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# Applications of four dimensional matrices to generating some roughly $I_2$-convergent double sequence spaces

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**Abstract** In [3] Dündar introduced the notion of rough $I_2$-convergence and obtained two criteria for rough $I_2$-convergence. In the present paper we introduce some new rough ideal convergent sequence spaces of Musielak-Orlicz functions and four dimensional bounded-regular matrices. We study some topological and algebraic properties of these spaces. We also establish some inclusion relations between these spaces. Finally, we examine that these spaces are normal as well as monotone and sequence algebras.

**Keywords** Ideal · Bounded-regular matrix · Musielak-Orlicz function · Rough $I_2$-convergence · Double difference sequence space

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## 1 Introduction

The notion of $I$-convergence was initially introduced by Kostyrko et al. [9] as a generalization of statistical convergence [5]. Later on, it was further investigated from the sequence spaces point of view and linked with the summability theory by Salat [26], Tripathy and Hazarika [29], Savas and Das [27] and many others. The concept of rough $I$-convergence of single sequences was introduced by Pal et al. [18] which is a generalization of the earlier concepts namely rough convergence [28] and rough statistical convergence [1] of single sequences. Recently, rough statistical convergence of double sequences has been introduced by Malik and Maity [12] as a generalization of rough convergence of double sequences [11] and investigated some basic properties of this type of convergence and also studied relation between the set of statistical cluster points and the set of rough limit points of a double sequence. Recently, the notion of rough $I_2$-convergence for double sequences has been introduced by Dündar [3]. So in view of the recent applications of ideals in the theory of convergence of sequences, it seems very natural to extend the interesting concept of rough convergence further

by using Orlicz functions.
Let $X$ be a non empty set. Then a family of sets $I \subseteq 2^X$ (Power set of $X$) is said to be an *ideal* if $I$ is additive, that is, $A, B \in I \Rightarrow A \cup B \in I$ and if $A \in I, B \subseteq A \Rightarrow B \in I$.
A non-empty family of sets $F \subset 2^X$ is said to be a *filter* on $X$ if and only if
$(i)$ $\phi \in F$
$(ii)$ for all $A, B \in F \Rightarrow A \cap B \in F$
$(iii)$ $A \in F$, $A \subset B \Rightarrow B \in F$.
An ideal $I \subseteq 2^X$ is called *non-trivial* if $I \neq 2^X$. A non-trivial ideal in $I$ is called *admissible* if and only if $I \supset \{\{x\} : x \in X\}$. A non-trivial ideal is *maximal* if there does not exist any non-trivial ideal $J \neq I$, containing $I$ as a subset. For each ideal $I$ there is a filter $F(I)$ corresponding to $I$, that is, $F(I) = \{K \subseteq \mathbb{N} : K^c \in I\}$, where $K^c = \mathbb{N} \setminus K$.

An Orlicz function $M : [0, \infty) \to [0, \infty)$ is a continuous, non-decreasing and convex such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$. If convexity of Orlicz function is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called *modulus function.* Let $w$ denote the set of all real or complex sequences. Lindenstrauss and Tzafriri [10] used the idea of Orlicz function to define the following sequence space,

$$\ell_M = \Big\{x = (x_k) \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty, \text{ for some } \rho > 0\Big\}$$

is known as an Orlicz sequence space. The space $\ell_M$ is a Banach space with the norm

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is said to be Musielak-Orlicz function (see [16], [17]). A Musielak-Orlicz function $\mathcal{M} = (M_k)$ is said to satisfy $\Delta_2$-*condition* if there exist constants $a$, $K > 0$ and a sequence $c = (c_k)_{k=1}^{\infty} \in l_+^1$ (the positive cone of $l^1$) such that the inequality

$$M_k(2u) \leq KM_k(u) + c_k$$

hold for all $k \in \mathbb{N}$ and $u \in \mathbb{R}^+$, whenever $M_k(u) \leq a$.

By $l_\infty, c$ and $c_0$ we denote the classes of all bounded, convergent and null sequence spaces, respectively. The notion of difference sequence spaces was introduced by Kızmaz [8] who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [4] by introducing the spaces $l_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$.
Let $m$ be a non-negative integer. Then for $Z = c, c_0$ and $l_\infty$, we have the following sequence spaces

$$Z(\Delta^m) = \{x = (x_k) \in w : (\Delta^m x_k) \in Z\},$$

where $\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}.$$

Taking $m = 1$, we get the spaces studied by Et and Çolak [4]. Similarly, we can define difference operators on double sequence spaces as:

$$\begin{aligned}\Delta x_{k,l} &= (x_{k,l} - x_{k,l+1}) - (x_{k+1,l} - x_{k+1,l+1}) \\ &= x_{k,l} - x_{k,l+1} - x_{k+1,l} + x_{k+1,l+1}\end{aligned}$$

and

$$\Delta^m x_{k,l} = \Delta^{m-1} x_{k,l} - \Delta^{m-1} x_{k,l+1} - \Delta^{m-1} x_{k+1,l} + \Delta^{m-1} x_{k+1,l+1}.$$

For more details about sequence spaces (see [13], [14], [15], [20], [22], [23], [24]) and references therein.

Let $X$ be a linear space. A function $p : X \to \mathbb{R}$ is called *paranorm* if
($i$) $p(x) \geq 0$ for each $x \in X$,
($ii$) $p(-x) = p(x)$ for any $x \in X$,
($iii$) $p(x + y) \leq p(x) + p(y)$ for any $x, y \in X$,
($iv$) if $(\lambda_n)$ is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and $(x_n)$ is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.
A paranorm $p$ for which $p(x) = 0$ implies $x = 0$ is called *total paranorm* on $X$ and the pair $(X, p)$ is called a *total paranormed space*. It is well known that the metric of any linear metric space is given by some total paranorm (see [30] Theorem 10.4.2, pp. 183).

