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Higher-order identities for the second-order sequence

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Abstract The objective of this article is to derive some higher-order identities concerning a general second-order recurrence sequence.

Keywords recurrence relation · second-order recurrence sequence · generating function

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1 Introduction

Let $\{A_n\}$ be a general second-order recurrence of the form

$$A_{n+2} = uA_{n+1} + vA_n \quad (n \geq 0)$$

with $A_0 = a_0$ and $A_1 = a_1$ (see, e.g., [1,2,8,9] and references therein). This sequence is the generalization of some known sequences like Fibonacci numbers $\{F_n\}$, Pell numbers $\{P_n\}$, balancing numbers $\{B_n\}$ etc. Here the parameters u and v are non-zero real numbers satisfying $\Delta = u^2 + 4v > 0$. In this paper, put $a_0 = 0$ and $a_1 = a \neq 0$ for simplicity. The generating function $f(x)$ of this sequence $\{A_n\}$ is given by

$$f(x) = \sum_{n=0}^{\infty} A_n x^n = \frac{ax}{1 - ux - vx^2}, \tag{1.1}$$

while its explicit form known as Binet formula is given by the identity

$$A_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where $\alpha = \frac{u+\sqrt{\Delta}}{2}$ and $\beta = \frac{u-\sqrt{\Delta}}{2}$.

Higher-order identities for Fibonacci, Cauchy and balancing numbers were studied in [3–5] by Komatsu et al. They have obtained some explicit expressions of the identities

$$\sum_{l=0}^{2r-3}\binom{2r-3}{l}\sum_{\substack{j_1+\cdots+j_r=n\\ j_1,\dots,j_r\ge 1}}F_{j_1}\dots F_{j_r}\quad\text{and}\quad\sum_{\substack{j_1+\cdots+j_r=n\\ j_1,\dots,j_r\ge 1}}F_{j_1}\dots F_{j_r}$$

for Fibonacci numbers in [4]. Whereas, the expressions

$$\sum_{l=0}^{2r-3}\binom{2r-3}{l}\sum_{\substack{j_1+\cdots+j_r=n\\ j_1,\dots,j_r\ge 1}}B_{j_1}\dots B_{j_r}\quad\text{and}\quad\sum_{\substack{j_1+\cdots+j_r=n\\ j_1,\dots,j_r\ge 1}}B_{j_1}\dots B_{j_r}$$

for balancing numbers found explicitly in [5]. In this article, we generalize all these identities by considering the sequence $\{A_n\}$. The results in this paper can be reduced to those in previous or classical papers.

2 Main Results

In this section we establish some higher-order identities concerning the sequence $\{A_n\}$.

Theorem 2.1 *For $n \ge 1$,*

$$nA_n = \frac{1}{a}\sum_{j=1}^{n}(A_jA_{n-j+1} + vA_{j-1}A_{n-j}).$$

Remark. When $a = v = 1$, Theorem 2.1 is reduced to Theorem 1.1 in [4]. When $a = 1$ and $v = -1$, Theorem 2.1 is reduced to Theorem 1 in [5]. Notice that the value u does not appear in the identity.

Proof. Differentiating both sides of (1.1) yields the identity

$$(1 + vx^2)f(x)^2 = ax^2f^{'}(x). \tag{2.1}$$

It follows that, $(1 + vx^2)\sum_{n=0}^{\infty}\sum_{j=0}^{n}A_jA_{n-j}x^n = ax^2\sum_{n=1}^{\infty}nA_nx^{n-1}$. Further,

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n}A_jA_{n-j}x^n + v\sum_{n=2}^{\infty}\sum_{j=0}^{n-2}A_jA_{n-j-2}x^n = a\sum_{n=1}^{\infty}(n-1)A_{n-1}x^n.$$

