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Essential spectra and Drazin invertibility of quasisimilar closed operators

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Abstract The purpose of this paper is to introduce several generalizations of the notion of similarity between closed operators on a Hilbert space, in particular, the notion of quasi-similarity and mutually quasi-similarity. Furthermore, we explore to what extend they preserve spectral properties, Drazin invertibility and hyperinvariant subspaces.

Keywords Similarity · quasi-similarity · mutually quasi-similarity · Fredholm spectrums · Drazin invertibility · hyperinvariant subspaces.

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1 Introduction and preliminaries

Similarity of operators is a weaker concept than unitary equivalence. Much of the study of similarity has been driven by a desire to characterize operators. It is easy to see that similarity preserves the spectrum and various parts of the spectrum and invariant and hyperinvariant subspaces. The notion of similarity between bounded operators is standard, it is too restrictive for applications, which naturally leads to introduce several generalizations. So some other concepts related to bounded operators may be studied via similarity as quasi-similarity, asymptotic similarity and compalece. Many authors have worked on this type of problems for bounded operators, we refer the interested reader to see e.g. [7], [17], [19] and [5].

Similarity and some weak notions of similarity for unbounded operators have also been introduced in several recent works [15], [6], [2], [3] and [4]. In the present article generalizing the work of [5], we establish in the setting of unbounded operators on a Hilbert space some new spectral results on weak notions of similarity : quasi-similarity and mutually quasi-similarity. The goal here is to study which spectral properties are preserved under such relations of similarity and weak similarity and the existence of hyperinvariant subspaces.

We examine first, in section 2, the notion of similarity, quasi-similarity and mutually quasi-similarity between closed densely defined linear operators. The defintions and properties of similarity, quasi-similarity and mutually quasi-similarity are often inspired of articles [2], [3] and [4]. In sections 3 and 4, we generalize and investigate some spectral results of unbounded operators which are quasi-similar or mutually quasi-similar. We show that the Drazin invertibility and some spectral and Fredholm properties as well as the closedness, the index,... are preserved under quasi-similarity or mutually quasi-similarity. We essentially establish several interesting connections between different Fredholm spectra. Finally, we prove in section 5, that an unbounded operator which is mutually quasi-similar to an operator with an hyperinvariant subspace has an hyperinvariant subspace itself. To our knowledge the concepts of quasi-similarity and mutually quasi-similarity are not well known in the unbounded case, these generalizations are original and the obtained results are particularly interesting especially in applications to problems of partial differential equations and those of quantum mechanics.

In the following we give some definitions and preliminary results in which our investigation will be need. Let H and K be complex Hilbert spaces. For A linear operator from H to K, the symbols $\mathcal{D}(A) \subset H$, $N(A) \subset H$ and $R(A) \subset K$ will denote the domain, null space and the range space of A, respectively. We denote the identity operator by I. The operator A is closed if and only if its graph is a closed subset of $H \times K$. The set of all closed and densely defined linear operators from H to K will be denoted by $\mathcal{C}(H, K)$. Denote by $\mathcal{B}(H, K)$ the Banach space of all bounded linear operators from H to K. If $H = K$, write $\mathcal{C}(H, H) = \mathcal{C}(H)$ and $\mathcal{B}(H, H) = \mathcal{B}(H)$. If A and B are two unbounded linear operators with domains $D(A)$ and $D(B)$ respectively, $A \subset B$ means that B is an extension of A, that is, $D(A) \subset D(B)$ and $Ax = Bx$ for all $x \in D(A)$. A linear operator $A : D(A) \longrightarrow K$ is closable if it has a closed extension $\overline{A}$ the closure of A, in other words for any sequence $(x_n)_{n\in\mathbb{N}} \subset D(A)$ such that, as $n \to \infty$, $x_n \to 0$ and $Ax_n \to y$, it follows that $y = 0$. If $A \in \mathcal{C}(H, K)$, the adjoint A^* of A exists, it is unique and $A^* \in \mathcal{C}(K, H)$. A is closable if and only if A^* is densely defined, in which case $\overline{A} = A^{**}$. Also that if A, B and AB are all densely defined, then we have $B^*A^* \subset (AB)^*$. There are cases where equality holds in the previous inclusion, namely if A is bounded or if B is boundedly invertible in $B(H)$. All relevant concepts from theory of closed linear operators can be found in [16].