By the convergence of double sequence $x = (x_{kl})$ we mean the convergence in the Pringsheim sense, i.e. a double sequence $x = (x_{kl})$ is said to converges to the limit $L$ in Pringsheim sense (denoted by, $P\text{-}\lim x = L$) provided that given $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $|x_{kl} - L| < \epsilon$ whenever $k, l > n$ (see [21]). We shall write more briefly as $P$-convergent. If, in addition, $x \in l^2_\infty$, then $x$ is said to be bounded $P$-convergent to $L$. We shall denote the space of all bounded convergent double sequences by $c^2_\infty$. A double sequence $x = (x_{kl})$ is said to be *bounded* if $\|x\|_{(\infty,2)} = \sup_{k,l} |x_{kl}| < \infty$. By $l^2_\infty$, we denote the space of all bounded double sequences. Throughout the paper we take $I_2$ as a nontrivial ideal in $\mathbb{N} \times \mathbb{N}$.

**Definition 1.1** *A double sequence $(x_{kl}) \in w$ is said to be $I_2$-convergent to a number $L$ if for every $\epsilon > 0$, $\{(k,l) \in \mathbb{N} \times \mathbb{N} : |x_{kl} - L| > \epsilon\} \in I_2$ and we write $I_2 - \lim x_{kl} = L$.*

**Definition 1.2** *A double sequence $(x_{kl}) \in w$ is said to be $I_2$-null if for every $\epsilon > 0$, $\{(k,l) \in \mathbb{N} \times \mathbb{N} : |x_{kl}| > \epsilon\} \in I_2$ and we write $I_2 - \lim x_{kl} = 0$.*

**Definition 1.3** *A double sequence $(x_{kl}) \in w$ is said to be $I_2$-bounded if there exists $M > 0$ such that $\{(k,l) \in \mathbb{N} \times \mathbb{N} : |x_{kl}| > M\} \in I_2$.*

**Definition 1.4** *A sequence space $E$ is said to be solid or normal if $(x_{kl}) \in E$ implies $\alpha_{kl} x_{kl}$ for all sequence of scalars $(\alpha_{kl})$ with $|\alpha_{kl}| < 1$ for all $k, l$.*

**Definition 1.5** *A sequence space $E$ is said to be convergence free if $(y_{kl}) \in E$, whenever $(x_{kl}) \in E$ and $x_{kl} = 0$ implies $y_{kl} = 0$ for all $k, l$.*

**Definition 1.6** *A sequence space $E$ is said to be a sequence algebra if $(x_{kl} y_{kl}) \in E$ whenever $(x_{kl}) \in E$ and $(y_{kl}) \in E$.*

**Definition 1.7** [19] *A double sequence $x = (x_{kl})$ is said to be rough convergent (r-convergent) to $x_*$ with the roughness degree $r$ which is denoted by $x_{kl} \xrightarrow{r} x_*$ provided that*

$$\forall \epsilon > 0 \; \exists \, k_\epsilon \in \mathbb{N} : k, l \geq k_\epsilon \Rightarrow \|x_{kl} - x_*\| < r + \epsilon,$$

*or equivalent, if*

$$\limsup \|x_{kl} - x_*\| < r.$$

**Definition 1.8** [3] *For some given real number $r \geq 0$, a double sequence $x = (x_{kl})$ is said to be rough $I_2$-convergent to $x_*$ with the roughness degree $r$ which is denoted by $x_{kl} \xrightarrow{r - I_2} x_*$ provided that*

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : \|x_{kl} - x_*\| \geq r + \epsilon\} \in I_2,$$

*for every $\epsilon > 0$ or equivalent, if the condition*

$$I_2 - \limsup \|x_{kl} - x_*\| < r$$

*is satisfied. In addition, $x_{kl} \xrightarrow{r - I_2} x_*$ if the inequality $\|x_{kl} - x_*\| < r + \epsilon$ holds for every $\epsilon > 0$ and almost all $k, l$.*

**Lemma 1.9** [7] *Every normal sequence space is monotone.*

Let $A = (a_{nmkl})$ be a four-dimensional infinite matrix of scalars. For all $m, n \in \mathbb{N}_0$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, the sum

$$y_{nm} = \sum_{k,l} a_{nmkl} x_{kl}$$

is called the $A$-*means* of the double sequence $(x_{kl})$. A double sequence $(x_{kl})$ is said to be $A$-*summable* to the limit $L$ if the $A$-means exist for all $m, n$ in the sense of Pringsheim's convergence:

$$P\text{-}\lim_{p,q \to \infty} \sum_{k,l=0,0}^{p,q} a_{nmkl} x_{kl} = y_{nm} \quad \text{and} \quad P\text{-}\lim_{n,m \to \infty} y_{nm} = L.$$

A four-dimensional matrix $A$ is said to be *bounded-regular* (or *RH-regular*) if every bounded $P$-convergent sequence is $A$-summable to the same limit and the $A$-means are also bounded.
The following is a four-dimensional analogue of the well-known Silverman-Toeplitz theorem [2].

