

## Wavelet frame characterization of Lebesgue spaces on local fields

Firdous A. Shah · Shallu Sharma ·  
M. Younus Bhat

**Abstract** Shah and Debnath [Tight wavelet frames on local fields, *Analysis*, 33 (2013), 293-307] have derived an explicit method for constructing tight wavelet frames on local fields using the machinery of unitary extension principle. Continuing our investigation of wavelet frames on local fields, this paper deals with the establishment of complete characterization of functions in Lebesgue spaces  $L^p(K)$ ,  $1 < p < \infty$ , in terms of their wavelet frame coefficients.

**Keywords** Wavelet frame, extension principle, Lebesgue space, maximal function, local field, Fourier transform.

**Mathematics Subject Classification (2010)** 42C15; 42C40, 42B35, 43A70, 11S85.

### 1 Introduction

Function spaces play an important role in both classical and modern analysis, ordinary and partial differential equations, and approximation theory. Representation of function spaces in terms of wavelet bases and other subsequent allied developments have been of significant importance and interest to mathematicians and signal analysts. The problem of characterizing function spaces using wavelet-type systems has been extensively studied by several authors. For example, Meyer [14] proved that wavelets with polynomial decay form an unconditional bases for Lebesgue spaces. On the other hand, Gripenberg [12] has given unconditional wavelet bases for the Lebesgue spaces  $L^p(\mathbb{R})$ ,  $1 < p < \infty$  without any smoothness of the mother wavelet. Similar results with weaker hypotheses were proved by Wojtaszczyk in [22]. These studies have been extended by Borup et al. [5], where they have established a complete characterization of Lebesgue and Sobolev spaces in terms of analysis coefficients associated with the wavelet frames generated through an extension principle. More results in this direction can also be found in [6, 8, 9, 11] and the references therein.

Considerable research has been carried out on the construction of wavelets on local fields or more generally on local fields of positive characteristic. For instance, J. Benedetto and R. Benedetto [3] built wavelet bases on local fields containing compact open subgroups. The notion of multiresolution analysis on local fields of positive characteristic was introduced by Jiang et al. [13]. They investigate certain properties of multiresolution subspaces which provides the quantitative criteria for the construction of MRA in  $L^2(K)$  and gave an algorithm for constructing wavelet basis on local fields. On other hand, an excellent construction of tight wavelet frames on local fields of positive characteristic was given by Shah and Debnath [19] by adapting the extension principles of Daubechies et al. [7] on the Euclidean spaces to the local fields. To be more precise, they provide a complete characterization of tight wavelet frames on local fields by virtue of the modulation matrix  $\mathcal{M}(\xi) = \{m_\ell(\xi + \mathfrak{p}u(k))\}_{\ell,k=0}^{q-1}$  formed by the wavelet masks  $m_\ell(\xi)$ ,  $\ell = 0, 1, \dots, N$ . These studies were proceeded by Shah and his associates in [15, 16, 17, 18], where they have given some algorithms for constructing periodic wavelet frames, wave packet frames, and semi-orthogonal wavelet frames on local fields. The characterization of wavelets and MRA wavelets on local fields has been completely discussed by Behera and Jahan in [2] by virtue of some basic equations in the Fourier domain. As far as the construction of wavelets in Lebesgue spaces of local fields is concerned, Behera [1] has described a general scheme for constructing Haar wavelets on local fields and proved that under some mild conditions, the Haar wavelet system form an unconditional basis for  $L^p(K)$ ,  $1 < p < \infty$ .

Drawing inspiration from the construction of wavelets and MRA based wavelet frames on local fields of positive characteristic [1, 19], our aim is to characterize the functions in Lebesgue spaces  $L^p(K)$  of local fields in terms of their frame wavelet coefficients.

The rest of this paper is organized as follows. In Section 2, we review some concepts and fix some notations and terminologies concerning local fields, multiresolution analysis and MRA based wavelet frames over local fields of positive characteristic. Section 3 focuses on characterization of Lebesgue spaces of local fields by means of framelets generated by the unitary extension principles.

## 2 Preliminaries and wavelet frames on local fields

Let  $K$  be a field and a topological space. Then  $K$  is called a *local field* if both  $K^+$  and  $K^*$  are locally compact Abelian groups, where  $K^+$  and  $K^*$  denote the additive and multiplicative groups of  $K$ , respectively. If  $K$  is any field and is endowed with the discrete topology, then  $K$  is a local field. Further, if  $K$  is connected, then  $K$  is either  $\mathbb{R}$  or  $\mathbb{C}$ . If  $K$  is not connected, then it is totally disconnected. Hence by a local field, we mean a field  $K$  which is locally compact, non-discrete and totally disconnected. The  $p$ -adic fields are examples of local fields. We use the notation of the book by Taibleson [21].

In the rest of this paper, we use the symbols  $\mathbb{N}$ ,  $\mathbb{N}_0$  and  $\mathbb{Z}$  to denote the sets of natural, non-negative integers and integers, respectively.

Let  $K$  be a local field. Let  $dx$  be the Haar measure on the locally compact Abelian group  $K^+$ . If  $\alpha \in K$  and  $\alpha \neq 0$ , then  $d(\alpha x)$  is also a Haar measure. Let  $d(\alpha x) = |\alpha|dx$ . We call  $|\alpha|$  the *absolute value* of  $\alpha$ . Moreover, the map  $x \rightarrow |x|$  has the following properties:

- (a)  $|x| = 0$  if and only if  $x = 0$ ;
- (b)  $|xy| = |x||y|$  for all  $x, y \in K$ ;
- (c)  $|x + y| \leq \max\{|x|, |y|\}$  for all  $x, y \in K$ .

Property (c) is called the *ultrametric inequality*. The set  $\mathfrak{D} = \{x \in K : |x| \leq 1\}$  is called the *ring of integers* in  $K$ . It is the unique maximal compact subring of  $K$ . Define  $\mathfrak{B} = \{x \in K : |x| < 1\}$ . The set  $\mathfrak{B}$  is called the *prime ideal* in  $K$ . The prime ideal in  $K$  is the unique maximal ideal in  $\mathfrak{D}$  and hence as result  $\mathfrak{B}$  is both principal and prime. Since the local field  $K$  is totally disconnected, so there exist an element of  $\mathfrak{B}$  of maximal absolute value. Let  $\mathfrak{p}$  be a fixed element of maximum absolute value in  $\mathfrak{B}$ . Such an element is called a *prime element* of  $K$ . Therefore, for such an ideal  $\mathfrak{B}$  in  $\mathfrak{D}$ , we have  $\mathfrak{B} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{D}$ . As it was proved in [21], the set  $\mathfrak{D}$  is compact and open. Hence,  $\mathfrak{B}$  is compact and open. Therefore, the residue space  $\mathfrak{D}/\mathfrak{B}$  is isomorphic to a finite field  $GF(q)$ , where  $q = p^c$  for some prime  $p$  and  $c \in \mathbb{N}$ .

