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# On the Stanley depth of a special class of Borel type ideals

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**Abstract** We give sharp bounds for the Stanley depth of a special class of monomial ideals of Borel type.

**Keywords** monomial ideals · ideals of Borel type · Stanley depth

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## Introduction

Let $K$ be a field and $S = K[x_1,\dots,x_n]$ the polynomial ring over $K$. Let $M$ be a $\mathbb{Z}^n$-graded $S$-module. A *Stanley decomposition* of $M$ is a direct sum $\mathscr{D} : M = \bigoplus_{i=1}^r m_iK[Z_i]$ of $\mathbb{Z}^n$-graded $K$-vector spaces, where $m_i \in M$ is homogeneous with respect to $\mathbb{Z}^n$-grading, $Z_i \subset \{x_1,\dots,x_n\}$ such that $m_iK[Z_i] = \{um_i : u \in K[Z_i]\} \subset M$ is a free $K[Z_i]$-submodule of $M$. We define $\operatorname{sdepth}(\mathscr{D}) = \min_{i=1,\dots,r}|Z_i|$ and $\operatorname{sdepth}(M) = \max\{\operatorname{sdepth}(\mathscr{D})|\ \mathscr{D}$ is a Stanley decomposition of $M\}$. The number $\operatorname{sdepth}(M)$ is called the *Stanley depth* of $M$.

Stanley conjectured in [17] that $\operatorname{sdepth}(M) \geq \operatorname{depth}(M)$ for any $M$. The conjecture was disproved in [9] for $M = S/I$, where $I \subset S$ is a monomial ideal, but remains open in the case $M = I$. Herzog, Vladoiu and Zheng showed in [13] that $\operatorname{sdepth}(M)$ can be computed in a finite number of steps if $M = I/J$, where $J \subset I \subset S$ are monomial ideals. In [16], Rinaldo gave a computer implementation for this algorithm, in the computer algebra system `CoCoA` [8]. For an introduction in the thematic of Stanley depth, we refer the reader to [10].

We say that a monomial ideal $I \subset S$ is of *Borel type*, see [12], if it satisfies the following condition: $(I : x_j^\infty) = (I : (x_1,\dots,x_j)^\infty)$, $(\forall)1 \leq j \leq n$. The *Mumford-Castelnuovo regularity* of $I$ is the number $\operatorname{reg}(I) = \max\{j - i : \beta_{ij}(I) \neq 0\}$, where $\beta_{ij}$'s are the graded Betti numbers.

The regularity of the ideals of Borel type was extensively studied, see for instance [12], [1] and [5]. We study the invariant $\operatorname{sdepth}(I)$, for an ideal of Borel type. In the general case, we note some bounds for $\operatorname{sdepth}(I)$, see Proposition 1.2 and we give some tighter ones, when $I$ has a special form, see Theorem 1.6.

## 1 Main results

First, we recall the construction of the sequential chain associated to a Borel type ideal $I \subset S$, see [12] for more details. Assume that $\operatorname{Ass}(S/I) = \{P_0, \dots, P_m\}$ with $P_i = (x_1, \dots, x_{n_i})$, where $n \geq n_0 > n_1 > \cdots > n_m \geq 1$. Also, assume that $I = \bigcap_{i=0}^m Q_i$ is the reduced primary decomposition of $I$, with $P_i = \sqrt{Q_i}$, for all $0 \leq i \leq m$.

We define $I_k := \bigcap_{j=k}^m Q_j$, for all $0 \leq k \leq m$. One can easily check that $I_i = (I_{i-1} : x_{n_{i-1}}^\infty)$, for all $1 \leq i \leq m$. The sequence of ideals $I = I_0 \subset I_1 \subset \cdots \subset I_m \subset I_{m+1} := S$ is called the *sequential sequence* of $I$. Let $J_i$ be the monomial ideal generated by $G(I_i)$ in $S_i := K[x_1, \dots, x_{n_i}]$, for all $0 \leq i \leq m$. Then, the saturation $J_i^{sat} = (J_i : (x_1, \dots, x_{n_i})^\infty) = J_{i+1} S_i$, for all $0 \leq i \leq m$, where $J_{m+1} := S_{m+1}$. One has $I_{i+1}/I_i \cong (J_i^{sat}/J_i)[x_{n_i+1}, \dots, x_n]$. If $M = \bigoplus_{t \geq 0} M_t$ is an Artinian graded $S$-module, we denote $s(M) = \max\{t \ : \ M_t \neq 0\}$. We recall the following result.