**Theorem 1.10** *(Robison* [25] *and Hamilton* [6]*) The four dimensional matrix $A$ is RH-regular if and only if*

$(RH_1)$ $P\text{-}\lim_{n,m} a_{nmkl} = 0$ *for each $k$ and $l$,*

$(RH_2)$ $P\text{-}\lim_{n,m} \sum_{k,l} |a_{nmkl}| = 1,$

$(RH_3)$ $P\text{-}\lim_{n,m} \sum_{k} |a_{nmkl}| = 0$ *for each $l$,*

$(RH_4)$ $P\text{-}\lim_{n,m} \sum_{l} |a_{nmkl}| = 0$ *for each $k$,*

$(RH_5)$ $\sum_{k,l} |a_{nmkl}| < \infty$ *for all $n, m \in \mathbb{N}_0$.*

Let $\mathcal{M} = (M_{kl})$ be a Musielak-Orlicz function, $p = (p_{kl})$ be a bounded double sequence of real numbers such that $p_{kl} > 0$ for all $k, l$, $u = (u_{kl})$ be a double sequence of strictly positive real numbers and let $A = (a_{nmkl})$ be a nonnegative four-dimensional bounded-regular matrix. In the present paper we define the following classes of sequence spaces:

$$w^2_{I_c}[A, \mathcal{M}, u, r, \Delta^m, p] =$$
$$\Bigg\{x = (x_{kl}) : \Bigg\{(k,l) \in \mathbb{N} \times \mathbb{N} : \sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m x_{kl} - x_*}{\rho}\Big\|\Big)\Big]^{p_{kl}}$$
$$\geq r + \epsilon\Bigg\} \in I_2, \text{ for some } \rho > 0 \text{ and } x_* \in \mathbb{R}\Bigg\},$$

$$w^2_{I_{c_0}}[A, \mathcal{M}, u, r, \Delta^m, p] =$$
$$\Bigg\{x = (x_{kl}) : \Bigg\{(k,l) \in \mathbb{N} \times \mathbb{N} : \sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m x_{kl}}{\rho}\Big\|\Big)\Big]^{p_{kl}}$$
$$\geq r + \epsilon\Bigg\} \in I_2, \text{ for some } \rho > 0 \Bigg\},$$

$$w^2_{I_{l_\infty}}[A,\mathcal{M},u,r,\Delta^m,p] =$$

$$\Bigg\{x=(x_{kl}):\exists K>0, \Bigg\{(k,l)\in\mathbb{N}\times\mathbb{N}: \sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m x_{kl}-x_*}{\rho}\Big\|\Big)\Big]^{p_{kl}}$$

$$\geq r+K\Bigg\}\in I_2, \text{ for some } \rho>0 \text{ and } x_*\in\mathbb{R}\Bigg\}$$

and

$$w^2_{l_\infty}[A,\mathcal{M},u,r,\Delta^m,p] =$$

$$\Bigg\{x=(x_{kl}): \Bigg\{(k,l)\in\mathbb{N}\times\mathbb{N}: \sup_{n,m}\sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m x_{kl}}{\rho}\Big\|\Big)\Big]^{p_{kl}} < \infty,$$

$$\text{for some } \rho>0\Bigg\}.$$

The following results are obtained for some special cases:

$(i)$ If we take $M_{kl}(x) = x$ for all $k,l$, then the above classes of sequences are denoted by $w^2_{I_c}[A,u,r,\Delta^m,p], w^2_{I_{c_0}}[A,u,r,\Delta^m,p], w^2_{I_{l_\infty}}[A,u,r,\Delta^m,p]$, respectively.

$(ii)$ If $(p_{kl}) = (1,1,1,...)$ for all $k,l$, then we denote the above sequences by $w^2_{I_c}[A,\mathcal{M},u,r,\Delta^m]$, $w^2_{I_{c_0}}[A,\mathcal{M},u,r,\Delta^m]$, $w^2_{I_{l_\infty}}[A,\mathcal{M},u,r,\Delta^m]$, respectively.

$(iii)$ If we take A = (C, 1, 1) in $w^2_{I_c}[A,\mathcal{M},u,r,\Delta^m,p], w^2_{I_{c_0}}[A,\mathcal{M},u,r,\Delta^m,p]$, $w^2_{I_{l_\infty}}[A,\mathcal{M},u,r,\Delta^m,p]$ then these sequences are reduced to $w^2_{I_c}[\mathcal{M},u,r,\Delta^m]$, $w^2_{I_{c_0}}[\mathcal{M},u,r,\Delta^m], w^2_{I_{l_\infty}}[\mathcal{M},u,r,\Delta^m]$.

$(iv)$ If $r=0$, we get $w^2_{I_c}[A,\ \mathcal{M},u,\ \Delta^m], w^2_{I_{c_0}}[A,\ \mathcal{M},\ u,\ \Delta^m], w^2_{I_{l_\infty}}[A,\ \mathcal{M},\ u,$ $\Delta^m]$, respectively.

Throughout the paper, we shall use the following inequality. Let $(a_{kl})$ and $(b_{kl})$ be two double sequences, then

$$|a_{kl}+b_{kl}|^{p_{kl}} \leq K(|a_{kl}|^{p_{kl}}+|b_{kl}|^{p_{kl}}), \tag{1.1}$$

where $K=\max(1,2^{H-1})$ and $\sup\limits_{k,l} p_{kl} = H$.

The main purpose of this paper is to introduce and study some rough $I_2$-convergent sequence spaces by using four dimensional bounded-regular (shortly, RH-regular) matrices and a Musielak-Orlicz functions in more general setting. We also make an effort to study some properties like linearity, paranorm, solidity and some interesting inclusion relations between these spaces.