Let  $\mathfrak{D}^* = \mathfrak{D} \setminus \mathfrak{B} = \{x \in K : |x| = 1\}$ . Then, it can be proved that  $\mathfrak{D}^*$  is a group of units in  $K^*$  and if  $x \neq 0$ , then we may write  $x = \mathfrak{p}^k x', x' \in \mathfrak{D}^*$ . For a proof of this fact we refer to [21]. Moreover, each  $\mathfrak{B}^k = \mathfrak{p}^k \mathfrak{D} = \{x \in K : |x| \leq q^{-k}\}$  is a compact subgroup of  $K^+$  and usually known as the *fractional ideals* of  $K^+$ . Let  $\mathcal{U} = \{a_i\}_{i=0}^{q-1}$  be any fixed full set of coset representatives of  $\mathfrak{B}$  in  $\mathfrak{D}$ , then every element  $x \in K$  can be expressed uniquely as  $x = \sum_{\ell=k}^{\infty} c_\ell \mathfrak{p}^\ell$  with  $c_\ell \in \mathcal{U}$ . Let  $\chi$  be a fixed character on  $K^+$  that is trivial on  $\mathfrak{D}$  but is non-trivial on  $\mathfrak{B}^{-1}$ . Therefore,  $\chi$  is constant on cosets of  $\mathfrak{D}$  so if  $y \in \mathfrak{B}^k$ , then  $\chi_y(x) = \chi(y, x), x \in K$ . Suppose that  $\chi_u$  is any character on  $K^+$ , then clearly the restriction  $\chi_u|_{\mathfrak{D}}$  is also a character on  $\mathfrak{D}$ . Therefore, if  $\{u(n) : n \in \mathbb{N}_0\}$  is a complete list of distinct coset representative of  $\mathfrak{D}$  in  $K^+$ , then, as it was proved in [21], the set  $\{\chi_{u(n)} : n \in \mathbb{N}_0\}$  of distinct characters on  $\mathfrak{D}$  is a complete orthonormal system on  $\mathfrak{D}$ .

**Definition 2.1** If  $f \in L^1(K)$ , then the Fourier transform of  $f$  is defined by

$$\mathcal{F}\{f(x)\} = \hat{f}(\xi) = \int_K f(x) \overline{\chi_\xi(x)} dx. \quad (2.1)$$

It is noted that

$$\hat{f}(\xi) = \int_K f(x) \overline{\chi_\xi(x)} dx = \int_K f(x) \chi(-\xi x) dx.$$

The properties of Fourier transform on local field  $K$  are much similar to those of on the real line. In fact, the Fourier transform have the following properties:

1. The map  $f \rightarrow \hat{f}$  is a bounded linear transformation of  $L^1(K)$  into  $L^\infty(K)$ , and  $\|\hat{f}\|_\infty \leq \|f\|_1$ .
2. If  $f \in L^1(K)$ , then  $\hat{f}$  is uniformly continuous.
3. If  $f \in L^1(K) \cap L^2(K)$ , then  $\|\hat{f}\|_2 = \|f\|_2$ .

The Fourier transform of a function  $f \in L^2(K)$  is defined by

$$\hat{f}(\xi) = \lim_{k \rightarrow \infty} \hat{f}_k(\xi) = \lim_{k \rightarrow \infty} \int_{|x| \leq q^k} f(x) \overline{\chi_\xi(x)} dx, \quad (2.2)$$

where  $f_k = f \Phi_{-k}$  and  $\Phi_k$  is the characteristic function of  $\mathfrak{B}^k$ . Furthermore, if  $f \in L^2(\mathfrak{D})$ , then we define the Fourier coefficients of  $f$  as

$$\hat{f}(u(n)) = \int_{\mathfrak{D}} f(x) \overline{\chi_{u(n)}(x)} dx. \quad (2.3)$$

We now impose a natural order on the sequence  $\{u(n)\}_{n=0}^\infty$ . We have  $\mathfrak{D}/\mathfrak{B} \cong GF(q)$  where  $GF(q)$  is a  $c$ -dimensional vector space over the field  $GF(p)$ . We choose a set  $\{1 = \zeta_0, \zeta_1, \zeta_2, \dots, \zeta_{c-1}\} \subset \mathfrak{D}^*$  such that  $\text{span} \{\zeta_j\}_{j=0}^{c-1} \cong GF(q)$ . For  $n \in \mathbb{N}_0$  satisfying

$$0 \leq n < q, \quad n = a_0 + a_1 p + \dots + a_{c-1} p^{c-1}, \quad 0 \leq a_k < p, \quad \text{and } k = 0, 1, \dots, c-1,$$

we define

$$u(n) = (a_0 + a_1 \zeta_1 + \dots + a_{c-1} \zeta_{c-1}) \mathfrak{p}^{-1}.$$

Also, for  $n = b_0 + b_1 q + b_2 q^2 + \dots + b_s q^s$ ,  $n \in \mathbb{N}_0$ ,  $0 \leq b_k < q$ ,  $k = 0, 1, 2, \dots, s$ , we set

$$u(n) = u(b_0) + u(b_1) \mathfrak{p}^{-1} + \dots + u(b_s) \mathfrak{p}^{-s}.$$

This defines  $u(n)$  for all  $n \in \mathbb{N}_0$ . In general, it is not true that  $u(m+n) = u(m) + u(n)$ . But, if  $r, k \in \mathbb{N}_0$  and  $0 \leq s < q^k$ , then  $u(rq^k + s) = u(r) \mathfrak{p}^{-k} + u(s)$ . Further, it is also easy to verify that  $u(n) = 0$  if and only if  $n = 0$  and  $\{u(\ell) + u(k) : k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$  for a fixed  $\ell \in \mathbb{N}_0$ . Hereafter we use the notation  $\chi_n = \chi_{u(n)}$ ,  $n \geq 0$ .

Let the local field  $K$  be of characteristic  $p > 0$  and  $\zeta_0, \zeta_1, \zeta_2, \dots, \zeta_{c-1}$  be as above. We define a character  $\chi$  on  $K$  as follows:

$$\chi(\zeta_\mu \mathfrak{p}^{-j}) = \begin{cases} \exp(2\pi i/p), & \mu = 0 \text{ and } j = 1, \\ 1, & \mu = 1, \dots, c-1 \text{ or } j \neq 1. \end{cases} \quad (2.4)$$

Since  $\bigcup_{j \in \mathbb{Z}} \mathfrak{p}^{-j} \mathfrak{D} = K$ , we can regard  $\mathfrak{p}^{-1}$  as the dilation and since  $\{u(n) : n \in \mathbb{N}_0\}$  is a complete list of distinct coset representatives of  $\mathfrak{D}$  in  $K$ , the set  $\Lambda = \{u(n) : n \in \mathbb{N}_0\}$  can be treated as the translation set. Note that  $\Lambda$  is a subgroup of  $K^+$  and is not a group for the local field of characteristic zero.