Comparing the coefficients of like power of x, we get

$$a(n-1)A_{n-1} = \sum_{j=1}^{n-1}(A_jA_{n-j} + vA_{j-1}A_{n-j-1}).$$

For $A_0 = 0$ and letting n by $n+1$, we obtain the following result that resembles the identity $A_{m+n} = A_mA_{n+1} + vA_{m-1}A_n$ (see, e.g., [6, Theorem 2.13]). □

Theorem 2.2 *For $n \geq 4$, we have*

$$\sum_{k=0}^{3} v^k \binom{3}{k} \sum_{\substack{r_1+r_2+r_3=n-2k \\ r_1,r_2,r_3 \geq 1}} A_{r_1} A_{r_2} A_{r_3} = a^2 \binom{n-1}{2} A_{n-2} + a^2 v \binom{n-4}{2} A_{n-4}.$$

Remark. When $a = v = 1$, Theorem 2.2 is reduced to Theorem 1.2 in [4]. When $a = 1$ and $v = -1$, Theorem 2.2 is reduced to Theorem 2 in [5].

Proof. Differentiating both sides of (2.1) with respect to x, we obtain

$$f(x)f'(x) = \frac{ax}{(1+vx^2)^2} f'(x) + \frac{ax^2}{2(1+vx^2)} f''(x). \tag{2.2}$$

Now the identity (2.1) together with the identity (2.2) gives the following

$$f(x)^3 = \frac{a^2x^3}{(1+vx^2)^3} f'(x) + \frac{a^2x^4}{2(1+vx^2)^2} f''(x).$$

It follows that

$$(1+vx^2)^3 f(x)^3 = a^2x^3 f'(x) + \frac{1}{2}a^2x^4(1+vx^2)f''(x).$$

Further simplification yields

$$\begin{aligned}(1+3vx^2+3v^2x^4+v^3x^6) &\sum_{n=0}^{\infty} \sum_{\substack{r_1+r_2+r_3=n \\ r_1,r_2,r_3 \geq 1}} A_{r_1}A_{r_2}A_{r_3}x^n \\ &= a^2x^3 \sum_{n=1}^{\infty} nA_n x^{n-1} + \frac{1}{2}a^2x^4 \sum_{n=2}^{\infty} n(n-1)A_n x^{n-2} \\ &\quad + \frac{1}{2}a^2vx^6 \sum_{n=2}^{\infty} n(n-1)A_n x^{n-2}.\end{aligned}$$

Furthermore,

$$\begin{aligned}\sum_{k=0}^{3} \sum_{n=2k}^{\infty} v^k \binom{3}{k} &\sum_{\substack{r_1+r_2+r_3=n-2k \\ r_1,r_2,r_3 \geq 1}} A_{r_1}A_{r_2}A_{r_3}x^n \\ &= a^2 \sum_{n=2}^{\infty} \frac{(n-1)(n-2)}{2} A_{n-2}x^n + a^2 v \sum_{n=4}^{\infty} \frac{(n-4)(n-5)}{2} A_{n-4}x^n,\end{aligned}$$

and the result follows. □

The following lemma is useful to derive the subsequent main results.

Lemma 2.3 *For $r \geq 2$, we have*

$$f(x)^r = \frac{a^{r-1}x^{2r-2}f^{(r-1)}(x)}{(r-1)!\,(1+vx^2)^{r-1}} \tag{2.3}$$

$$+\sum_{k=1}^{r-2}\frac{\sum_{j=0}^{k-1}a^{r-1}(-v)^j\binom{k}{j}\binom{r-2}{k-j-1}x^{2r-k-2+2j}}{k(r-k-2)!\,(1+vx^2)^{r+k-1}}f^{(r-k-1)}(x). \tag{2.4}$$

Remark. When $a = v = 1$, Lemma 2.3 is reduced to Lemma 2.2 in [4]. When $a = 1$ and $v = -1$, Lemma 2.3 is reduced to Lemma 1 in [5].