For an unbounded linear operator A from H to K, let as usual $\rho(A)$, $\sigma(A)$, $\sigma_p(A)$, $\sigma_c(A)$, $\sigma_r(A)$, $\sigma_{ap}(A)$, $\sigma_{su}(A)$ denote the resolvent set, the spectrum, the point spectrum, the continuous spectrum, the residual spectrum, the approximate point spectrum and the surjective spectrum of A, respectively. Recall that a linear operator $A \in \mathcal{C}(H)$, is Fredholm if $R(A)$ is closed and both $\alpha(A) = \dim N(A)$ and $\beta(A) = \dim N(A^*) = \dim H/R(A)$ are finite. An operator $A \in \mathcal{C}(H)$ is semi-Fredholm if $R(A)$ is closed and at least one of $\alpha(A)$ and $\beta(A)$ is finite. For such an operator we define the index of A by $ind(A) = \alpha(A) - \beta(A)$. A is called Weyl operator if it is Fredholm of index zero. The sets of all Fredholm operators, upper semi-Fredholm operators and lower semi-Fredholm operators on H are respectively defined by:

$$\begin{aligned}\Phi(H) &= \{A \in \mathcal{C}(H) : R(A) \text{ is closed, } \alpha(A) < \infty \text{ and } \beta(A) < \infty\}\,,\\ \Phi_+(H) &= \{A \in \mathcal{C}(H) : R(A) \text{ is closed and } \alpha(A) < \infty\}\,,\\ \Phi_-(H) &= \{A \in \mathcal{C}(H) : R(A) \text{ is closed and } \beta(A) < \infty\}\,.\end{aligned}$$

The sets of all Weyl operators, upper semi-Weyl operators, lower semi-Weyl operators on H are respectively defined by:

$$\begin{aligned}&\mathcal{W}(H) = \{A \in \Phi(H) : ind(A) = 0\}\,,\\ &\mathcal{W}_+(H) = \{A \in \Phi_+(H) : ind(A) \le 0\}\,,\ \mathcal{W}_-(H) = \{A \in \Phi_-(H) : ind(A) \ge 0\}\,.\end{aligned}$$

To define ascent and descent we consider the case in which $D(A)$ and $R(A)$ are in the same Hilbert space H. If $A \in \mathcal{C}(H)$, it is well known that $N(A^n) \subset N(A^{n+1})$ and $R(A^{n+1}) \subset R(A^n)$, for all $n \in \mathbb{N}$.
The ascent $a(A)$ of A is defined to be the smallest nonnegative integer k (if it exists) which satisfies that $N(A^k) = N(A^{k+1})$. If such k does not exist, then the ascent of A is defined as infinity. Similarly, the descent $d(A)$ of A is defined as the smallest nonnegative integer k (if it exists) for which $R(A^k) = R(A^{k+1})$ holds. If such k does not exist, then $d(A)$ is defined as infinity, too. For $A \in \mathcal{C}(H)$ we define the hyper-kernel of A by $N^\infty(A) = \bigcup\limits_{n=1}^{\infty} N(A^n)$ and the hyper-range of A by $R^\infty(A) = \bigcap\limits_{n=1}^{\infty} R(A^n)$. If $N^\infty(A) = N(A^p)$ for some p, then $a(A)$ is finite and the ascending sequence $N(A^n)$ terminates. If $R^\infty(A) = R(A^p)$ for some p, then $d(A)$ is finite and the descending sequence $R(A^n)$ terminates. If the ascent and the descent of A are finite, then they are equal, on the other hand if $a(A) < \infty$ then $\alpha(A) \le \beta(A)$. If $d(A) < \infty$ then $\beta(A) \le \alpha(A)$. If $a(A) = d(A) < \infty$ then $\alpha(A) = \beta(A)$. If $\alpha(A) = \beta(A) < \infty$ and if either $a(A) < \infty$ or $d(A) < \infty$ then $a(A) = d(A)$. For more details on ascent and descent see e.g. [1].