## 2 Main results

**Theorem 2.1** *Let $\mathcal{M} = (M_{kl})$ be a Musielak-Orlicz function, $p = (p_{kl})$ be a bounded sequence of strictly positive real numbers and $u = (u_{kl})$ be a sequence of strictly positive real numbers. Then the spaces $w^2_{I_c}[A, \mathcal{M}, u, r, \Delta^m, p]$, $w^2_{I_{c_0}}[A, \mathcal{M}, u, r, \Delta^m, p]$ and $w^2_{I_{l_\infty}}[A, \mathcal{M}, u, r, \Delta^m, p]$ are linear spaces over the complex field $\mathbb{C}$.*

*Proof.* We shall prove the result for the space $w^2_{I_{c_0}}[A, \mathcal{M}, u, r, \Delta^m, p]$ only and the others can be proved in a similar way. Let $x = (x_{kl})$ and $y = (y_{kl})$ belongs to $w^2_{I_{c_0}}[A, \mathcal{M}, u, r, \Delta^m, p]$. Then there exists $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$A_{\frac{\epsilon}{2}} = \Big\{(k,l) \in \mathbb{N}\times\mathbb{N} : \sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m x_{kl}}{\rho_1}\Big\|\Big)\Big]^{p_{kl}} \geq r + \frac{\epsilon}{2}\Big\} \in I_2$$

and

$$B_{\frac{\epsilon}{2}} = \Big\{(k,l) \in \mathbb{N}\times\mathbb{N} : \sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m y_{kl}}{\rho_2}\Big\|\Big)\Big]^{p_{kl}} \geq r + \frac{\epsilon}{2}\Big\} \in I_2.$$

Let $\alpha$, $\beta$ be any scalars. Then by using the inequality (1.1) and continuity of the function $\mathcal{M} = (M_{kl})$, we have

$$\begin{aligned}
&\sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m(\alpha x_{kl}+\beta y_{kl})}{|\alpha|\rho_1+|\beta|\rho_2}\Big\|\Big)\Big]^{p_{kl}} \\
&\qquad \leq D\sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[\frac{|\alpha|}{|\alpha|\rho_1+|\beta|\rho_2}M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m x_{kl}}{\rho_1}\Big\|\Big)\Big]^{p_{kl}} \\
&\qquad\quad +D\sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[\frac{|\beta|}{|\alpha|\rho_1+|\beta|\rho_2}M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m y_{kl}}{\rho_2}\Big\|\Big)\Big]^{p_{kl}} \\
&\qquad \leq DK\sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m x_{kl}}{\rho_1}\Big\|\Big)\Big]^{p_{kl}} \\
&\qquad\quad +DK\sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m y_{kl}}{\rho_2}\Big\|\Big)\Big]^{p_{kl}},
\end{aligned}$$

where $K=\max\left\{1,\left(\frac{|\alpha|}{|\alpha|\rho_1+|\beta|\rho_2}\right)^H,\left(\frac{|\beta|}{|\alpha|\rho_1+|\beta|\rho_2}\right)^H\right\}$.
From the above relation we obtain the following:

$$\left\{(k,l)\in\mathbb{N}\times\mathbb{N}:\sum_{k,l=1,1}^{\infty,\infty}a_{nmkl}\Big[M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m(\alpha x_{kl}+\beta y_{kl})}{|\alpha|\rho_1+|\beta|\rho_2}\Big\|\Big)\Big]^{p_{kl}}\geq r+\epsilon\right\}\subseteq$$

$$\left\{(k,l)\in\mathbb{N}\times\mathbb{N}:DK\sum_{k,l=1,1}^{\infty,\infty}a_{nmkl}\Big[M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m x_{kl}}{\rho_1}\Big\|\Big)\Big]^{p_{kl}}\geq r+\frac{\epsilon}{2}\right\}$$

$$\cup\left\{(k,l)\in\mathbb{N}\times\mathbb{N}:DK\sum_{k,l=1,1}^{\infty,\infty}a_{nmkl}\Big[M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m y_{kl}}{\rho_2}\Big\|\Big)\Big]^{p_{kl}}\geq r+\frac{\epsilon}{2}\right\}\in I_2.$$

This completes the proof. □

**Theorem 2.2** *Let $\mathcal{M}=(M_{kl})$ be a Musielak-Orlicz function, $p=(p_{kl})$ be a bounded sequence of strictly positive real numbers and $u=(u_{kl})$ be a sequence of strictly positive real numbers. Then the spaces $w^2_{I_c}[A,\mathcal{M},u,r,\Delta^m,p]$, $w^2_{I_{c_0}}[A,\mathcal{M},u,r,\Delta^m,p]$ and $w^2_{I_{l_\infty}}[A,\mathcal{M},u,r,\Delta^m,p]$ are paranormed spaces with the paranorm $g$ defined by*

$$g(x)=\inf\left\{(\rho)^{\frac{p_{kl}}{H}}:\left(\sum_{k,l=1,1}^{\infty,\infty}a_{nmkl}\Big[M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m x_{kl}}{\rho}\Big\|\Big)\Big]^{p_{kl}}\right)^{\frac{1}{H}}\leq r,\right.$$

$$\left.\text{for some }\rho>0\right\},$$

*where $H=\max\{r,\sup_{k,l}p_{kl}\}$.*

*Proof.* Clearly, $g(-x)=g(x)$ and $g(0)=0$. Let $x=(x_{kl})$ and $y=(y_{kl})$ belongs to $w^2_{I_{c_0}}[A,\mathcal{M},u,r,\Delta^m,p]$. Then for every $\rho>0$ we write

$$A_1=\left\{\rho_1>0:\left(\sum_{k,l=1,1}^{\infty,\infty}a_{nmkl}\Big[M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m x_{kl}}{\rho_1}\Big\|\Big)\Big]^{p_{kl}}\right)^{\frac{1}{H}}\leq r\right\}$$

and

$$A_2=\left\{\rho_2>0:\left(\sum_{k,l=1,1}^{\infty,\infty}a_{nmkl}\Big[M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m y_{kl}}{\rho_2}\Big\|\Big)\Big]^{p_{kl}}\right)^{\frac{1}{H}}\leq r\right\}.$$