Proof. The method of induction on r is used to prove this result. The basis step is clear as the identity (2.3) holds for $r = 2$. Assume that the result holds for some $r \geq 2$. Now differentiating both sides of (2.3) with respect to x, we obtain

$$rf(x)^{r-1}f'(x) = \frac{a^{r-1}x^{2r-2}}{(r-1)!\,(1+vx^2)^{r-1}}f^{(r)}(x) + \frac{(2r-2)a^{r-1}x^{2r-3}}{(r-1)!\,(1+vx^2)^{r}}f^{(r-1)}(x)$$

$$+\sum_{k=1}^{r-2}\frac{\sum_{j=0}^{k-1}a^{r-1}(-v)^j\binom{k}{j}\binom{r-2}{k-j-1}x^{2r-k-2+2j}}{k(r-k-2)!\,(1+vx^2)^{r+k-1}}f^{(r-k)}(x)$$

$$+\sum_{k=1}^{r-2}\frac{\sum_{j=0}^{k-1}(2r-k-2+2j)a^{r-1}(-v)^j\binom{k}{j}\binom{r-2}{k-j-1}x^{2r-k-3+2j}}{k(r-k-2)!\,(1+vx^2)^{r+k}}\times$$

$$f^{(r-k-1)}(x) + \sum_{k=1}^{r-2}\frac{\sum_{j=0}^{k-1}(3k-2j)a^{r-1}(-v)^{j+1}\binom{k}{j}\binom{r-2}{k-j-1}x^{2r-k-1+2j}}{k(r-k-2)!\,(1+vx^2)^{r+k}}\times$$

$$f^{(r-k-1)}(x).$$

Further simplification leads to

$$
\begin{aligned}
&rf(x)^{r-1}f'(x)\\
&= \frac{a^{r-1}x^{2r-2}}{(r-1)!\,(1+vx^2)^{r-1}}f^{(r)}(x) + \frac{2a^{r-1}x^{2r-3}}{(r-2)!\,(1+vx^2)^{r}}f^{(r-1)}(x)\\
&+ \sum_{k=1}^{r-2}\frac{\sum_{j=0}^{k-1} a^{r-1}(-v)^j\binom{k}{j}\binom{r-2}{k-j-1}x^{2r-k-2+2j}}{k(r-k-2)!\,(1+vx^2)^{r+k-1}}f^{(r-k)}(x)\\
&+ \sum_{k=2}^{r-1}\frac{\sum_{j=0}^{k-2} a^{r-1}(-v)^j(2r-k-1+2j)\binom{k-1}{j}\binom{r-2}{k-j-2}x^{2r-k-2+2j}}{(k-1)(r-k-1)!\,(1+vx^2)^{r+k-1}}f^{(r-k)}(x)\\
&+ \sum_{k=2}^{r-1}\frac{\sum_{j=1}^{k-1} a^{r-1}(-v)^j(3k-2j-1)\binom{k-1}{j-1}\binom{r-2}{k-j-1}x^{2r-k-2+2j}}{(k-1)(r-k-1)!\,(1+vx^2)^{r+k-1}}f^{(r-k)}(x)\\
&= \frac{a^{r-1}x^{2r-2}f^{(r)}(x)}{(r-1)!\,(1+vx^2)^{r-1}}\\
&\qquad\qquad + r\sum_{k=1}^{r-1}\frac{\sum_{j=0}^{k-1} a^{r-1}(-v)^j\binom{k}{j}\binom{r-1}{k-j-1}x^{2r-k-2+2j}}{k(r-k-1)!\,(1+vx^2)^{r+k-1}}f^{(r-k)}(x).
\end{aligned}
$$