The sets of all Browder operators, upper semi-Browder operators, lower semi-Browder operators on H are respectively defined by:

$$\begin{aligned}&Br(H) = \{A \in \Phi(H) : a(A) = d(A) < \infty\ \}\,,\\ &Br_+(H) = \{A \in \Phi_+(H) : a(A) < \infty\ \}\,,\ Br_-(H) = \{A \in \Phi_-(H) : d(A) < \infty\ \}\,.\end{aligned}$$

By the help of above set classes, for $A \in \mathcal{C}(H)$, we can define its corresponding spectra as following:

Definition 1.1 *Let*

$\sigma_{ec}(A)$=$\{\lambda \in \mathbb{C}$:$R(A-\lambda I)$ *is not closed in* $H\}$,
$\sigma_{ef}(A)$=$\{\lambda \in \mathbb{C}$:$(A-\lambda I) \notin \Phi(H)\}$, $\sigma_{uf}(A)$=$\{\lambda \in \mathbb{C} : (A-\lambda I) \notin \Phi_+(H)\}$,
$\sigma_{lf}(A)$=$\{\lambda \in \mathbb{C}$:$(A-\lambda I) \notin \Phi_-(H)\}$,
$\sigma_{sf}(A)$=$\{\lambda \in \mathbb{C}$:$(A-\lambda I) \notin \Phi_+(H) \cup \Phi_-(H)\}$,
$\sigma_{ew}(A)$=$\{\lambda \in \mathbb{C}$:$(A-\lambda I) \notin \mathcal{W}(H)\}$, $\sigma_{uw}(A)$=$\{\lambda \in \mathbb{C} : (A-\lambda I) \notin \mathcal{W}_+(H)\}$,
$\sigma_{lw}(A)$=$\{\lambda \in \mathbb{C}$:$(A-\lambda I) \notin \mathcal{W}_-(H)\}$, $\sigma_{eb}(A)$=$\{\lambda \in \mathbb{C} : (A-\lambda I) \notin Br(H)\}$,
$\sigma_{ub}(A)$=$\{\lambda \in \mathbb{C}$:$(A-\lambda I) \notin Br_+(H)\}$, $\sigma_{lb}(A)$=$\{\lambda \in \mathbb{C} : (A-\lambda I) \notin Br_-(H)\}$,

are respectively the Goldberg spectrum, the Fredholm spectrum, the upper semi-Fredholm spectrum, the lower semi-Fredholm spectrum, the semi-Fredholm spectrum, the Weyl spectrum, the upper semi-Weyl spectrum, the lower semi-Weyl spectrum, the Browder spectrum, the upper semi-Browder spectrum and the lower semi-Browder spectrum. $\rho_e(A) = \mathbb{C} \setminus \sigma_{ef}(A)$ *is the essential resolvent set of* A.

Fialkow [11] proved that $\sigma_{ef}(A) \cap \sigma_{ef}(B) \neq \emptyset$ if A and B are quasi-similar bounded operators. Several authors have raised the following natural question: Is it also true that each component of $\sigma_{ef}(A)$ intersects $\sigma_{ef}(B)$ and viceversa? The main purpose of the present article is to solve this question in the case of unbounded operators and for the various above mentioned parts of the spectrum.

2 Similar, quasi-similar and mutually quasi-similar operators

In this section, we present some basic definitions and discuss properties of similarity, quasi-similarity and mutually quasi-similarity of operators in Hilbert spaces. We use essentially paper [4] to introduce the notions of similarity, quasi-similarity and mutually quasi-similarity of unbounded operators in a general case.

Definition 2.1 *Let* $A \in \mathcal{C}(H)$, $B \in \mathcal{C}(K)$ *and* $T \in \mathcal{C}(H,K)$. T *is called an intertwining operator for* A *and* B *if* $D(TA) = D(A) \subset D(T)$, $T\ (D(A)) \subset D(B)$, *and* $BTx = TAx$ *for all* $x \in D(A)$.