For  $1 \leq p < \infty$ , we define  $L^p(K)$  as the Lebesgue space of all Lebesgue measurable functions  $f : K \rightarrow \mathbb{C}$  together with the norm

$$\|f\|_{L^p(K)} = \left( \int_K |f(x)|^p dx \right)^{1/p} < \infty. \quad (2.5)$$

Next, we define the dilation  $\delta_j$  and the translation operators  $\tau_y$  as follows:

$$\delta_j f(x) = q^{j/2} f(\mathfrak{p}^{-j} x) \quad \text{and} \quad \tau_y f(x) = f(x - y), \quad f \in L^2(K).$$

For given  $\Psi := \{\psi^1, \psi^2, \dots, \psi^N\} \subset L^2(K)$ , define the wavelet system

$$X(\Psi) = \left\{ \psi_{j,k}^\ell := q^{j/2} \psi^\ell(\mathfrak{p}^{-j} x - u(k)), j \in \mathbb{Z}, k \in \mathbb{N}_0, \ell = 1, 2, \dots, N \right\}. \quad (2.6)$$

The wavelet system  $X(\Psi)$  is called a *wavelet frame*, if there exist positive numbers  $A$  and  $B$  with  $0 < A \leq B < \infty$  such that

$$A \|f\|_2^2 \leq \sum_{\ell=1}^N \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{N}_0} \left| \langle f, \psi_{j,k}^\ell \rangle \right|^2 \leq B \|f\|_2^2, \quad \text{for all } f \in L^2(K). \quad (2.7)$$

The largest  $A$  and the smallest  $B$  for which (2.7) holds are called *wavelet frame bounds*. A *tight wavelet frame* refers to the case when  $A = B$ , and a *normalized* or *Parseval wavelet frame* refers to the case when  $A = B = 1$ .

A generalization of classical theory of multiresolution analysis on local fields of positive characteristic was considered by Jiang et al. [13]. Analogous to the Euclidean case, following is a definition of *multiresolution analysis* on local field  $K$  of positive characteristic.

**Definition 2.2** *Let  $K$  be a local field of positive characteristic and  $\mathfrak{p}$  be a prime element of  $K$ . A multiresolution analysis (MRA) of  $L^2(K)$  is a sequence of closed subspaces  $\{V_j : j \in \mathbb{Z}\}$  of  $L^2(K)$  satisfying the following properties:*

- (a)  $V_j \subset V_{j+1}$  for all  $j \in \mathbb{Z}$ ;
- (b)  $\bigcup_{j \in \mathbb{Z}} V_j$  is dense in  $L^2(K)$ ;
- (c)  $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$ ;
- (d)  $f(\cdot) \in V_j$  if and only if  $f(\mathfrak{p}^{-1} \cdot) \in V_{j+1}$  for all  $j \in \mathbb{Z}$ ;

(e) there is a function  $\phi \in V_0$ , called the scaling function, such that

$$\{\phi(x - u(k)) : k \in \mathbb{N}_0\}$$

forms an orthonormal basis for  $V_0$ .

**Definition 2.3** We say that a function  $\psi$  defined on local field  $K$  belongs to the regularity class  $\mathcal{R}(K)$  if there exists constants  $C_1, C_2, \gamma$  and  $\varepsilon > 0$  such that

$$(a) \quad \hat{\psi}(0) = \int_K \psi(x) dx = 0$$

$$(b) \quad |\hat{\psi}(\xi)| \leq C_2(1 + |\xi|)^{-1-\varepsilon}, \quad \forall \xi \in K.$$

An MRA is said to be regular if the subspace  $V_0$  has an orthonormal basis of the form  $\{\phi(x - u(k)) : k \in \mathbb{N}_0\}$  for some regular scaling function  $\phi$ .

Given such a regular MRA, as in the case of  $\mathbb{R}^n$ , we can find a collection of smooth (of class  $\mathcal{R}(K)$ ) functions  $\{\psi^1, \psi^2, \dots, \psi^N\} \subset V_1$ , such that their translates and dilations form an regular orthonormal basis of  $W_j$ , where  $W_j, j \in \mathbb{Z}$  are the direct complementary subspace of  $V_j$  in  $V_{j+1}$  defined by

$$W_j = \overline{\text{span}} \left\{ q^{j/2} \psi^\ell(\mathfrak{p}^{-j}x - u(k)) : 1 \leq \ell \leq N, k \in \mathbb{N}_0 \right\}, \quad j \in \mathbb{Z}. \quad (2.8)$$

The most widely recognized technique for building wavelet frames relies on unitary extension principles (UEP) presented by Ron and Shen [20] and subsequently extended by Daubechies et al. [7] in the form of the oblique extension principle (OEP). In contrast with other wavelet frame characterizations, the conditions showed in the two principles are essentially simple to check, which makes the construction of wavelet frames painless. Following the unitary extension principle [19], one often begins with a refinable function  $\phi$  or even with a refinement mask to construct desired wavelet frames.

Consider a refinement mask  $m_0(\xi)$  of the form

$$m_0(\xi) = \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} a_k \overline{\chi_k(\xi)}, \quad (2.9)$$

such that

$$\hat{\phi}(\xi) = m_0(\mathfrak{p}\xi)\hat{\phi}(\mathfrak{p}\xi). \quad (2.10)$$

Given a regular MRA  $\{V_j : j \in \mathbb{Z}\}$  generated by a refinable  $\phi(x)$ , one can construct a set of basic framelets  $\Psi := \{\psi^1, \psi^2, \dots, \psi^N\} \subset V_1$  satisfying

$$\hat{\psi}_\ell(\omega) = m_\ell(\mathfrak{p}\omega)\hat{\phi}(\mathfrak{p}\omega), \quad (2.11)$$

where

$$m_\ell(\omega) = \frac{1}{\sqrt{q}} \sum_{n \in \mathbb{N}_0} b_n^\ell \overline{\chi_n(\omega)}, \quad \ell = 1, \dots, N = q - 1 \quad (2.12)$$

are the integral periodic functions in  $L^2(\mathfrak{D})$  and are called wavelet masks. With  $\mu_\ell(\omega), \ell = 0, 1, \dots, N, N > q - 1$ , as the wavelet masks, we formulate the matrix  $\mathcal{M}(\omega)$  as:

$$\mathcal{M}(\xi) = \begin{pmatrix} m_0(\xi) & m_0(\xi + \mathfrak{p}u(1)) & \dots & m_0(\xi + \mathfrak{p}u(q-1)) \\ m_1(\xi) & m_1(\xi + \mathfrak{p}u(1)) & \dots & m_1(\xi + \mathfrak{p}u(q-1)) \\ \vdots & \vdots & \ddots & \vdots \\ m_N(\xi) & m_N(\xi + \mathfrak{p}u(1)) & \dots & m_N(\xi + \mathfrak{p}u(q-1)) \end{pmatrix}_{N+1 \times N+1} \quad (2.13)$$