Using the fact $\frac{2}{(r-2)!} + \frac{1}{(r-3)!} = \frac{r}{(r-2)!}$ and the identity (2.1), we have

$$
\begin{aligned}
f(x)^{r+1} &= \frac{ax^2}{(1+vx^2)}f(x)^{r-1}f'(x)\\
&= \frac{ax^2}{(1+vx^2)}\frac{a^{r-1}x^{2r-2}f^{(r)}(x)}{r!\,(1+vx^2)^{r-1}}\\
&\qquad + \sum_{k=1}^{r-1}\frac{\sum_{j=0}^{k-1} a^{r-1}(-v)^j\binom{k}{j}\binom{r-1}{k-j-1}x^{2r-k-2+2j}}{k(r-k-1)!\,(1+vx^2)^{r+k-1}}f^{(r-k)}(x)\\
&= \frac{a^rx^{2r}f^{(r)}(x)}{r!\,(1+vx^2)^r} + \sum_{k=1}^{r-1}\frac{\sum_{j=0}^{k-1} a^{r}(-v)^j\binom{k}{j}\binom{r-1}{k-j-1}x^{2r-k+2j}}{k(r-k-1)!\,(1+vx^2)^{r+k}}f^{(r-k)}(x).
\end{aligned}
$$

This ends the proof. □

Theorem 2.4 *Let $r \geq 2$. Then for $n \geq 3r-5$,*

$$\sum_{l=0}^{2r-3} v^l \binom{2r-3}{l} \sum_{\substack{j_1+\cdots+j_r=n-2l \\ j_1,\dots,j_r\geq 1}} A_{j_1}\dots A_{j_r}$$
$$= a^{r-1}\sum_{k=1}^{r-1} v^{k-1}\frac{n-2k-r+3}{r-1}\binom{n-2k+1}{r-k-1}\binom{n-k-2r+3}{k-1}A_{n-2k-r+3}.$$

Remark. When $a=v=1$, Theorem 2.4 is reduced to Theorem 2.1 in [4]. When $a=1$ and $v=-1$, Theorem 2.4 is reduced to Theorem 3 in [5].

Proof. By virtue of Lemma 2.3, we have

$$\begin{aligned}&(1+vx^2)^{2r-3}f(x)^r\\ &= (1+vx^2)^{r-2}\frac{a^{r-1}x^{2r-2}f^{(r-1)}(x)}{(r-1)!} + \sum_{k=1}^{r-2}(1+vx^2)^{r-k-2}\\ &\quad\times\frac{\sum_{j=0}^{k-1} a^{r-1}(-v)^j\binom{k}{j}\binom{r-2}{k-j-1}x^{2r-k-2+2j}}{k(r-k-2)!}f^{(r-k-1)}(x).\end{aligned}\tag{2.5}$$

Since $A_0=0$, the left hand side of (2.5) becomes

$$\begin{aligned}&(1+vx^2)^{2r-3}\sum_{n=0}^{\infty}\sum_{\substack{j_1+\cdots+j_r=n \\ j_1,\dots,j_r\geq 1}} A_{j_1}\dots A_{j_r}x^n\\ &= \sum_{l=0}^{2r-3}\sum_{n=2l}^{\infty} v^l\binom{2r-3}{l}\sum_{\substack{j_1+\cdots+j_r=n-2l \\ j_1,\dots,j_r\geq 1}} A_{j_1}\dots A_{j_r}x^n.\end{aligned}$$

Further, the first term of the right hand side of (2.5) simplifies to

$$\begin{aligned}&(1+vx^2)^{r-2}\frac{a^{r-1}x^{2r-2}f^{(r-1)}(x)}{(r-1)!}\\ &= \sum_{t=0}^{r-2} v^t\binom{r-2}{t}x^{2t}a^{r-1}\frac{x^{2r-2}}{(r-1)!}\sum_{n=r-1}^{\infty}\frac{n!}{(n-r+1)!}A_n x^{n-r+1}\\ &= \frac{a^{r-1}}{(r-1)!}\sum_{t=0}^{r-2} v^t\binom{r-2}{t}\sum_{n=2r+2t-2}^{\infty}\frac{(n-r-2t+1)!}{(n-2r-2t+2)!}A_{n-r-2t+1}x^n.\end{aligned}$$