Let $\rho_1 \in A_1$ and $\rho_2 \in A_2$. If $\rho = \rho_1 + \rho_2$, then we get the following:

$$\begin{aligned}
&\Bigg( \sum_{k,l=1,1}^{\infty,\infty} a_{nmkl} \Big[ M_{kl}\Big( \Big\| \frac{u_{kl}\Delta^m (x_{kl}+y_{kl})}{\rho} \Big\| \Big) \Big]^{p_{kl}} \Bigg) \\
&\qquad \le \Big( \frac{\rho_1}{\rho_1+\rho_2} \Big) \Bigg( \sum_{k,l=1,1}^{\infty,\infty} a_{nmkl} \Big[ M_{kl}\Big( \Big\| \frac{u_{kl}\Delta^m x_{kl}}{\rho_1} \Big\| \Big) \Big]^{p_{kl}} \Bigg) \\
&\qquad\quad + \Big( \frac{\rho_2}{\rho_1+\rho_2} \Big) \Bigg( \sum_{k,l=1,1}^{\infty,\infty} a_{nmkl} \Big[ M_{kl}\Big( \Big\| \frac{u_{kl}\Delta^m y_{kl}}{\rho_2} \Big\| \Big) \Big]^{p_{kl}} \Bigg).
\end{aligned}$$

Thus, we have

$$\sum_{k,l=1,1}^{\infty,\infty} a_{nmkl} \Big[ M_{kl}\Big( \Big\| \frac{u_{kl}\Delta^m (x_{kl}+y_{kl})}{\rho} \Big\| \Big) \Big]^{p_{kl}} \le r$$

and

$$\begin{aligned}
g(x+y) &= \inf\{(\rho_1+\rho_2)^{\frac{p_{kl}}{H}} : \rho_1 \in A_1, \rho_2 \in A_2\} \\
&\le \inf\{(\rho_1)^{\frac{p_{kl}}{H}} : \rho_1 \in A_1\} + \inf\{(\rho_2)^{\frac{p_{kl}}{H}} : \rho_2 \in A_2\} \\
&= g(x) + g(y).
\end{aligned}$$

Let $t_{kl}^m \to t$, where $t_{kl}^m,\ t \in \mathbb{C}$ and let $g(x_{kl}^m - x_{kl}) \to 0$ as $m \to \infty$. To prove that $g(t_{kl}^m x_{kl}^m - t x_{kl}) \to 0$ as $m \to \infty$. Let $t_{kl} \to t$, where $t_{kl},\ t \in \mathbb{C}$ and $g(x_{kl}^m - x_{kl}) \to 0$ as $m \to \infty$. We have

$$A_3 = \Bigg\{ \rho_k > 0 : \sum_{k,l=1,1}^{\infty,\infty} a_{nmkl} \Big[ M_{kl}\Big( \Big\| \frac{u_{kl}\Delta^m x_{kl}^m}{\rho_k} \Big\| \Big) \Big]^{p_{kl}} \le r \Bigg\}$$

and

$$A_4 = \Bigg\{ \rho_k' > 0 : \sum_{k,l=1,1}^{\infty,\infty} a_{nmkl} \Big[ M_{kl}\Big( \Big\| \frac{u_{kl}\Delta^m (x_{kl}^m - x)}{\rho_k'} \Big\| \Big) \Big]^{p_{kl}} \le r \Bigg\}.$$

If $\rho_k \in A_3$ and $\rho_k' \in A_4$ then by inequality (1.1) and continuity of the function $\mathcal{M} = (M_{kl})$, we have

$$\begin{aligned}
M_{kl}\Big( \Big\| \frac{u_{kl}\Delta^m (t^m x_{kl}^m - tx)}{|t^m - t|\rho_k + |t|\rho_k'} \Big\| \Big) &\le M_{kl}\Big( \Big\| \frac{u_{kl}\Delta^m (t^m x_{kl}^m - t x_{kl})}{|t^m - t|\rho_k + |t|\rho_k'} \Big\| \Big) \\
&\quad + M_{kl}\Big( \Big\| \frac{u_{kl}\Delta^m (t x_{kl} - tx)}{|t^m - t|\rho_k + |t|\rho_k'} \Big\| \Big) \\
&\le \frac{|t^m - t|\rho_k}{|t^m - t|\rho_k + |t|\rho_k'} M_{kl}\Big( \Big\| \frac{u_{kl}\Delta^m x_{kl}^m}{\rho_k} \Big\| \Big) \\
&\quad + \frac{|t|\rho_k'}{|t^m - t|\rho_k + |t|\rho_k'} M_{kl}\Big( \Big\| \frac{u_{kl}\Delta^m (x_{kl}^m - x_{kl})}{\rho_k'} \Big\| \Big).
\end{aligned}$$