The matrix  $\mathcal{M}(\xi)$  is called the *modulation matrix*. The characterization of the wavelet system (2.6) to be an orthonormal basis over the Vilenkin groups in terms of the modulation matrices has been studied by Farkov et al. [10] where as an explicit algorithm for the construction of the unitary modulation matrix  $\mathcal{M}(\xi)$  over local fields of positive characteristic has been obtained by Berdnikov et al. [4]. In [19], Shah and Debnath built up a strategy to produce wavelet frames on local fields using extension principles and established a complete characterization of such frames by virtue of the modulation matrix  $\mathcal{M}(\xi)$ . More precisely, they demonstrated that the wavelet system  $X(\Psi)$  given by (2.6) constitutes a tight wavelet frame for  $L^2(K)$  if the modulation matrix  $\mathcal{M}(\xi)$  given by (2.13) satisfies the UEP condition

$$\mathcal{M}(\xi)\mathcal{M}^*(\xi) = I_{N+1}, \quad \text{for } a.e. \xi \in \sigma(V_0), \quad (2.14)$$

where  $\sigma(V_0) := \{\xi \in \mathfrak{D} : \sum_{k \in \mathbb{N}_0} |\hat{\phi}(\xi + u(k))|^2 \neq 0\}$ .

In order to prove the main result to be presented in next section, we need the following lemma whose proof can be found in [14].

**Lemma 2.4** *Let  $\{a_k : k = 1, 2, \dots, M\}$  be a scalar sequence. Then*

$$\sum_{k=1}^M |a_k|^p \leq \left( \sum_{k=1}^M |a_k| \right)^p \leq M^{p-1} \sum_{k=1}^M |a_k|^p, \quad p \geq 1,$$

$$M^{p-1} \sum_{k=1}^M |a_k|^p \leq \left( \sum_{k=1}^M |a_k| \right)^p \leq \sum_{k=1}^M |a_k|^p, \quad 0 < p < 1.$$

With Lemma 2.4 at hand, we can obtain the following result:

**Lemma 2.5** *Let  $1 < p < \infty$ . Suppose that  $\{f_k\}_{k=1}^M$  is a non-negative function sequence in  $L^p(K)$ . Then*

$$\sum_{k=1}^M \|f_k\|_{L^p(K)} \leq M^{2-2/p} \left\| \sum_{k=1}^M f_k \right\|_{L^p(K)}. \quad (2.15)$$

*Proof.* Using Lemma 2.4 for  $0 < p < 1$ , we have

$$\begin{aligned} \sum_{k=1}^M \|f_k\|_{L^p(K)} &= \sum_{k=1}^M \left\{ \int_K |f_k(x)|^p dx \right\}^{1/p} \\ &\leq M^{1-1/p} \sum_{k=1}^M \left\{ \int_K |f_k(x)|^p dx \right\}^{1/p}. \end{aligned}$$

Again applying Lemma 2.4 and using the fact that the sum is finite, we obtain

$$\begin{aligned} \sum_{k=1}^M \|f_k\|_{L^p(K)} &\leq M^{1-1/p} \left\{ M^{1-1/p} \left\{ \sum_{k=1}^M \int_K |f_k(x)|^p dx \right\}^{1/p} \right\} \\ &= M^{2-2/p} \left\{ \int_K \sum_{k=1}^M |f_k(x)|^p dx \right\}^{1/p} \\ &= M^{2-2/p} \left\| \sum_{k=1}^M f_k \right\|_{L^p(K)}. \end{aligned}$$

□

### 3 Wavelet frame characterization of Lebesgue spaces

In this section, we establish our main results concerning the characterization of Lebesgue spaces  $L^p(K)$ ,  $1 < p < \infty$  on local fields of positive characteristic by means of the framelets generated by the so-called extension principles. For any  $t \in K^*$ , we consider the function  $\phi_t(x) = \frac{1}{|t|} \phi(x/t)$  such that  $\phi$  belongs to the regular class  $\mathcal{R}(K)$ . Then, for this choice  $\phi(x)$ , we define the *Littlewood-Paley function*  $Pf(x)$  on local fields of positive characteristic as:

$$Pf(x) = \left\{ \int_K |\phi_t * f(x)|^2 \frac{dt}{t} \right\}^{1/2} \quad \text{for all } f \in \bigcup_{p>1} L^p(K) \quad (3.1)$$

where  $\phi_t * f$  denotes convolution of  $\phi_t$  and  $f$ . The discrete version of equality (3.1) can be estimated as

$$\begin{aligned} [Pf(x)]^2 &= \sum_{j \in \mathbb{Z}} \int_{\mathfrak{p}^j(1+\mathfrak{D})} |\phi_t * f(x)|^2 \frac{dt}{t} \\ &\approx \sum_{j \in \mathbb{Z}} |\phi_{\mathfrak{p}^j} * f(x)|^2 \int_{\mathfrak{p}^j(1+\mathfrak{D})} \frac{dt}{t} \\ &= C \sum_{j \in \mathbb{Z}} |\phi_{\mathfrak{p}^j} * f(x)|^2. \end{aligned}$$

For brevity, we shall use the discrete version of (3.1) in the following way:

$$F(f)(x) = \left\{ \sum_{j \in \mathbb{Z}} |\phi_{\mathfrak{p}^j} * f(x)|^2 \right\}^{1/2} \quad (3.2)$$

Moreover, from now we consider  $F(f)$  as the  $l^2(\mathbb{N}_0)$ -norm of the vector-valued function whose value at  $x$  is the sequence  $(H_\phi f)(x) \equiv \{\phi_{\mathfrak{p}^j} * f(x) : j \in \mathbb{Z}\}$ ; that is

$$F(f)(x) = \left\| \{\phi_{\mathfrak{p}^j} * f(x)\}_{j \in \mathbb{Z}} \right\|_{l^2(\mathbb{N}_0)} = \|(H_\phi f)(x)\|_{l^2(\mathbb{N}_0)}.$$

Similar to the standard Littlewood-Paley theory on the Euclidean space  $\mathbb{R}^n$  (see [11], Theorem 5.3, pp. 505), one can prove the following result.