A and B are called similar, and write $A \overset{s}{\sim} B$, *if there exists a bounded intertwining operator T for A and B with bounded inverse* $T^{-1} : K \longrightarrow H$, *intertwining for B and A. That is,* $A = T^{-1}BT$. *When T is unitary, A and B are unitarily equivalent, in which case we write* $A \overset{u}{\sim} B$.

We say again that A is quasi-similar to B, and write $A \overset{q.s}{\sim} B$, *if there exists a (possibly unbounded) intertwining operator T for A and B which is invertible, with inverse* T^{-1} *densely defined.*

Remark 2.1 If T is bounded, then the condition

$$D(TA) = D(A) \subset D(T) = H$$

is satisfied automatically.

If T is an intertwining bounded operator for A and B, then $T^* : K \longrightarrow H$ is an intertwining operator for B^* and A^*.

A is quasi-similar to B if and only if $A \subset T^{-1}BT$, where T is a closed densely defined operator which is injective and has dense range but not necessarily bounded. Notice that $A \overset{q.s}{\sim} B$ does not imply $B^* \overset{q.s}{\sim} A^*$ in general for an unbounded intertwining operator. In particular, if $A \overset{q.s}{\sim} B$ and the corresponding intertwining operator T is invertible with bounded inverse then $B^* \overset{q.s}{\sim} A^*$ with intertwining operator T^*.

Often, the intertwining operator T is considered bounded on H with inverse T^{-1} densely defined but not necessarily bounded.

Similarity preserves the closedness and invertibility of operators and various usual parts of the spectrum, namely, σ_p, σ_c, σ_r, σ_{ap} and σ_{su}.

Theorem 2.2 *([3]) Let A and B be densely defined linear operators in H and K, respectively. The following statements hold.*

1) $A \overset{s}{\sim} B$ if and only if $A^* \overset{s}{\sim} B^*$.

2) If $A \overset{s}{\sim} B$, then $A \in \mathcal{C}(H)$ if and only if $B \in \mathcal{C}(K)$.

3) If $A \overset{s}{\sim} B$, then A^{-1} exists if and only if B^{-1} exists. Moreover, $A^{-1} \overset{s}{\sim} B^{-1}$.

4) If $A \in \mathcal{C}(H)$, $B \in \mathcal{C}(K)$ and $A \overset{s}{\sim} B$, then $\sigma(A) = \sigma(B)$ and $\sigma_i(A) = \sigma_i(B)$, for $i = p, c, r, ap, su$.

Remark 2.2 The equalities, in assertion (4), of approximate point spectra of A and B and that of the surjective spectra of A and B does not appear in the paper [3]. The proof of this new result is automatic. Indeed, recall that $\sigma_{ap}(A) = \sigma(A) \setminus \sigma_{r_1}(A)$ where:
$\sigma_{r_1}(A) = \left\{ \lambda \in \sigma_r(A) \ : \ (A - \lambda I)^{-1} \text{exists and is bounded} \right\}.$
Thus, $\sigma_{r_1}(A) = \sigma_{r_1}(B)$ and $\sigma_{ap}(A) = \sigma_{ap}(B)$ since $\sigma(A) = \sigma(B)$. As $\sigma(A) = \sigma_{ap}(A) \cup \sigma_{su}(A)$ and $\sigma_{ap}(A) = \sigma_{ap}(B)$, we deduce that $\sigma_{su}(A) = \sigma_{su}(B)$.

It should be noted that, in general, the similarity does not preserve symmetry and a fortiori the selfadjointness of operators. But, under certain conditions, an operator similar to its adjoint is automatically selfadjoint [9], [17], [18].

Now we consider the relationship between the spectra of quasi-similar closed densely defined linear operators.

Theorem 2.3 *([3], [4]) Let $A \in \mathcal{C}(H)$ and $B \in \mathcal{C}(K)$ such that $A \overset{q.s}{\sim} B$ with intertwining operator $T \in \mathcal{C}(H, K)$. Then,*

1) If $\rho(A) \neq \emptyset$, T is necessarely bounded.

2) $\sigma_p(A) \subseteq \sigma_p(B)$.

