $r(\theta, r_*)$ or (γ_*) refer both to this curve solution. To show that it is a periodic solution, we have to show that

• the function $\theta \to g(\theta)$, where in this case

$$g(\theta) = 1 + \frac{\exp(-\theta)}{1 + \frac{1}{r_*^2 - 1} - \exp(-\theta) + f(\theta)}, \tag{1.17}$$

is 2π-periodic,

• $g(\theta) > 0$ for all $\theta \in [0, 2\pi[$.

The last condition ensures that $r(\theta, r_*)$ is well defined for all $\theta \in [0, 2\pi[$ and the periodic solution do not pass through the unique equilibrium point $(0, 0)$ of system (1.3).

Periodicity. Let $\theta \in [0, 2\pi[$, then

$$g(\theta + 2\pi) = 1 + \frac{\exp(-\theta - 2\pi)}{1 + \frac{1}{r_*^2 - 1} - \exp(-\theta - 2\pi) + f(\theta + 2\pi)}. \tag{1.18}$$

But

$$\begin{aligned} f(\theta + 2\pi) &= \int_0^{\theta+2\pi} \tfrac{e^{-s}}{a + b\cos 2s} ds \\ &= f(2\pi) + \int_{2\pi}^{\theta+2\pi} \tfrac{e^{-s}}{a + b\cos 2s} ds. \end{aligned}$$

In the integral $\int_{2\pi}^{\theta+2\pi} \frac{e^{-s}}{a+b\cos 2s} ds$, we make the change of variable $u = s - 2\pi$, we obtain

$$\begin{aligned} f(\theta + 2\pi) &= f(2\pi) + \int_0^{\theta} \tfrac{e^{-u-2\pi}}{a + b\cos 2(u+2\pi)} du \\ &= f(2\pi) + e^{-2\pi} f(\theta). \end{aligned}$$

A last return to (1.18), gives $g(\theta + 2\pi) = g(\theta)$, hence g is 2π-periodic.

Strict positivity of $g(\theta)$ **for all** $\theta \in [0, 2\pi[$**.** Let $\phi(\theta) = \frac{e^{2\pi a}}{1-e^{2\pi a}} f(2\pi) + f(\theta)$. Since $\frac{d\phi}{d\theta}(\theta) = \exp(-a\theta - b\cos 2\theta) > 0$ for all $\theta \in [0, 2\pi[$, the function $\theta \to \phi(\theta)$ is strictly increasing with $\phi(0) = \frac{e^{2\pi a}}{1-e^{2\pi a}} f(2\pi)$ and $\phi(2\pi) = \frac{1}{1-e^{2\pi a}} f(2\pi)$. Since $a > 0$, then $\phi(2\pi) < 0 \Longrightarrow \phi(\theta) < 0$ for all $\theta \in [0, 2\pi[$, thus $\exp(a\theta + b\cos 2\theta)\,\phi(\theta) < 0$, hence $g(\theta) > 0$ for all $\theta \in [0, 2\pi[$.

Finally, it remains to show that (γ_*) is in fact a limit cycle. For that we consider (1.14), and introduce the Poincaré return map $r_0 \to \Pi(2\pi; r_0) = r(2\pi; r_0)$, with the positive x-axis as Poincaré section. We compute $\frac{d\Pi}{dr_0}(2\pi; r_0)$ at the value $r_0 = r_*$:

$$\left.\frac{d\Pi}{dr_0}(2\pi; r_0)\right|_{r_0 = r_*} = r_* e^{-2\pi} \frac{\left(\left(f(2\pi) + 1 - e^{-2\pi}\right) r_*^2 + e^{-2\pi} - f(2\pi)\right)^{-2}}{\sqrt{\frac{(f(2\pi) + 1)\, r_*^2 - f(2\pi)}{\left(f(2\pi) + 1 - e^{-2\pi}\right) r_*^2 + e^{-2\pi} - f(2\pi)}}}.$$

By replacing r_* by its value given by (1.16), we get

$$\left.\frac{d\Pi}{dr_0}(2\pi; r_0)\right|_{r_0=r_*} = e^{2\pi} > 1, \tag{1.19}$$

Consequently the limit cycle of the differential equation (1.10) is unstable and hyperbolic (see [6], section 1.6 for more details), so it is a stable hyperbolic limit cycle for system (1.4). Since the Poincaré return map do not possess other fixed points, the system (1.3) admits exactly two limit cycles. □

2 Example

As an example let us take $a = \frac{3}{2}$, $b = \frac{1}{2}$, the system (1.3) becomes

$$\begin{aligned} x' &= x - \tfrac{7}{2}x^3 + 4x^2y - \tfrac{7}{2}xy^2 + 2y^3 + \tfrac{5}{2}x^5 - 2x^4y + 5x^3y^2 - 2x^2y^3 + \tfrac{5}{2}xy^4, \\ y' &= y - 2x^3 - \tfrac{7}{2}x^2y - \tfrac{7}{2}y^3 + \tfrac{5}{2}x^4y - 2x^3y^2 + 5x^2y^3 - 2xy^4 + \tfrac{5}{2}y^5. \end{aligned} \tag{2.1}$$

Then we have $r_* \simeq 0.782\,27$. It is easy to verify that all conditions of the theorem are satisfied. We conclude that system (2.1) possesses two limit cycles plotted on the Poincaré disc as shown in figure bellow :

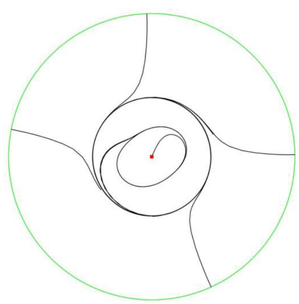

Fig. 1 Limit cycles of system (2.1).

Conclusion. In this work, we have improved the result obtained in [8] by reducing considerably the degree of the differential system from $n = 9$ to $n = 5$. The method adopted is more intuitive. Obtaining interesting results of this kind becomes more and more difficult for lower values of n. Nevertheless it is interesting to study the following problems :

- The coexistence of two explicit non-algebraic limit cycles for quintic systems.

- The coexistence of explicit algebraic and non-algebraic limit cycles for cubic systems.
- Obtain a quadratic system with exact non-algebraic limit cycle (this question has been raised before by Benterki and Llibre [5]).

Acknowledgements The authors would like to express their gratitude to the anonymous referee for his valuable remark and for his interest to this work.

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Received: 20.XII.2016 / Accepted: 8.II.2017

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