**Lemma 3.1** *Let  $\psi$  be an integrable function on  $K$  such that*

$$\hat{\psi}(0) = \int_K \psi(x) dx = 0$$

*and assume that, for some  $\alpha > 0$ , it verifies*

$$|\psi(x)| \leq C(1+|x|)^{-1-\alpha}, \quad \text{and} \quad \int_K |\psi(x+h) - \psi(x)| dx \leq C|h|^\alpha, \quad h \in K.$$

*Then, the operator  $F(f)(x)$  defined by (3.2) is bounded on  $L^p(K)$ ,  $1 < p < \infty$ .*

**Theorem 3.2** *Let  $\{\psi^1, \psi^2, \dots, \psi^N\}$  be the generators of a tight wavelet frame for  $L^2(K)$ . If  $m_0$  and  $\hat{\phi}$  are continuous at the origin and the frame generators  $\psi^\ell, \ell = 1, 2, \dots, N$  belongs to the regular class  $\mathcal{R}(K)$ . Then, there exist positive numbers  $A_p, B_p > 0$  such that*

$$A_p \|f\|_{L^p(K)} \leq \|F_\Psi(f)\|_{L^p(K)} \leq B_p \|f\|_{L^p(K)}, \quad (3.3)$$

holds for every  $f \in L^p(K)$ ,  $1 < p < \infty$ , where

$$F_{\Psi}(f)(x) = \sum_{\ell=1}^N F_{\psi^\ell}(f)(x) = \left\{ \sum_{j \in \mathbb{Z}} |\psi_{\mathbf{p}^j}^\ell * f(x)|^2 \right\}^{1/2}. \quad (3.4)$$

*Proof.* We first prove the right hand inequality in (3.3). By the continuity property of  $\hat{\phi}$  at the origin and the fact that  $\chi_k(0) = \hat{\phi}(0) = 1$ , we observe from (2.10) and (2.11) that  $m(0) = 1$  and  $m_\ell(0) = 0$ ,  $\ell = 1, 2, \dots, N$ . By Definition 2.3, it follows that each function  $\psi^1, \psi^2, \dots, \psi^N$  belongs to  $L^1(K)$ , which implies that each  $\hat{\psi}^\ell(\xi)$  is continuous on  $K$ . By (2.11), we observe that  $\hat{\psi}^\ell(0) = 0$ ,  $1 \leq \ell \leq N$ . Hence  $\hat{\psi}^\ell \in \mathcal{R}(K)$ , for every  $\ell = 1, 2, \dots, N$ . Therefore, by Lemma 3.1, we infer that the associated Littlewood-Paley function  $F_{\Psi}(f)$  is a bounded operator on  $L^p(K)$ ,  $1 < p < \infty$ . More precisely, we can say that there exists a positive constant  $C_\ell > 0$  (depending only on  $\ell$ ) such that

$$\|F_{\psi^\ell}(f)\|_{L^p(K)} \leq C_\ell \|f\|_{L^p(K)}, \quad 1 < p < \infty. \quad (3.5)$$

Thus, for every  $f \in L^p(K)$ , we have

$$\begin{aligned} \|F_{\Psi}(f)\|_{L^p(K)} &= \left\| \sum_{\ell=1}^N F_{\psi^\ell}(f) \right\|_{L^p(K)} \\ &\leq \sum_{\ell=1}^N \|F_{\psi^\ell}(f)\|_{L^p(K)} \leq \sum_{\ell=1}^N C_\ell \|f\|_{L^p(K)} = C \|f\|_{L^p(K)}. \end{aligned} \quad (3.6)$$

We now turn to the left hand inequality in (3.3). By implementing the Plancherel's theorem, we obtain

$$\begin{aligned} \|f\|_{L^2(K)}^2 &= \sum_{\ell=1}^N \|F_{\psi^\ell}(f)\|_{L^2(K)}^2 \\ &= \sum_{\ell=1}^N \left\| \|(H_{\psi^\ell} f)\|_{l^2(\mathbb{N}_0)} \right\|_{L^2(K)}^2, \text{ for all } f \in L^2(K). \end{aligned} \quad (3.7)$$

Since  $\{\psi^1, \psi^2, \dots, \psi^N\}$  generates a tight wavelet frame for  $L^2(K)$  if and only if

$$\sum_{\ell=1}^N \sum_{j \in \mathbb{Z}} \left| \hat{\psi}^\ell(\mathbf{p}^{-j}\xi) \right|^2 = 1 \quad \text{for a.e. } \xi \in K$$

$$\sum_{\ell=1}^N \sum_{j=0}^{\infty} \hat{\psi}^\ell(\mathbf{p}^{-j}\xi) \overline{\hat{\psi}^\ell(\mathbf{p}^{-j}(\xi + u(s)))} = 0 \quad \text{for a.e. } \xi \in K, s \in \mathbb{N}_0 \setminus q\mathbb{N}_0$$

(See Behera and Jehan [2], Theorem 3.3). Using this result in the following estimate, we get

$$\begin{aligned} \sum_{\ell=1}^N \left\| F_{\psi^\ell}(f) \right\|_{L^2(K)}^2 &= \sum_{\ell=1}^N \int_K \sum_{j \in \mathbb{Z}} \left| \psi_{\mathbf{p}^j}^\ell * f(x) \right|^2 dx \\ &= \sum_{\ell=1}^N \sum_{j \in \mathbb{Z}} \int_K \left| \hat{\psi}^\ell(\mathbf{p}^{-j}\xi) \hat{f}(\xi) \right|^2 d\xi \\ &= \int_K \sum_{\ell=1}^N \sum_{j \in \mathbb{Z}} \left| \hat{\psi}^\ell(\mathbf{p}^{-j}\xi) \right|^2 \left| \hat{f}(\xi) \right|^2 d\xi \\ &= \int_K \left| \hat{f}(\xi) \right|^2 d\xi \\ &= \|f\|_{L^2(K)}^2. \end{aligned}$$