3) If $T(D(A)) = D(B)$ and T^{-1} is bounded, then $\sigma_p(A) = \sigma_p(B)$.

4) If $T^{-1} \in \mathcal{B}(K, H)$, then $\sigma_r(B) \subseteq \sigma_r(A)$ and $\rho(A) \setminus \sigma_p(B) \subseteq \rho(B)$.

Furthermore, it can be seen from Example 3.31 of [3] that the quasi-similarity does not preserve some relevant parts of the spectra. The asymmetry of quasi-similarity allows to introduce the concept of mutually quasi-similar unbounded operators.

Definition 2.4 *We say that $A \in \mathcal{C}(H)$ and $B \in \mathcal{C}(K)$ are mutually quasi-similar if both $A \overset{q.s}{\sim} B$ and $B \overset{q.s}{\sim} A$, which we denote by $A \overset{m.q.s}{\sim} B$.*

This definition implies easily that A is mutually quasi-similar to B if, and only if, $A \subset T_{AB}^{-1} B T_{AB}$ and $B \subset T_{BA}^{-1} A T_{BA}$ where T_{AB} and T_{BA} are closed densely defined linear operators which are injective and have dense ranges. Clearly, similarity implies the mutually quasi-similarity but not the converse. $\overset{m.q.s}{\sim}$ is an equivalence relation, and that $A \overset{m.q.s}{\sim} B$ if and only if $A^* \overset{m.q.s}{\sim} B^*$.

Proposition 2.5 *An operator A defined on H mutually quasi-similar to a closable densely defined operator B on K is also a closable densely defined operator, and moreover the closure $\overline{A}$ of A is quasi-similar to the closure $\overline{B}$ of B.*

Proof. Let T_{AB} and T_{BA} be quasi-invertible (everywhere defined) transformations from H into K and from K into H respectively such that $T_{AB}A \subset BT_{AB}$ and $T_{BA}B \subset AT_{BA}$. It is easily seen, by the relation $T_{BA}D(B) \subset D(A)$, that A has a dense domain. Suppose a sequence $(x_n)_{n\in\mathbb{N}}$ tends to 0 and the sequence $(Ax_n)_{n\in\mathbb{N}}$ tends to some y in H. Then one has $T_{AB}x_n \longrightarrow 0$ and $BT_{AB}x_n = T_{AB}Ax_n \longrightarrow T_{AB}y$ as $n \to \infty$. Since B is closable, it follows that $T_{AB}y = 0$ and so $y = 0$. Hence A is closable, in addition $A^{**} = \overline{A} \overset{m.q.s}{\sim} B^{**} = \overline{B}$. □

We deduce directly from Theorem 2.5 and [15], the following results:

Theorem 2.6 *Let $A \in \mathcal{C}(H)$ and $B \in \mathcal{C}(K)$ and assume that $A \overset{m.q.s}{\sim} B$, with possibly unbounded intertwining operators $T_{AB} : D(A) \longrightarrow D(B)$ and $T_{BA} : D(B) \longrightarrow D(A)$.*

1) Then $\sigma_p(A) = \sigma_p(B)$. If in addition the operators T_{AB} and T_{BA} have a bounded inverse, then $\sigma(A) = \sigma(B)$, hence $\sigma_c(A) \cup \sigma_r(A) = \sigma_c(B) \cup \sigma_r(B)$.

2) If A and B are normal operators (in particular, selfadjoint), then $A \overset{u}{\sim} B$.

3) If A is symmetric and B is selfadjoint, then A is selfadjoint and $A \overset{u}{\sim} B$.

4) If $B = A^*$ where A is symmetric, then A is selfadjoint.

3 Drazin invertibility and mutually quasi-similarity

Now we recall the definition of the Drazin inverse and we study the stability of Drazin invertibility under mutually quasi-similarity of closed densely defined linear operators. We show that mutually quasi-similarity preserves

Drazin invertibility which constitutes a double generalization of the assertion (3) of Theorem 2.3.