Furthermore, the second term of the right hand side of (2.5) becomes

$$\sum_{k=1}^{r-2}\left(\sum_{t=0}^{r-k-2} v^t \binom{r-k-2}{t} x^{2t}\right) \frac{1}{k(r-k-2)!} \sum_{j=0}^{k-1} a^{r-1}(-v)^j \binom{r-2}{k-j-1}$$
$$\times x^{2r-k-2+2j} \sum_{n=r-k-1}^{\infty} \frac{n!}{(n-r+k+1)!} A_n x^{n-r+k+1}$$
$$= \sum_{t=0}^{r-3}\sum_{j=0}^{r-t-3}\sum_{k=j}^{r-t-3} \frac{(-1)^j a^{r-1} v^{t+j}}{(k+1)(r-k-3)!\,(n-2r+k-2t-2j+3)!}$$
$$\times \binom{r-k-3}{t}\binom{k+1}{t}\binom{r-2}{k-j} \sum_{n=2r+2t+2j-k-3} (n-r-2t-2j+1)!$$
$$\times A_{n-r-2t-2j+1} x^n$$
$$= \sum_{t=0}^{r-3}\sum_{\delta=t+1}^{r-2}\sum_{k=\delta-t-1}^{r-t-3} \frac{(-1)^{\delta-t-1} a^{r-1} v^{\delta-1}}{(k+1)(r-k-3)!\,(n-2r+k-2\delta+5)!} \binom{r-k-3}{t}$$
$$\times \binom{k+1}{\delta-t-1}\binom{r-2}{k-\delta+t+1} \sum_{n=2r+2\delta-k-5} (n-r-2\delta+3)!\, A_{n-r-2\delta+3} x^n.$$

Now the right-hand side of (2.5) becomes

$$\frac{a^{r-1}v^{k-1}}{(r-1)!}\binom{r-2}{k-1}\frac{(n-r-2k+3)!}{(n-2r-2k+4)!}$$
$$+\sum_{t=0}^{k-1}\sum_{l=k-t-1}^{r-t-3} \frac{(-1)^{k-t-1} a^{r-1} v^{k-1}}{(l+1)(r-l-3)!\,(n-2r+l-2k+5)!}$$
$$\times \binom{r-l-3}{t}\binom{l+1}{l-t-1}\binom{r-2}{l-k+t+1}(n-r-2k+3)!$$
$$= \frac{a^{r-1}v^{k-1}(n-r-2k+3)}{(r-1)}\binom{n-2k+1}{r-k-1}\binom{n-k-2r+3}{k-1},$$

which completes the proof. □

Theorem 2.5 *For* $n \geq 2$, $\sum_{j=0}^{n} A_j A_{n-j} = a \sum_{m=1}^{n-1} v^{\frac{(n-m-1)}{2}} \cos\frac{(n-m-1)\pi}{2} m A_m$.

Remark. When $a = v = 1$, Theorem 2.5 is reduced to Theorem 4.1 in [4]. When $a = 1$ and $v = -1$, Theorem 2.5 is reduced to Theorem 4 in [5].

Proof. From (2.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{j=0}^{n} A_j A_{n-j} x^n &= \left(\sum_{n=0}^{\infty} A_n x^n\right)\left(\sum_{m=0}^{\infty} A_m x^m\right)\\ &= ax^2\left(\sum_{j=0}^{\infty}(-v)^j x^{2j}\right)\left(\sum_{m=1}^{\infty} mA_m x^{m-1}\right)\\ &= ax^2\left(\sum_{j=0}^{\infty}\gamma_j v^{j/2} x^j\right)\left(\sum_{m=0}^{\infty}(m+1)A_{m+1} x^m\right)\\ &= a\sum_{n=0}^{\infty}\sum_{m=0}^{n-1} mA_m \gamma_{n-m-1} v^{(n-m-1)/2} x^n,\end{aligned}$$

where $\gamma_j = \cos\frac{j\pi}{2}$ for $j \geq 0$. Comparing the coefficients of like power of x from both the sides of above equation, we obtain the desired result. □