We now use the just established equality (3.7) to prove the first inequality in (3.3). Applying the polarization identity to (3.7), we obtain

$$\begin{aligned} \left| \int_K f(x)h(x) dx \right| &= \left| \sum_{\ell=1}^N \int_K \langle (H_{\psi^\ell} f)(x), (H_{\psi^\ell} h)(x) \rangle dx \right| \\ &\leq \sum_{\ell=1}^N \int_K \left\| (H_{\psi^\ell} f)(x) \right\|_{l^2(\mathbb{N}_0)} \left\| (H_{\psi^\ell} h)(x) \right\|_{l^2(\mathbb{N}_0)} dx \\ &= \sum_{\ell=1}^N \int_K F_{\psi^\ell}(f)(x) F_{\psi^\ell}(h)(x) dx \\ &\leq \sum_{\ell=1}^N \left\| F_{\psi^\ell}(f) \right\|_{L^p(K)} \left\| F_{\psi^\ell}(h) \right\|_{L^{p'}(K)}, \end{aligned}$$

for every  $f, h \in L^p(K) \cap L^2(K)$ , where  $p'$  is the conjugate exponent to  $p$ . Taking the supremum over all  $h$  such that  $\|h\|_{p'} \leq 1$  and using (3.5) with  $p$  replaced by  $p'$ , we infer that

$$\begin{aligned}
\|f\|_{L^p(K)} &= \sup_{\|h\|_{p'} \leq 1} \left| \int_K f(x)h(x) dx \right| \\
&\leq \sup_{\|h\|_{p'} \leq 1} \sum_{\ell=1}^N \left\| F_{\psi^\ell}(f) \right\|_{L^p(K)} \left\| F_{\psi^\ell}(h) \right\|_{L^{p'}(K)} \\
&\leq \sup_{\|h\|_{p'} \leq 1} \sum_{\ell=1}^N \left\| F_{\psi^\ell}(f) \right\|_{L^p(K)} C_\ell \|h\|_{L^{p'}(K)} \\
&\leq C \sum_{\ell=1}^N \left\| F_{\psi^\ell}(f) \right\|_{L^p(K)}, \quad C = \max \{C_\ell : \ell = 1, 2, \dots, N\} \\
&\leq CN^{2-2/p} \left\| \sum_{\ell=1}^N F_{\psi^\ell}(f) \right\|_{L^p(K)} \quad (\text{by Lemma 2.5}) \\
&\leq CN^{2-2/p} \left\| F_\Psi(f) \right\|_{L^p(K)}. \tag{3.8}
\end{aligned}$$

By combining (3.6) and (3.8), we get the desired inequalities. This completes the proof of the theorem.

We now give another characterization of the Lebesgue space  $L^p(K)$ ,  $1 < p < \infty$  by virtue of the tight wavelet frame coefficients.

For any real number  $\lambda > 0$  and  $g$  defined on  $K$ , we define the *maximal function* as

$$h_\lambda^*(x) = \sup_{y \in K} \frac{|h(x-y)|}{(1+|y|)^\lambda}, \quad x \in K. \tag{3.9}$$

The *Hardy-Littlewood maximal function* of  $f \in L^1(K)$  is defined by

$$Mf(x) = \sup_{k \in \mathbb{Z}} \frac{1}{q^k} \int_{x+\mathfrak{B}^k} |f(y)| dy. \tag{3.10}$$

The following result establishes an inequality between two maximal functions defined by (3.9) and (3.10). The proof of this Lemma can be found in [21].  $\square$

**Lemma 3.3** *Suppose  $h_\lambda^*(x)$  be the maximal function defined on  $K$  such that  $h_\lambda^*(x) < \infty$  for all  $x \in K$ . Then, there exists a constant  $C_\lambda$  such that*

$$h_\lambda^*(x) \leq C_\lambda \left[ M \left( |h|^{1/\lambda} \right) (x) \right]^\lambda, \quad x \in K. \tag{3.11}$$

Moreover, if  $\{f_j\}_{j=1}^{\infty}$  is a sequence of integrable functions defined on  $K$ , then there exists a constant  $C_{p,p'}$  such that

$$\left\| \left\{ \sum_{j=1}^{\infty} (Mf_j)^{p'} \right\}^{1/p'} \right\|_{L^p(K)} \leq C_{p,p'} \left\| \left\{ \sum_{j=1}^{\infty} (f_j)^{p'} \right\}^{1/p'} \right\|_{L^p(K)}. \quad (3.12)$$

Analogous to this result, we have the following inequality.

**Lemma 3.4** *Let  $\phi(x)$  be a compactly supported function and  $f \in L^p(K)$ ,  $0 < p \leq \infty$  such that  $\phi_{\mathfrak{p}^j} * f \in L^p(K)$  for all  $j \in \mathbb{Z}$ , then there exists a constant  $C_{\lambda}$  for any real  $\lambda > 0$  such that*

$$(\Phi_{j,\lambda}f)(x) \leq C_{\lambda} \left\{ M \left( |\phi_{\mathfrak{p}^j} * f|^{1/\lambda} \right) (x) \right\}^{\lambda}, \quad (3.13)$$

where

$$(\Phi_{j,\lambda}f)(x) = \sup_{y \in K} \frac{|(\phi_{\mathfrak{p}^j} * f)(x - y)|}{(1 + q^j|y|)^{\lambda}}. \quad (3.14)$$

*Proof.* Let  $h(x) = (\phi_{\mathfrak{p}^j} * f)(\mathfrak{p}^j x)$ , so that  $h \in L^p(K)$  and  $h_{\lambda}^*(x) < \infty$ , where  $h_{\lambda}^*(x)$  is defined by (3.9). Since  $(\phi_{\mathfrak{p}^j})^{\wedge}(\xi) = \hat{\phi}(\mathfrak{p}^j \xi)$  and  $\phi(x)$  is compactly supported on  $K$ , which in turn implies that inequality (3.11) holds for  $h(x)$ . On the other hand, we have

$$\begin{aligned} h_{\lambda}^*(x) &= \sup_{y \in K} \frac{|h(x - y)|}{(1 + |y|)^{\lambda}} \\ &= \sup_{y \in K} \frac{|(\phi_{\mathfrak{p}^j} * f)(\mathfrak{p}^j x - \mathfrak{p}^j y)|}{(1 + |y|)^{\lambda}} \\ &= \sup_{z \in K} \frac{|(\phi_{\mathfrak{p}^j} * f)(\mathfrak{p}^j x - z)|}{(1 + q^j|z|)^{\lambda}} \\ &= (\phi_{j,\lambda}f)(\mathfrak{p}^j x), \end{aligned}$$

and

$$\left[ M \left( |h|^{1/\lambda} \right) (x) \right]^{\lambda} = \left[ M \left( |\phi_{\mathfrak{p}^j} * f|^{1/\lambda} \right) (\mathfrak{p}^j x) \right]^{\lambda}.$$

Hence, the desired result follows immediately from the inequality (3.11).