Definition 3.1 *([13]) Let $A \in \mathcal{C}(H)$. A is called generalized Drazin invertible if it can be expressed in the form $A = A_1 \oplus A_2$ where A_1 is bounded and quasinilpotent and A_2 is closed and boundedly invertible on H. Thus, $A_2^{-1} \in \mathcal{B}(H)$, the operators $A^{GD} = 0 \oplus A_2^{-1}$ is the generalized Drazin inverse of A. The Drazin index $i(A)$ is defined to be $i(A) = 0$ if A is invertible, $i(A) = q$ if A is not invertible and A_1 is nilpotent of index q, and $i(A) = \infty$ otherwise.*

Generalized Drazin invertible operators include closed invertible and quasinilpotent operators when $A_1 = 0$ and $A_2 = 0$ respectively and projections. We deduce directly from Definition 3.1, that if $A \in \mathcal{C}(H)$ is Drazin invertible with Drazin inverse A^{GD}, then $A^{GD} \in \mathcal{B}(H)$ is unique, $R(A^{GD}) \subset D(A)$, $R(I - AA^{GD}) \subset D(A)$, $A^{GD}AA^{GD} = A^{GD}$, $AA^{GD} = A^{GD}A$, we also have $(AA^{GD})^2 = AA^{GD}$, other useful results on generalized Drazin inverse can be found in [13]. The generalized Drazin spectrum of A is defined by:

$$\sigma_{gD}(A) = \{\lambda \in \mathbb{C} : (A - \lambda I) \text{ is not generalized Drazin invertible}\}.$$

It should be noted that, the mutually quasi-similarity preserves generalized Drazin invertibility. Indeed, we have the following first main result:

Theorem 3.2 *An operator A defined on H mutually quasi-similar to a closed densely defined generalized Drazin invertible operator B on K is also a closed densely defined generalized Drazin invertible operator, $i(A) = i(B)$ and moreover the generalized Drazin inverse A^{GD} of A is mutually quasi-similar to the generalized Drazin inverse B^{GD} of B.*

Proof. Let $A \in \mathcal{C}(H)$ and T_{AB} and T_{BA} be quasi-invertible (everywhere-defined) transformations from H into K and from K into H respectively such that $T_{AB}A \subset BT_{AB}$ and $T_{BA}B \subset AT_{BA}$. It is easily seen, by Proposition 2.7, that $B \in \mathcal{C}(K)$. Moreover, $B \subset T_{BA}^{-1}A_1T_{BA} \oplus T_{BA}^{-1}A_2T_{BA}$ where $A = A_1 \oplus A_2$, A_1 is bounded and quasinilpotent and A_2 is closed boundedly invertible on H. Thus, $B = B_1 \oplus B_2$, where $B_1 = T_{BA}^{-1}A_1T_{BA|D(B)}$ is bounded and quasinilpotent since $B_1^m = T_{BA}^{-1}A_1^mT_{BA|D(B)}$ for all positive integer m and $B_2 = T_{BA}^{-1}A_2T_{BA|D(B)}$ is closed boundedly invertible on K. We also have $B^{GD} = 0 \oplus B_2^{-1} = T_{BA}^{-1}(0 \oplus A_2^{-1})T_{BA|D(B)}$ and $A^{GD} = 0 \oplus A_2^{-1} = T_{AB}^{-1}(0 \oplus B_2^{-1})T_{AB|D(A)}$, so A^{GD} and B^{GD} are mutually quasi-similar. It also follows that A and B have the same Drazin index. □

Corollary 3.3 *Let $A, B \in \mathcal{C}(H)$ such that $A \overset{m.q.s}{\sim} B$, then $\sigma_{gD}(A) = \sigma_{gD}(B)$.*

Remark 3.1 The results of this section can be generalized to left and right generalized Drazin invertible operators introduced in [14].