From the above inequality it follows that

$$\sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m(t^m x_{kl}^m - tx)}{|t^m - t|\rho_k + |t|\rho_k'}\Big\|\Big)\Big]^{p_k} \le r$$

and consequently,

$$\begin{aligned} g(t^m x_{kl}^m + tx) &= \inf\{(|t^m - t|\rho_k + |t|\rho_k')\}^{\frac{p_{kl}}{H}} : \rho_k \in A_3, \rho_k' \in A_4\} \\ &\le |t^m - t|^{\frac{p_{kl}}{H}} \inf\{(\rho_k)^{\frac{p_{kl}}{H}} : \rho_k \in A_3\} + |t|^{\frac{p_{kl}}{H}} \inf\{(\rho_k')^{\frac{p_{kl}}{H}} : \rho_k' \in A_4\} \\ &\le \max\{1, |t^m - t|^{\frac{p_{kl}}{H}}\} g(x_{kl}^m) + \max\{1, |t|^{\frac{p_{kl}}{H}}\} g(x_{kl}^m - x) \\ &\to 0 \text{ as } m \to \infty. \end{aligned}$$

This completes the proof. □

**Theorem 2.3** *(i) Let* $0 < \inf p_{kl} \le p_{kl} \le 1$. *Then* $w_{I_c}^2[A, \mathcal{M}, u, r, \Delta^m, p] \subseteq w_{I_c}^2[A, \mathcal{M}, u, r, \Delta^m]$, $w_{I_{c_0}}^2[A, \mathcal{M}, u, r, \Delta^m, p] \subseteq w_{I_{c_0}}^2[A, \mathcal{M}, u, r, \Delta^m]$.
*(ii) Let* $1 \le p_{kl} \le \sup p_{kl} < \infty$. *Then* $w_{I_c}^2[A, \mathcal{M}, u, r, \Delta^m] \subseteq w_{I_c}^2[A, \mathcal{M}, u, r, \Delta^m, p]$, $w_{I_{c_0}}^2[A, \mathcal{M}, u, r, \Delta^m] \subseteq w_{I_{c_0}}^2[A, \mathcal{M}, u, r, \Delta^m, p]$.

*Proof.* (*i*) Let $x = (x_{kl}) \in w_{I_c}^2[A, \mathcal{M}, u, r, \Delta^m, p]$. Since $0 < \inf p_{kl} \le p_{kl} \le 1$, we have

$$\sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m x_{kl} - x_*}{\rho}\Big\|\Big)\Big]$$

$$\le \sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m x_{kl} - x_*}{\rho}\Big\|\Big)\Big]^{p_{kl}}.$$

Therefore,

$$\Big\{(k,l) \in \mathbb{N} \times \mathbb{N} : \sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m x_{kl} - x_*}{\rho}\Big\|\Big)\Big] \ge r + \epsilon\Big\}$$

$$\subseteq \Big\{(k,l) \in \mathbb{N} \times \mathbb{N} : \sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m x_{kl} - x_*}{\rho}\Big\|\Big)\Big]^{p_{kl}} \ge r + \epsilon\Big\} \in I_2.$$

The other part can be proved in a similar way.

(*ii*) Let $x = (x_{kl}) \in w^2_{I_c}[A, \mathcal{M}, u, r, \Delta^m]$. Since $1 \le p_{kl} \le \sup p_{kl} < \infty$. Then, for each $0 < \epsilon < 1$ there exists positive integer $n_0$ such that

$$\sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m x_{kl} - x_*}{\rho}\Big\|\Big)\Big] \le \epsilon < 1 \text{ for all } n, m \ge n_0.$$

This implies that

$$\sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m x_{kl} - x_*}{\rho}\Big\|\Big)\Big]^{p_{kl}}$$

$$\le \sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m x_{kl} - x_*}{\rho}\Big\|\Big)\Big].$$

Therefore, we have

$$\Big\{(k,l) \in \mathbb{N}\times\mathbb{N} : \sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m x_{kl} - x_*}{\rho}\Big\|\Big)\Big]^{p_{kl}} \ge r + \epsilon\Big\}$$

$$\subseteq \Big\{(k,l) \in \mathbb{N}\times\mathbb{N} : \sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m x_{kl} - x_*}{\rho}\Big\|\Big)\Big] \ge r + \epsilon\Big\} \in I_2.$$

The other part can be proved in a similar way. This completes the proof. □

**Theorem 2.4** *Let $\mathcal{M} = (M_{kl})$ be a Musielak-Orlicz function, $p = (p_{kl})$ be a bounded sequence of strictly positive real numbers and $u = (u_{kl})$ be a sequence of strictly positive real numbers. Then*

$$w^2_{I_{c_0}}[A, \mathcal{M}, u, r, \Delta^m, p] \subset w^2_{I_c}[A, \mathcal{M}, u, r, \Delta^m, p] \subset w^2_{I_{l_\infty}}[A, \mathcal{M}, u, r, \Delta^m, p].$$

*Proof.* The inclusion $w^2_{I_{c_0}}[A, \mathcal{M}, u, r, \Delta^m, p] \subset w^2_{I_c}[A, \mathcal{M}, u, r, \Delta^m, p]$ is obvious. Let $x = (x_{kl}) \in w^2_{I_c}[A, \mathcal{M}, u, r, \Delta^m, p]$. Then there is some number $x_*$ such that

$$\Big\{(k,l) \in \mathbb{N}\times\mathbb{N} : \sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m x_{kl} - x_*}{\rho}\Big\|\Big)\Big]^{p_{kl}} \ge r + \epsilon\Big\} \in I_2.$$