□

**Theorem 3.5** Let  $\{\psi^1, \psi^2, \dots, \psi^N\}$  be the generators of a tight wavelet frame for  $L^2(K)$  and each  $\psi^\ell, 1 \leq \ell \leq N$  satisfies the conditions as in Theorem 3.2. Then, there exist constants  $0 < A_p \leq B_p < \infty$  such that

$$A_p \|f\|_{L^p(K)} \leq \|\mathcal{W}_\Psi(f)\|_{L^p(K)} \leq B_p \|f\|_{L^p(K)}, \quad (3.15)$$

for every  $f \in L^p(K)$ , where

$$\mathcal{W}_\Psi(f)(x) = \left\{ \sum_{\ell=1}^N \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{N}_0} \left| \langle f, \psi_{j,k}^\ell \rangle \right|^2 |\Omega| \chi_\Omega(x) \right\}^{1/2}, \quad (3.16)$$

and  $\Omega = \mathfrak{p}^j(\mathfrak{D} + k)$ ,  $|\Omega|$  denoting the measure of the sphere  $\Omega$ .

*Proof.* We first consider the right hand inequality in (3.15). For  $f \in L^2(K)$ , we have

$$\begin{aligned} \left| \langle f, \psi_{j,k}^\ell \rangle \right| &= q^{-j/2} \left| \int_K f(x) \overline{\psi_{\mathfrak{p}^j}^\ell(x - \mathfrak{p}^j u(k))} dx \right| \\ &= q^{-j/2} \left| (\tilde{\psi}_{\mathfrak{p}^j}^\ell * f)(\mathfrak{p}^j u(k)) \right| \\ &\leq q^{-j/2} \sup_{y \in \Omega} \left| (\tilde{\psi}_{\mathfrak{p}^j}^\ell * f)(y) \right|, \quad \text{where } \tilde{\psi}_{\mathfrak{p}^j}^\ell(y) = \overline{\psi_{\mathfrak{p}^j}^\ell(-y)}. \end{aligned}$$

For any fixed  $\ell, 1 \leq \ell \leq N$  and  $j \in \mathbb{Z}$ , we have

$$\begin{aligned} \sum_{k \in \mathbb{N}_0} \left| \langle f, \psi_{j,k}^\ell \rangle \right|^2 q^j |\Omega| \chi_\Omega(x) &\leq \sum_{k \in \mathbb{N}_0} \left\{ \sup_{y \in \Omega} \left| (\tilde{\psi}_{\mathfrak{p}^j}^\ell * f)(y) \right| \right\}^2 \chi_\Omega(x) \\ &\leq \left\{ \sup_{|z| \leq q^{-j}} \left| (\tilde{\psi}_{\mathfrak{p}^j}^\ell * f)(x - z) \right| \right\}^2 \\ &\leq \left\{ \sup_{|z| \leq q^{-j}} \left| \frac{(\tilde{\psi}_{\mathfrak{p}^j}^\ell * \tilde{f})(-x + z)}{(1 + q^j |z|)^\lambda} (1 + q^j |z|)^\lambda \right| \right\}^2 \\ &\leq q^{2\lambda} \left\{ (\Psi_{j,\lambda}^\ell \tilde{f})(-x) \right\}^2 \\ &\leq C_\lambda q^{2\lambda} \left\{ M \left( \left| \psi_{\mathfrak{p}^j}^\ell * \tilde{f} \right|^{1/\lambda} \right) (-x) \right\}^{2\lambda}, \end{aligned}$$

where  $\Psi_{j,\lambda}^\ell$  has been defined in a similar fashion as that of (3.14).

Applying Lemmas 2.5 and 3.4, we obtain

$$\begin{aligned}
\|\mathcal{W}_\Psi(f)\|_{L^p(K)} &\leq C_\lambda q^\lambda \left\| \left\{ \sum_{\ell=1}^N \sum_{j \in \mathbb{Z}} \left[ M \left( |\psi_{\mathbf{p}^j}^\ell * \tilde{f}|^{1/\lambda} \right) \right]^{2\lambda} \right\}^{1/2} \right\|_{L^p(K)} \\
&\leq C_\lambda \sum_{\ell=1}^N \left\| \left\{ \sum_{j \in \mathbb{Z}} \left[ M \left( |\psi_{\mathbf{p}^j}^\ell * \tilde{f}|^{1/\lambda} \right) \right]^{2\lambda} \right\}^{1/2} \right\|_{L^p(K)} \\
&= C_\lambda \sum_{\ell=1}^N \left\| \left\{ \sum_{j \in \mathbb{Z}} \left[ M \left( |\psi_{\mathbf{p}^j}^\ell * \tilde{f}|^{1/\lambda} \right) \right]^{2\lambda} \right\}^{1/2} \right\|_{L^{p\lambda}(K)}^\lambda \\
&\leq C_{p,\lambda} \sum_{\ell=1}^N \left\| \left\{ \sum_{j \in \mathbb{Z}} \left[ M \left( |\psi_{\mathbf{p}^j}^\ell * \tilde{f}|^{1/\lambda} \right) \right]^{2\lambda} \right\}^{1/2} \right\|_{L^{p\lambda}(K)}^\lambda \\
&\leq C_{p,\lambda} \left\| \sum_{\ell=1}^N \left\{ \sum_{j \in \mathbb{Z}} \left[ M \left( |\psi_{\mathbf{p}^j}^\ell * \tilde{f}|^{1/\lambda} \right) \right]^{2\lambda} \right\}^{1/2} \right\|_{L^p(K)} \\
&\leq C_{p,\lambda} \|F_\Psi(\tilde{f})\|_{L^p(K)} \\
&\leq B_p C_{p,\lambda} \|f\|_{L^p(K)}. \tag{3.17}
\end{aligned}$$

Next we prove the first inequality of (3.15). Let

$$(\mathcal{S}_\Psi f)(x) = \left\{ \langle f, \psi_{j,k}^\ell \rangle |\Omega|^{1/2} \chi_\Omega(x) : 1 \leq \ell \leq N, j \in \mathbb{Z}, k \in \mathbb{N}_0 \right\}.$$

Then, we have

$$(\mathcal{W}_\Psi f)(x) = \left\{ (\mathcal{S}_\Psi f)(x) (\mathcal{S}_\Psi f)(x) \right\}^{1/2}.$$

Since the wavelet system  $X(\Psi)$  is a tight wavelet frame for  $L^2(K)$ , we have

$$\begin{aligned} \int_K (\mathcal{S}_\Psi f)(x)(\mathcal{S}_\Psi f)(x) dx &= \left\| \mathcal{W}_\Psi f \right\|_{L^2(K)}^2 \\ &= \int_K \sum_{\ell=1}^N \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{N}_0} \left| \langle f, \psi_{j,k}^\ell \rangle \right|^2 |\Omega| \chi_\Omega(x) dx \\ &= \sum_{\ell=1}^N \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{N}_0} \left| \langle f, \psi_{j,k}^\ell \rangle \right|^2 \\ &= \|f\|_{L^2(K)}^2. \end{aligned}$$