**Proposition 1.1** *[12, Corollary 2.7]* $\operatorname{reg}(I) = \max\{s(J_0^{sat}/J_0), \dots, s(J_m^{sat}/J_m)\} + 1.$

**Proposition 1.2** *With the above notations, the following assertions hold:*
(1) $\operatorname{sdepth}(S/I_i) = \operatorname{depth}(S/I_i) = n - n_i$, *for all* $0 \leq i \leq m$.
(2) $\operatorname{sdepth}(I_0) \leq \operatorname{sdepth}(I_1) \leq \cdots \leq \operatorname{sdepth}(I_m)$.
(3) $\operatorname{depth}(I_i) = n - n_i + 1 \leq \operatorname{sdepth}(I_i) \leq \operatorname{sdepth}(P_i) = n - \left\lfloor \frac{n_i}{2} \right\rfloor$, $(\forall) 0 \leq i \leq m$.

*Proof.* (1) From [13, Lemma 3.6] it follows that $\operatorname{sdepth}(S/I_i) = \operatorname{sdepth}(S_i/J_i) + n - n_i$. Also, we have $\operatorname{depth}(S/I_i) = \operatorname{depth}(S_i/J_i) + n - n_i$.

Since $P_i S_i = (x_1, \dots, x_{n_i}) S_i \in Ass(S_i/J_i)$, it follows that $\operatorname{depth}(S_i/J_i) = 0$ and thus, by [6, Theorem 1.4] or [10, Proposition 18], we get $\operatorname{sdepth}(S_i/J_i) = \operatorname{depth}(S_i/J_i) = 0$.

(2) Since $I_i = (I_{i-1} : x_{n_{i-1}}^\infty)$, by [14, Proposition 1.3] (see arXiv version), we get $\operatorname{sdepth}(I_{i-1}) \leq \operatorname{sdepth}(I_i)$, for all $1 \leq i \leq m$.

(3) Since $I_i = J_i S$, by [13, Lemma 3.6], it follows that $\operatorname{sdepth}(I_i) = n - n_i + \operatorname{sdepth}_{S_i}(J_i) \geq n - n_i + 1$. Since $P_i \in \operatorname{Ass}(I_i)$, it follows that there exists a monomial $v \in S$, such that $P_i = (I_i : v)$. Therefore, by [14, Proposition 1.3] (see arXiv version), it follows that $\operatorname{sdepth}(P_i) \geq \operatorname{sdepth}(I_i)$. On the other hand, $P_i$ is generated by variables. Thus, by [13, Lemma 3.6] and [2, Theorem 1.1], it follows that $\operatorname{sdepth}(P_i) = n - \left\lfloor \frac{ht(P_i)}{2} \right\rfloor = n - \left\lfloor \frac{n_i}{2} \right\rfloor$. □

**Lemma 1.3** *Let $r \leq n$ and $a_1, \dots, a_r$ be positive integers. If $Q = (x_1^{a_1}, \dots, x_r^{a_r}) \subset S$, then $reg(Q) = a_1 + \cdots + a_r - r + 1$.*

*Proof.* Let $\bar{Q} = Q \cap S' \subset S'$, where $S' = K[x_1, \dots, x_r]$. As a particular case of Proposition 1.1, we get $\operatorname{reg}(Q) = \operatorname{reg}(\bar{Q}) = s(S'/\bar{Q}) + 1 = a_1 + \cdots + a_r - r + 1$. □

We recall the following result from [1].

**Proposition 1.4** *[1, Corollary 3.17] If $I \subset S$ is an ideal of Borel type with the irredundant irreducible decomposition $I = \bigcap_{i=1}^{r} C_i$, then* $\operatorname{reg}(I) = \max\{\operatorname{reg}(C_i) : 1 \leq i \leq r\}$.

Let $n \geq n_0 > n_1 > \cdots > n_m \geq 1$ be some integers. Let $a_{ij}$ be some positive integers, where $0 \leq i \leq m$ and $1 \leq j \leq n_i$. We consider the monomial irreducible ideals $Q_i = (x_1^{a_{i1}}, \dots, x_{n_i}^{a_{in_i}})$, for $0 \leq i \leq m$. Let $I_i := \bigcap_{j=i}^{m} Q_j$ and denote $I = I_0$. Since $P_i = (x_1, \dots, x_{n_i}) = \sqrt{Q_i}$ for all $0 \leq i \leq m$, by [11, Proposition 5.2] or [5, Corollary 1.2], it follows that $I$ is an ideal of Borel type. As a direct consequence of Lemma 1.3 and Proposition 1.4, we get the following corollary.

**Corollary 1.5** *If $a_{ij} \geq a_{i+1,j}$ for all $j \leq n_{i+1}$ and $i < m$, then* $\operatorname{reg}(I_i) = \operatorname{reg}(Q_i) = a_{i1} + a_{i2} + \cdots + a_{in_i} - n_i + 1$, *for all* $0 \leq i \leq m$.