Let C_n be the n-th Catalan number [7], defined by

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Theorem 2.6 *For $n \geq r \geq 2$, we have*

$$\begin{aligned}&\sum_{\substack{j_1+\cdots+j_r=n\\ j_1,\ldots,j_r\geq 1}} A_{j_1}\ldots A_{j_r}\\ &\quad= \frac{a^{r-1}C_{r-2}}{(2r-4)!\,2^{2r-4}}\sum_{m=1}^{n-r+1} v^{\frac{n-m-r+1}{2}}\frac{(n+m+r-3)!!\,(n-m+r-3)!!}{(n+m-r+1)!!\,(n-m-r+1)!!}\\ &\qquad\times mA_m\cos\frac{(n-m-r+1)\pi}{2}.\end{aligned}$$

Remark. When $a = v = 1$, Theorem 2.6 is reduced to Theorem 4.2 in [4]. When $a = 1$ and $v = -1$, Theorem 2.6 is reduced to Theorem 5 in [5].

Proof. Using binomial expansion, we have

$$\begin{aligned}\frac{1}{(1+vx^2)^{r-1}} &= \sum_{i=0}^{\infty}(-v)^i\binom{i+r-2}{r-2}x^{2i}\\ &= \sum_{k=0}^{\infty}\frac{v^{k/2}}{(r-2)!\,2^{r-2}}\frac{(k+2r-4)!!}{k!!}\cos\frac{k\pi}{2}x^k. \qquad (2.6)\end{aligned}$$

Since

$$f^{(r-1)}(x)=\sum_{m=0}^{\infty}\frac{(m+r-1)!}{m!}A_{m+r-1}x^m, \tag{2.7}$$

the first term on the right-hand side of (2.3) becomes

$$\begin{aligned}
&\frac{a^{r-1}x^{2r-2}f^{(r-1)}(x)}{(r-1)!\,(1+vx^2)^{r-1}}\\
&=\frac{a^{r-1}x^{2r-2}}{(r-1)!}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{v^{(n-m)/2}}{(r-2)!\,2^{r-2}}\frac{(n-m+2r-4)!\,!}{(n-m)!\,!}\\
&\qquad\times\cos\frac{(n-m)\pi}{2}\frac{(m+r-1)!}{m!}A_{m+r-1}x^n\\
&=\frac{a^{r-1}}{(r-1)!\,(r-2)!\,2^{r-2}}\sum_{n=2r-2}^{\infty}\sum_{m=0}^{n-2r+2}v^{\frac{(n-2r+2-m)}{2}}\\
&\quad\times\frac{(n-m-2)!\,!}{(n-m-2r-2)!\,!}\cos\frac{(n-m-2r+2)\pi}{2}\frac{(m+r-1)!}{m!}A_{m+r-1}x^n\\
&=\frac{a^{r-1}}{(r-1)!\,(r-2)!\,2^{r-2}}\sum_{n=2r-2}^{\infty}\sum_{m=r-1}^{n-r+1}v^{\frac{(n-m-r+1)}{2}}\frac{(n-m+r-3)!\,!}{(n-m-r+1)!\,!}\\
&\qquad\times\cos\frac{(n-m-r+1)\pi}{2}\frac{m!}{(m-r+1)!}A_mx^n.
\end{aligned}$$