From this equality, the polarization identity and a density argument, we obtain

$$\int_K f(x)h(x) dx = \int_K (\mathcal{S}_\Psi f)(x)(\mathcal{S}_\Psi h)(x) dx$$

for all  $f \in L^p(K)$  and  $h \in L^{p'}(K)$ . By implementing the duality argument together with Hölder's inequality and equation (3.17) for  $L^{p'}(K)$ , we deduce that

$$\begin{aligned} \|f\|_{L^p(K)} &= \sup_{\|h\|_{p'}} \left| \int_K f(x)h(x) dx \right| \\ &\leq \sup_{\|h\|_{p'}} \left\| \mathcal{W}_\Psi f \right\|_{L^p(K)} \left\| \mathcal{W}_\Psi h \right\|_{L^{p'}(K)} \\ &\leq B_{p'} \left\| \mathcal{W}_\Psi f \right\|_{L^p(K)}. \end{aligned} \tag{3.18}$$

Combining (3.17) and (3.18), we get the desired inequalities. This completes the proof of Theorem 3.5.

□

**Acknowledgements** We are deeply indebted to the referee for his/her valuable suggestions which greatly improved the presentation of this paper. The first author is supported by NBHM, Department of Atomic Energy, Government of India. (Grant No. 2/48(8)/2016/NBHM(R.P)/R&D II/13924).

## References

1. BEHERA, B. – *Haar wavelets on the Lebesgue spaces of local fields of positive characteristic*, Collq. Mathematicum, **136** (2014), no. 2, 149-168.
2. BEHERA, B.; JAHAN, Q. – *Characterization of wavelets and MRA wavelets on local fields of positive characteristic*, Collect. Math., **66** (2015), no. 1, 33-53.
3. BENEDETTO, J.J.; BENEDETTO, R.L. – *A wavelet theory for local fields and related groups*, J. Geom. Anal., **14** (2004), no. 3, 423-456.

4. BERDNIKOV, G.; KRUSS, I.; LUKOMSKII, S. – *How to construct wavelets on local fields of positive characteristic*, Lobachevskii J. Math., **38** (2017), no. 4, 615-621.
5. BORUP, L.; GRIBONVAL, R.; NIELSEN, M. – *Tight wavelet frames in Lebesgue and Sobolev spaces*, J. Function Spaces Appl., **2** (2004), no. 3, 227-252.
6. CHUONG, N.M.; DUONG, D.V. – *Wavelet bases in the Lebesgue spaces on the field of  $p$ -adic numbers*,  $p$ -Adic Numbers Ultrametric Anal. Appl., **5** (2013), no. 2, 106-121.
7. DAUBECHIES, I.; HAN, B.; RON, A.; SHEN, Z. – *Framelets: MRA-based constructions of wavelet frames*, Appl. Comput. Harmonic Anal., **14** (2003), no. 1, 1-46.
8. DAYONG, L.; QIBIN, F. – *Characterizations of  $L_p(\mathbb{R})$  using tight wavelet frames*, Wuhan University Journal of Natural Sciences, **15** (2010), no. 6, 461-466.
9. DEBNATH, L.; SHAH, F.A. – *Wavelet Transforms and Their Applications. Second edition*, Birkhäuser/Springer, New York, 2015, xvi+553 pp.
10. FARKOV, Y.; LEBEDEVA, E.; SKOPINA, M. – *Wavelet frames on Vilenkin groups and their approximation properties*, Int. J. Wavelets, Multiresolut. Informat. Process., **13** (2015), no. 5, 1550036 (19 pages).
11. GARCÍA-CUERVA, J.; RUBIO DE FRANCIA, J.L. – *Weighted norm inequalities and related topics*, North-Holland Mathematics Studies, 116. Notas de Matemática [Mathematical Notes], 104, North-Holland Publishing Co., Amsterdam, 1985, x+604 pp.
12. GRIPENBERG, G. – *Wavelet bases in  $L^p(\mathbb{R})$* , Studia Math., **106** (1993), no. 2, 175-187.
13. JIANG, H.; LI, D.; JIN, N. – *Multiresolution analysis on local fields*, J. Math. Anal. Appl., **294** (2004), no. 2, 523-532.
14. MEYER, Y. – *Wavelets and Operators*, Translated from the 1990 French original by D. H. Salinger, Cambridge Studies in Advanced Mathematics, 37, Cambridge University Press, Cambridge, 1992, xvi+224 pp.
15. SHAH, F.A. – *Periodic wavelet frames on local fields of positive characteristic*, Numer. Funct. Anal. Optimizat., **37** (2016), no. 5, 603-627.
16. SHAH, F.A.; ABDULLAH – *Wave packet frames on local fields of positive characteristic*, Appl. Math. Comput., **249** (2014), 133-141.
17. SHAH, F.A.; ABDULLAH – *A characterization of tight wavelet frames on local fields of positive characteristic*, translated from Izv. Nats. Akad. Nauk Armenii Mat. **49** (2014), no. 6, 122-132, J. Contemp. Math. Anal. **49** (2014), no. 6, 251-259.
18. SHAH, F.A.; BHAT, M.Y. – *Semi-orthogonal wavelet frames on local fields*, Analysis (Berlin), **36** (2016), no. 3, 173-181.
19. SHAH, F.A.; DEBNATH, L. – *Tight wavelet frames on local fields*, Analysis (Berlin), **33** (2013), no. 3, 293-307.
20. RON, A.; SHEN, Z. – *Affine systems in  $L_2(\mathbb{R}^d)$ : the analysis of the analysis operator*, J. Funct. Anal., **148** (1997), no. 2, 408-447.
21. TAIBLESON, M.H. – *Fourier analysis on local fields*, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1975, xii+294 pp.
22. WOJTASZCZYK, P. – *Wavelets as unconditional bases in  $L_p(\mathbb{R})$* , J. Fourier Anal. Appl., **5** (1999), no. 1, 73-85.

Received: 21.IX.2017 / Revised: 7.II.2018 / Accepted: 1.III.2018

#### AUTHORS

FIRDOUS A. SHAH (Corresponding author),  
 Department of Mathematics,  
 University of Kashmir, South Campus, Anantnag-192101,  
 Jammu and Kashmir, India,  
*E-mail*: fashah79@gmail.com

SHALLU SHARMA,  
 Department of Mathematics,  
 University of Jammu, Jammu-180006  
 Jammu and Kashmir, India,  
*E-mail*: shallujamwal09@gmail.com

M. YOUNUS BHAT,  
 Department of Mathematics,  
 National Institute of Technology, Srinagar-190006,  
 Jammu and Kashmir, India,  
*E-mail*: gyounusg@gmail.com