Now by inequality (1.1), we have

$$\sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m x_{kl}}{\rho}\Big\|\Big)\Big]^{p_{kl}}$$

$$\leq D\sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m x_{kl}-x_*}{\rho}\Big\|\Big)\Big]^{p_{kl}}$$

$$+D\sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[M_{kl}\Big(\Big\|\frac{x_*}{\rho}\Big\|\Big)\Big]^{p_{kl}}.$$

This implies that $x=(x_{kl})\in w^2_{I_{l_\infty}}[A,\mathcal{M},u,r,\Delta^m,p]$. This completes the proof. □

**Theorem 2.5** *Let $\mathcal{M}=(M_{kl})$ and $\mathcal{M}'=(M'_{kl})$ be Musielak-Orlicz functions. Then*

$$w^2_{I_c}[A,\mathcal{M},u,r,\Delta^m,p]\cap w^2_{I_c}[A,\mathcal{M}',u,r,\Delta^m,p]\subset w^2_{I_c}[A,\mathcal{M}+\mathcal{M}',u,r,\Delta^m,p].$$

*Proof.* Let $x=(x_{kl})\in w^2_{I_c}[A,\mathcal{M},u,r,\Delta^m,p]\cap w^2_{I_c}[A,\mathcal{M}',u,r,\Delta^m,p]$ using the inequality (1.1), we have

$$\sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[(M_{kl}+M'_{kl})\Big(\Big\|\frac{u_{kl}\Delta^m x_{kl}-x_*}{\rho}\Big\|\Big)\Big]^{p_{kl}}$$

$$=\sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m x_{kl}-x_*}{\rho}\Big\|\Big)+M'_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m x_{kl}-x_*}{\rho}\Big\|\Big)\Big]^{p_{kl}}$$

$$\leq D\Big\{\sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m x_{kl}-x_*}{\rho}\Big\|\Big)\Big]^{p_{kl}}$$

$$+\sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[M'_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m x_{kl}-x_*}{\rho}\Big\|\Big)\Big]^{p_{kl}}\Big\}.$$

Thus, $x=(x_{kl})\in w^2_{I_c}[A,\mathcal{M}+\mathcal{M}',u,r,\Delta^m,p]$. This completes the proof. □

**Theorem 2.6** *Let $\mathcal{M}=(M_{kl})$ be a Musielak-Orlicz function which satisfies the $\Delta_2$-condition. Then $w^2_{I_c}[A,u,r,\Delta^m,p]\subseteq w^2_{I_c}[A,\mathcal{M},u,r,\Delta^m,p]$.*

*Proof.* Let $x=(x_{kl})\in w^2_{I_c}[A,u,r,\Delta^m,p]$, that is,

$$\Big\{(k,l)\in\mathbb{N}\times\mathbb{N}:\sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[\Big(\Big\|\frac{u_{kl}\Delta^m x_{kl}-x_*}{\rho}\Big\|\Big)\Big]^{p_{kl}}\geq r+\epsilon\Big\}\in I_2.$$

Let $\epsilon > 0$ and choose $\delta$ with $0 < \delta < 1$ such that $M_{kl}(t) < \epsilon$ for $0 \le t \le \delta$.

Write $y_{kl} = (\|\frac{u_{kl}\Delta^m x_{kl} - x_*}{\rho}\|)$ and consider

$$\sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[M_{kl}(y_{kl})\Big]^{p_{kl}}$$

$$= \sum_{k,l:|y_{kl}|\le\delta} a_{nmkl}\Big[M_{kl}(y_{kl})\Big]^{p_{kl}} + \sum_{k,l:|y_{kl}|>\delta} a_{nmkl}\Big[M_{kl}(y_{kl})\Big]^{p_{kl}}$$

$$= \epsilon \sum_{k,l:|y_{kl}|\le\delta} a_{nmkl} + \sum_{k,l:|y_{kl}|>\delta} a_{nmkl}\Big[M_{kl}(y_{kl})\Big]^{p_{kl}}.$$

For $y_{kl} > \delta$, we use the fact that $y_{kl} < \frac{y_{kl}}{\delta} < 1 + \frac{y_{kl}}{\delta}$. Hence,

$$M_{kl}(y_{kl}) < M_{kl}\Big(1 + \frac{y_{kl}}{\delta}\Big) < \frac{M_{kl}(2)}{2} + \frac{1}{2}M_{kl}\Big(2\frac{y_{kl}}{\delta}\Big).$$

Since $\mathcal{M}$ satisfies the $\Delta_2$-condition, we have

$$M_{kl}(y_{kl}) < K\frac{y_{kl}}{2\delta}M_{kl}(2) + K\frac{y_{kl}}{2\delta}M_{kl}(2) = K\frac{y_{kl}}{\delta}M_{kl}(2),$$

and hence,

$$\sum_{k,l:|y_{kl}|>\delta} a_{nmkl}\Big[M_{kl}(y_{kl})\Big]^{p_{kl}}$$

$$\le K\frac{M_{kl}}{\delta}(2)\sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[\Big(\Big\|\frac{u_{kl}\Delta^m x_{kl} - x_*}{\rho}\Big\|\Big)\Big]^{p_{kl}}.$$

Since $A$ is $RH$-regular and $x \in w^2_{I_c}[A, u, r, \Delta^m, p]$, we get $x \in w^2_{I_c}[A, \mathcal{M}, u, r, \Delta^m, p]$. □