4 Essential spectra of mutually quasi-similar operators

Since invertible operators with bounded inverse are Weyl operators, it follows that the Fredholm property as well as the index are preserved under similarity. Herrero showed in [12] that each component of $\sigma_{ef}(A)$ intersects $\sigma_{ef}(B)$, and vice versa where A and B are bounded quasi-similar operators. This result admits a natural extension to closed linear operators as follows:

Theorem 4.1 *Let $A \in \mathcal{C}(H)$ and $B \in \mathcal{C}(K)$ be mutually quasi-similar. Then*

1) $\alpha(A-\lambda I) = \alpha(B-\lambda I)$, $\alpha((A-\lambda I)^*) = \alpha((B-\lambda I)^*)$, $\beta(A-\lambda I) = \beta(B-\lambda I)$ for all $\lambda \in \mathbb{C}$.

2) If in addition $R(B)$ is closed in H,

$$A \in \Lambda(H) \Longleftrightarrow B \in \Lambda(K),$$

where $\Lambda(X) = \Phi(X), \Phi_+(X), \Phi_-(X), \mathcal{W}(X), \mathcal{W}_+(X), \mathcal{W}_-(X)$, $X = H, K$.

3) $\rho_e(A) = \rho_e(B)$ and $ind(A-\lambda I) = ind(B-\lambda I)$ for all $\lambda \in \rho_e(A)$.

4) $\sigma_{ef}(A)$ intersects $\sigma_{ef}(B)$ and vice versa.

Proof. 1) $A \subset T_{AB}^{-1}BT_{AB}$ and $B \subset T_{BA}^{-1}AT_{BA}$ where T_{AB} and T_{BA} are closed densely defined linear operators which are injective and have dense ranges, then it follows that $T_{AB}f(A) = T_{AB}f(B)$ on $D(A)$ for every rational function f with poles of $\sigma(A) \cup \sigma(B)$. In particular, T_{AB} maps $N(A-\lambda I)$ into $N(B-\lambda I)$, thus $\alpha(A-\lambda I) \leq \alpha(B-\lambda I)$. Reversing the role of A and B, yields equality. By taking adjoints, we see that $A^* \overset{m.q.s}{\sim} B^*$, so that $\alpha((A-\lambda I)^*) = \alpha((B-\lambda I)^*)$. Thus, $\alpha((A-\lambda I)^*) = \dim R(A-\lambda I)^\perp = \beta(A-\lambda I) = \dim R(B-\lambda I)^\perp = \beta(B-\lambda I)$ for all $\lambda \in \mathbb{C}$.

The assertions (2)-(4) follow from (1) that if both A and B are semi-Fredholm, then they have the same index. □

A decade later, Djordjević investigated in [10] the connection between the semi-Browder essential spectra of quasi-similar operators. The main result of this section is to study this correspondance and generalize some important Djordjević's results to the case of unbounded mutually quasi-similar linear operators.

Theorem 4.2 *If $A \in \mathcal{C}(H)$ and $B \in \mathcal{C}(K)$ are mutually quasi-similar, then*

1) $\sigma_i(A) \setminus \sigma_{ec}(A) \subseteq \sigma_i(B)$ and $\sigma_i(B) \setminus \sigma_{ec}(B) \subseteq \sigma_i(A)$ for $i = ub; lb; uf; lf; uw; lw$.

2) $\sigma_i(A) \setminus \sigma_{ec}(A) \subseteq \sigma_i(B)$, so every component of $\sigma_i(A)$ intersects $\sigma_i(B)$ for $i = eb; sf; ew$.

Proof. 1) Let $\lambda \in \sigma_{ub}(A) \setminus \sigma_{ec}(A)$ and $\lambda \notin \sigma_{ub}(B)$. Since $A \overset{q.s}{\sim} B$ and $B \overset{q.s}{\sim} A$ with interwining operators T_{AB} and T_{BA}, it follows that $(A-\lambda I)^n \subset$

$T_{AB}^{-1}(B-\lambda I)^n T_{AB}$ and $(B-\lambda I)^n \subset T_{BA}^{-1}(A-\lambda I)^n T_{BA}$ for all positive integers n. Moreover,

$$T_{AB}\left(\bigcup_{n=0}^{\infty} N\left((A-\lambda I)^n\right)\right) \subset \bigcup_{n=0}^{\infty} N((B-\lambda I)^n) = N((B-\lambda I)^p),$$

since $a(B-\lambda I)$ is finite. As well as $\alpha((B-\lambda I)^p) \leq p\alpha(B-\lambda I) < \infty$ and T_{AB} is injective, then $\dim\left(\bigcup_{n=0}^{\infty} N((A-\lambda I)^n)\right) < \infty$. So $\alpha(A-\lambda I)$ and $a(A-\lambda I)$ are finite, which contradicts the assumption $\lambda \in \sigma_{ub}(A) \setminus \sigma_{ec}(A)$. The other inclusions arise in the same way by using the algebraic properties of the parameters $\alpha(.)$, $\beta(.)$, $a(.)$ and $d(.)$.