In a similar use of the expressions (2.6) and (2.7) gives

$$\begin{aligned}
&\frac{\sum_{j=0}^{k-1}a^{r-1}(-v)^j\binom{k}{j}\binom{r-2}{k-j-1}x^{2r-k-2+2j}}{(1+vx^2)^{r+k-1}}f^{(r-k-1)}(x)\\
&=\sum_{j=0}^{k-1}a^{r-1}(-v)^j\binom{k}{j}\binom{r-2}{k-j-1}x^{2r-k+2j-2}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{v^{(n-m)/2}}{(r+k-2)!\,2^{r+k-2}}\\
&\quad\times\frac{(n-m+2r+2k-4)!\,!}{(n-m)!\,!}\cos\frac{(n-m)\pi}{2}\frac{(m+r-k-1)!}{m!}A_{m+r-k-1}x^n\\
&=\frac{a^{r-1}}{(r+k-2)!\,2^{r+k-2}}\sum_{j=0}^{k-1}(-v)^j\binom{k}{j}\binom{r-2}{k-j-1}\sum_{n=2r-k-2+2j}^{\infty}\sum_{m=0}^{n-2r+k+2-2j}\\
&\quad\times v^{\frac{n-m-2r+k+2-2j}{2}}\frac{(n-m+3k-2-2j)!\,!}{(n-m-2r+k+2-2j)!\,!}\\
&\quad\times\cos\frac{(n-m+k-2r+2-2j)\pi}{2}\frac{(m+r-k-1)!}{m!}A_{m+r-k-1}x^n.
\end{aligned}$$

$$
\begin{aligned}
&= \frac{a^{r-1}}{(r+k-2)!\,2^{r+k-2}} \sum_{j=0}^{k-1} (-v)^j \binom{k}{j} \binom{r-2}{k-j-1} \\
&\quad \times \sum_{n=2r-k-2}^{\infty} \sum_{m=0}^{n-2r+k+2} v^{\frac{n-m+k-2r+2-2j}{2}} \frac{(n-m+3k-2-2j)!!}{(n-m-2r+k+2-2j)!!} \\
&\quad \times \cos\frac{(n-m+k-2r+2-2j)\pi}{2} \frac{(m+r-k-1)!}{m!} A_{m+r-k-1} x^n \\
&= \frac{a^{r-1}}{(r+k-2)!\,2^{r+k-2}} \sum_{n=2r-k-2}^{\infty} \sum_{m=r-k-1}^{n-r+1} \sum_{j=0}^{k-1} (-v)^j \binom{k}{j} \binom{r-2}{k-j-1} \\
&\quad \times v^{\frac{n-m+k-2r+2-2j}{2}} \frac{(n-m+r+2k-3-2j)!!}{(n-m-r+k+3-2j)!!} \\
&\quad \times \cos\frac{(n-m+k-2r+2-2j)\pi}{2} \frac{m!}{(m-r+k+1)!} A_m x^n.
\end{aligned}
$$

Since

$$
\frac{(n-m+r+2k-3-2j)!!}{(n-m-r+k+3-2j)!!} = 0,
$$

for $m = n-2r+k+2-2j$ $(j = 1, 2, \dots, k-2)$, this is equal to

$$
\begin{aligned}
&\frac{a^{r-1}}{(r+k-2)!\,2^{r+k-2}} \sum_{n=2r-k-2}^{\infty} \sum_{m=r-k-1}^{n-r+1} v^{\frac{n-m-r+1}{2}} \frac{(n-m+r-1)!!}{(n-m-r+1)!!} \\
&\quad \times \binom{r+k-2}{k-1} \frac{(n-m+r-3)!!}{(n-m+r-2k-1)!!} \cos\frac{(n-m-r+1)\pi}{2} \\
&\quad \times \frac{m!}{(m-r+k+1)!} A_m x^n.
\end{aligned}
$$