**Theorem 2.7** *Let $\mathcal{M} = (M_{kl})$ be a Musielak-Orlicz function, $A = (a_{nmkl})$ be a nonnegative four-dimensional RH-regular matrix such that* $\sup_{n,m}\sum_{k,l=0,0}^{\infty,\infty} a_{nmkl} < \infty$ *and* $\beta = \lim_{t\to\infty}\frac{M_{kl}(t)}{t} < \infty$. *Then*

$$w^2_{I_c}[A, u, r, \Delta^m, p] = w^2_{I_c}[A, \mathcal{M}, u, r, \Delta^m, p].$$

*Proof.* In order to prove that $w^2_{I_c}[A, u, r, \Delta^m, p] = w^2_{I_c}[A, \mathcal{M}, u, r, \Delta^m, p]$. It is sufficient to show that $w^2_{I_c}[A, \mathcal{M}, u, r, \Delta^m, p] \subset w^2_{I_c}[A, u, r, \Delta^m, p]$. Now, let $\beta > 0$. By definition of $\beta$, we have $M_{kl}(t) \ge \beta t$ for all $t \ge 0$. Since $\beta > 0$, we have $t \le \frac{1}{\beta}M_{kl}(t)$ for all $t \ge 0$. Let $x = (x_{kl}) \in w^2_{I_c}[A, \mathcal{M}, u, r, \Delta^m, p]$.

Thus, we have

$$\sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[\Big(\Big\|\frac{u_{kl}\Delta^m x_{kl}-x_*}{\rho}\Big\|\Big)\Big]^{p_{kl}}$$

$$\leq \frac{1}{\beta}\sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m x_{kl}-x_*}{\rho}\Big\|\Big)\Big]^{p_{kl}}$$

which implies that $x=(x_{kl})\in w^2_{I_c}[A,u,r,\Delta^m,p]$. This completes the proof. □

**Theorem 2.8** *The spaces* $w^2_{I_{c_0}}[A,\mathcal{M},u,r,\Delta^m,p]$ *and* $w^2_{I_c}[A,\mathcal{M},u,r,\Delta^m,p]$ *are normal as well as monotone.*

*Proof.* We shall prove the result for $w^2_{I_{c_0}}[A,\mathcal{M},u,r,\Delta^m,p]$. Let $x=(x_{kl})\in w^2_{I_{c_0}}[A,\mathcal{M},u,r,\Delta^m,p]$, then we have

$$\Big\{(k,l)\in\mathbb{N}\times\mathbb{N}:\sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m x_{kl}}{\rho}\Big\|\Big)\Big]^{p_{kl}}\geq r+\epsilon\Big\}\in I_2.$$

Let $(\alpha_{kl})$ be a sequence scalar with $|\alpha_{kl}|\leq 1$ for all $k,l$. Then we get

$$\sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[M_{kl}\Big(\Big\|\frac{\alpha_{kl}u_{kl}\Delta^m x_{kl}}{\rho}\Big\|\Big)\Big]^{p_{kl}}\Big]$$

$$\leq |\alpha_{kl}|\sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m x_{kl}}{\rho}\Big\|\Big)\Big]^{p_{kl}}$$

$$\leq \sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m x_{kl}}{\rho}\Big\|\Big)\Big]^{p_{kl}}.$$

The spaces $w^2_{I_{c_0}}[A,\mathcal{M},u,r,\Delta^m,p]$ is normal and hence monotone follows from Lemma 1.9. Similarly, we can prove other. □

**Theorem 2.9** *The sequence spaces* $w^2_{I_c}[A,\mathcal{M},u,r,\Delta^m,p]$ *and* $w^2_{I_{c_0}}[A,\mathcal{M},u,r,\Delta^m,p]$ *are sequence algebras.*

*Proof.* Let $x=(x_{kl})$ and $y=(y_{kl})$ belongs to $w^2_{I_c}[A,\mathcal{M},u,r,\Delta^m,p]$. Let $\epsilon>0$ be given. Then

$$A(\epsilon)=$$

$$\Big\{(k,l)\in\mathbb{N}\times\mathbb{N}:\sum_{k,l=1,1}^{\infty,\infty} a_{nmkl}\Big[M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m x_{kl}-x_*}{\rho}\Big\|\Big)\Big]^{p_{kl}}\geq r+\epsilon\Big\}\in I_2,$$

$B(\epsilon) =$

$$\left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \sum_{k,l=1,1}^{\infty,\infty} a_{nmkl} \Big[M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m y_{kl} - x_*}{\rho}\Big\|\Big)\Big]^{p_{kl}} \geq r + \epsilon\right\} \in I_2$$

and

$$C(\epsilon) = \left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \sum_{k,l=1,1}^{\infty,\infty} a_{nmkl} \Big[M_{kl}\Big(\Big\|\frac{u_{kl}\Delta^m (x_{kl} y_{kl}) - x_*}{\rho}\Big\|\Big)\Big]^{p_{kl}} \geq r + \epsilon\right\} \in I_2.$$

Now it can be easily shown that $C(\epsilon) \subset A(\epsilon) \cup B(\epsilon)$ and $C(\epsilon) \in w^2_{I_c}[A, \mathcal{M}, u, r, \Delta^m, p]$. □

**Theorem 2.10** *The spaces* $w^2_{I_c}[A, \mathcal{M}, u, r, \Delta^m, p]$ *and* $w^2_{I_{c_0}}[A, \mathcal{M}, u, r, \Delta^m, p]$ *are not convergence free in general.*

*Proof.* The proof is easy so we omit it. □

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