**Theorem 1.6** *If $a_{ij} \geq a_{i+1,j}$ for all $j \leq n_{i+1}$ and $i < m$, then for all $0 \leq i \leq m$, it holds that*

$$n - \left\lfloor \frac{n_i}{2} \right\rfloor \geq \operatorname{sdepth}(I_i) \geq n + \left\lceil \frac{n_m}{2} \right\rceil - n_i.$$

*Proof.* The first inequality follows from Proposition 1.2(3). In order to prove the second one, we use induction on $i \leq m$. If $i = m$, then $I_m = Q_m$ is an irreducible ideal, and therefore, by [7, Theorem 1.3], $\operatorname{sdepth}(I_m) = n - \lfloor \frac{n_m}{2} \rfloor = n + \lceil \frac{n_m}{2} \rceil - n_m$.

Assume $i < m$. We can write $Q_i = U_i + V_i$, where $U_i = (x_1^{a_{i1}}, \dots, x_{n_{i+1}}^{a_{in_{i+1}}})$ and $V_i = (x_{n_{i+1}+1}^{a_{i,n_{i+1}+1}}, \dots, x_{n_i}^{a_{in_i}})$. Since $a_{ij} \geq a_{i+1,j}$ for all $j \leq n_{i+1}$, it follows that $U_i \subset Q_{i+1}$. Therefore, $I_i = (U_i + V_i) \cap I_{i+1} = (U_i \cap I_{i+1}) + (V_i \cap I_{i+1})$. Note that $J := U_i \cap I_{i+1} = U_i \cap I_{i+2}$ is a Borel type ideal with the irreducible irredundant decomposition $J = U_i \cap Q_{i+2} \cap \cdots \cap Q_m$, and, therefore, of the same class as $I_{i+1}$. Thus, by induction hypothesis, it follows that $\operatorname{sdepth}(J) \geq n + \lceil \frac{n_m}{2} \rceil - n_{i+1}$.

On the other hand, by [4, Remark 1.3] and the induction hypothesis, $\operatorname{sdepth}(V_i \cap I_{i+1}) \geq \operatorname{sdepth}(V_i) + \operatorname{sdepth}(I_{i+1}) - n = \operatorname{sdepth}(I_{i+1}) - \left\lfloor \frac{n_i - n_{i+1}}{2} \right\rfloor$.

Let $\bar{V}_i \subset S' = K[x_{n_{i+1}+1}, \dots, x_{n_i}]$ be the monomial ideal generated by $G(V_i)$ and let $\bar{J} \subset S'' = K[x_1, \dots, x_{n_{i+1}}, x_{n_i+1}, \dots, x_n]$ be the monomial ideal generated by $G(J)$. Since $J \subset I_{i+1}$, it follows that $I_i = (\bar{J} \otimes_K (S''/\bar{V}_i)) \oplus (V_i \cap I_i)$. By [3, Proposition 2.10] and [15, Lemma 2.2], we get:

$$\operatorname{sdepth}(I_i) \geq \min\{\operatorname{sdepth}(J) - n_i + n_{i+1}, \operatorname{sdepth}(I_{i+1}) - \left\lfloor \frac{n_i - n_{i+1}}{2} \right\rfloor\} \geq n + \left\lceil \frac{n_m}{2} \right\rceil - n_i,$$

as required. □

**Question**: What can we say about the case when the condition $a_{ij} \geq a_{i+1,j}$ is removed? Of course, the method used in the proof of the Theorem 1.6 does not work. However, our computer experiments in `Cocoa` [8] suggested that the conclusion of the Theorem 1.6 might be true. Unfortunately, we are not able to give either a proof, or a counterexample.

The next example shows that the bounds given in Theorem 1.6 are sharp.

*Example 1.1* *Let* $I = Q_0 \cap Q_1$, *where* $Q_0 = (x_1^3, x_2^2, x_3^2, x_4, x_5)$, $Q_1 = (x_1, x_2, x_3, x_4)$ *are ideals in* $S = K[x_1, \dots, x_5]$. *Then* $I_1 = Q_1$ *and* $\operatorname{sdepth}(I_1) = 5 - \left\lfloor \frac{4}{2} \right\rfloor = 3$. *Also* $n = 5$, $n_0 = 5$ *and* $n_1 = 4$. *Using* `CoCoa`, *we get* $\operatorname{sdepth}(I) = 2 = n - \left\lceil \frac{n_1}{2} \right\rceil - n_0$. *Let* $Q_0' = (x_1^2, x_2^2, x_3, x_4, x_5) \subset S$ *and* $I' = Q_0' \cap Q_1$. *Using* `CoCoa` *[8], we get* $\operatorname{sdepth}(I') = 3 = n - \left\lfloor \frac{n_0}{2} \right\rfloor$.

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