Therefore, the equation (2.3) becomes

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{\substack{j_1+\cdots+j_r=n \\ j_1,\dots,j_r \ge 1}} A_{j_1} \dots A_{j_r} x^n \\
&= \frac{a^{r-1}}{(r-1)!\,(r-2)!\,2^{r-2}} \sum_{n=2r-2}^{\infty} \sum_{m=r-1}^{n-r+1} v^{\frac{n-m-r+1}{2}} \frac{(n-m+r-3)!!}{(n-m-r+1)!!} \\
&\quad \times \cos\frac{(n-m-r+1)\pi}{2} \frac{m!}{(m-r+1)!} A_m x^n + \sum_{k=1}^{r-2} \frac{1}{k(r-k-2)!}
\end{aligned}
$$

$$
\begin{aligned}
&\times \frac{a^{r-1}}{(r+k-2)!\,2^{r+k-2}} \sum_{n=2r-k-2}^{\infty} \sum_{m=r-k-1}^{n-r+1} v^{\frac{n-m-r+1}{2}} \frac{(n-m+r-1)!\,!}{(n-m-r+1)!\,!} \\
&\times \binom{r+k-2}{k-1} \frac{(n-m+r-3)!\,!}{(n-m+r-2k-1)!\,!} \cos\frac{(n-m-r+1)\pi}{2} \\
&\times \frac{m!}{(m-r+k+1)!} A_m x^n \\
= &\sum_{n=r-1}^{\infty} \sum_{m=1}^{r-2} \Bigg(\frac{a^{r-1}}{(r-1)!\,2^{r-2}} \sum_{k=r-m-1}^{r-2} \frac{1}{k!\,(r-k-2)!\,2^k} \frac{(n-m+r-1)!\,!}{(n-m-r+1)!\,!} \\
&\times \frac{(n-m+r-3)!\,!}{(n-m+r-2k-1)!\,!} \frac{m!}{(m-r+k+1)!} \Bigg) v^{\frac{n-m-r+1}{2}} \\
&\times \cos\frac{(n-m-r+1)\pi}{2} A_m x^n + \sum_{n=r-1}^{\infty} \sum_{m=r-1}^{n-r+1} \Bigg(\frac{a^{r-1}}{(r-1)!\,(r-2)!\,2^{r-2}} \\
&\times \frac{(n-m+r-3)!\,!}{(n-m-r+1)!\,!} \frac{m!}{(m-r+1)!} + \frac{a^{r-1}}{(r-1)!\,2^{r-2}} \\
&\times \sum_{k=r-m-1}^{r-2} \frac{1}{k!\,(r-k-2)!\,2^k} \frac{(n-m+r-1)!\,!}{(n-m-r+1)!\,!} \frac{(n-m+r-3)!\,!}{(n-m+r-2k-1)!\,!} \\
&\times \frac{m!}{(m-r+k+1)!} \Bigg) v^{\frac{n-m-r+1}{2}} \cos\frac{(n-m-r+1)\pi}{2} A_m x^n.
\end{aligned}
$$

For $1 \le m \le r-2$ and $r-1 \le m \le n-r+1$, the above expression reduces to

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{\substack{j_1+\cdots+j_r=n \\ j_1,\ldots,j_r \ge 1}} A_{j_1} \ldots A_{j_r} x^n \\
&= \Bigg[\sum_{n=0}^{\infty} \sum_{m=1}^{r-2} \Bigg(\frac{a^{r-1} m}{(r-1)!\,(r-2)!\,2^{2r-4}} \frac{(n+m+r-3)!\,!}{(n+m-r+1)!\,!} \frac{(n-m+r-3)!\,!}{(n-m-r+1)!\,!} \Bigg) \\
&+ \sum_{n=0}^{\infty} \sum_{m=r-1}^{n-r+1} \Bigg(\frac{a^{r-1}}{(r-1)!\,(r-2)!\,2^{2r-4}} \frac{(n+m+r-3)!\,!}{(n+m-r+1)!\,!} \\
&\quad \times \frac{(n-m+r-3)!\,!}{(n-m-r+1)!\,!} m \Bigg) \Bigg] v^{\frac{n-m-r+1}{2}} \cos\frac{(n-m-r+1)\pi}{2} A_m x^n,
\end{aligned}
$$

which ends the proof. □

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