2) is a simple consequence of Theorem 4.1. Suppose that $\lambda \in \sigma_{ew}(A) \setminus \sigma_{ec}(A)$. It follows that $R(A-\lambda I)$ is closed and one of the following two cases may occur $\alpha(A-\lambda I) \neq \alpha((A-\lambda I)^*)$ or $\alpha(A-\lambda I) = \infty$ and $\alpha((A-\lambda I)^*) = \infty$. We conclude that, $\lambda \in \sigma_{ew}(B)$. □

5 Invariant subspaces and mutually quasi-similarity

Let A be a linear operator on H with domain $D(A)$ and M a subspace of H. M is called invariant for A if A maps $D(A)\cap M$ into M or $A(D(A)\cap M) \subset M$. Further, the commutant of the operator A is defined by:

$$\{A\}^{'} = \{C \in \mathcal{B}(H) : CA \subseteq AC\}.$$

That is, an operator $C \in \mathcal{B}(H)$ is in the commutant $\{A\}^{'}$ if and only if C maps the domain $D(A)$ into itself and $CAx = ACx$ for all $x \in D(A)$. M is called hyperinvariant for A if it is invariant for every bounded operator which commutes with A as well as for A.

Remark 5.1 Given a densely defined operator A on H, the core of A is defined by $\mathcal{D} = \bigcap_{n=1}^{\infty} D(A^n)$. It follows immediately that $A\mathcal{D} \subset \mathcal{D}$, i.e. A leaves $\mathcal{D}$ invariant and, by definition, it must be the largest invariant set. Therefore, any definition for an invariant subspace of A should be restricted to its core $\mathcal{D}$. It may very well happen that $\mathcal{D}$ is the trivial zero invariant subspace for A; however, one would naturally like to have the other extreme. In other words, one wishes to work with densely defined operators such that their cores are also dense in the Hilbert spaces. Of course, this question could only make sense when a particular operator is concerned, see e.g. the operator of multiplication by the independent variable z in the Fock space of all entire functions on the complex plane $\mathbb{C}$ such that their modulus are square integrable with respect to the gaussian measure $d\mu(z) = \frac{e^{-|z|^2}}{\pi}dxdy$ where $dxdy$ is the Lebesgue area measure ([8]).

We show here that an operator which is mutually quasi-similar to an operator with a nontrivial hyperinvariant subspace has a nontrivial hyperinvariant subspace itself.

Theorem 5.1 *If $A, B \in \mathcal{C}(H)$ are mutually quasi-similar with bounded interwining operators and if B has a nontrivial hyperinvariant subspace, then A has a nontrivial hyperinvariant subspace.*

Proof. Let $A = T_{AB}^{-1} B T_{AB}$ on $D(A)$ and $B = T_{BA}^{-1} A T_{BA}$ on $D(B)$ where T_{AB} and T_{BA} are bounded injective and have dense ranges. Let N be a nontrivial invariant subspace for B. Define

$$M = \bigcup \left\{ C T_{BA} H : C \in \{A\}^{'} \right\}.$$

Clearly, M is A-hyperinvariant and $M \neq \{0\}$ because $T_{BA} N \subset M$. Moreover, $M \neq H$ because

$$T_{AB} M = T_{AB} \left\{ \bigcup \left\{ C T_{BA} H : C \in \{A\}^{'} \right\} \right\} \subset T_{AB} \left\{ DN : D \in \{B\}^{'} \right\} \subset N$$

Since $N \neq H$ Thus, M is nontrivial. □

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