An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)
Tomul LXIV, 2018, f. 2

On differential identities involving commutator and anti-commutator in prime and semiprime rings

Mohammad Ashraf · Sajad Ahmad Pary · Mohd Arif Raza

Abstract In the present paper, we investigate the action of derivation α in a semi(-prime) ring $\mathcal{R}$ which satisfies differential identities involving commutator and anti-commutator for a nonzero ideal $\mathcal{I}$ of R. In fact, we obtain commutativity of a prime ring $\mathcal{R}$ which satisfies either of the identities $\alpha(x \circ_m y) = [\alpha(x), \alpha(y)]_m$ for all $x, y \in \mathcal{I}$ or $\alpha([x, y]_m) = \alpha(x) \circ_m \alpha(y)$ for all $x, y \in \mathcal{I}$, where α is a nonzero derivation. Finally, the results have also been extended to the semiprime ring.

Keywords (Semi)-prime ring · derivation · Martindale ring of quotients

Mathematics Subject Classification (2010) 16N20 · 16W25

1 Introduction

In what follows $\mathcal{R}$ is a semi(-prime) ring with the center $\mathcal{Z}(\mathcal{R})$, $\mathcal{Q}$ is the Martindale quotient ring of $\mathcal{R}$ and $\mathcal{U}$ is the Utumi quotient ring of $\mathcal{R}$. The center of $\mathcal{U}$, denoted by $\mathcal{C}$ is called the extended centroid of $\mathcal{R}$ (we refer the reader to [3], for the definitions and related properties of these objects). For any $x, y \in \mathcal{R}$, the symbol $[x, y]$ and $x \circ y$ stands for the commutator $xy - yx$ and anti-commutator $xy + yx$, respectively. We set $x \circ_0 y = x,\ x \circ_1 y = x \circ y = xy + yx$, and inductively $x \circ_m y = (x \circ_{m-1} y) \circ y$ for $m > 1$. Again we set $[x, y]_0 = x,\ [x, y]_1 = [x, y] = xy - yx$ and inductively $[x, y]_m = [[x, y]_{m-1}, y]$ for $m > 1$.

Recall that a ring $\mathcal{R}$ is prime if $x\mathcal{R}y = (0)$ implies either $x = 0$ or $y = 0$, and $\mathcal{R}$ is semiprime if $x\mathcal{R}x = (0)$ implies $x = 0$. An additive mapping $\alpha : \mathcal{R} \to \mathcal{R}$ is called a derivation on R if $\alpha(xy) = \alpha(x)y + x\alpha(y)$ holds for all $x, y \in R$. In particular, α is an inner derivation induced by an element $\mathsf{q} \in \mathcal{R}$ if $\alpha(x) = [\mathsf{q}, x]$ holds for all $x \in \mathcal{R}$.

During the past few decades, there has been an ongoing interest concerning the relationship between the commutativity of a ring and the existence of certain specific types of derivations (see [2], where further references can be found). In [11] Herstein proved if $\mathcal{R}$ is a prime ring with characteristic different from 2 and $\mathcal{R}$ admits a nonzero derivation α such that $[\alpha(x), \alpha(y)] = 0$ for all $x, y \in \mathcal{R}$, then $\mathcal{R}$ is commutative. In 1992, Daif and Bell [8] proved that if $\mathcal{R}$ is a semiprime ring and α is a nonzero derivation of $\mathcal{R}$ such that $\alpha([x, y]) = [x, y]$ for all $x, y \in \mathcal{R}$, then $\mathcal{R}$ is commutative. In [2], Ashraf and Rehman proved that if $\mathcal{R}$ is a prime ring, $\mathcal{I}$ is a nonzero ideal of $\mathcal{R}$ and α is a nonzero derivation of $\mathcal{R}$ such that $\alpha(x \circ y) = x \circ y$ for all $x, y \in \mathcal{I}$, then $\mathcal{R}$ is commutative. In [1], Argaç and Inceboz generalized the above result and proved that, if a prime ring $\mathcal{R}$ admits a nonzero derivation α with the property $(\alpha(x \circ y))^n = x \circ y$ for all $x, y \in \mathcal{I}$, a nonzero ideal of $\mathcal{R}$, where n is a fixed positive integer, then $\mathcal{R}$ is commutative.

If $\mathcal{R}$ is a ring and $\mathcal{S} \subseteq \mathcal{R}$, a mapping $f : \mathcal{S} \to \mathcal{R}$ is called strong commutativity-preserving (scp) on $\mathcal{S}$ if $[f(x), f(y)] = [x, y]$ for all $x, y \in \mathcal{S}$. In 1994, Bell and Daif [5] initiated the study of strong commutativity-preserving maps and proved that a nonzero right ideal $\mathcal{I}$ of a semiprime ring $\mathcal{R}$ is central if $\mathcal{R}$ admits a derivation which is scp on $\mathcal{I}$. More precisely, they proved that if $\mathcal{R}$ is a semiprime ring, $\mathcal{I}$ is a nonzero ideal of $\mathcal{R}$ and $\mathcal{R}$ admits a derivation α such that $[\alpha(x), \alpha(y)] = [x, y]$ for all $x, y \in \mathcal{I}$, then $\mathcal{I} \subseteq \mathcal{Z}(\mathcal{R})$. In an attempt to generalize the above theorem, Deng and Ashraf [9] proved that if $\mathcal{R}$ is a semiprime ring and $\mathcal{I}$ a nonzero ideal of $\mathcal{R}$ and $\mathcal{R}$ admits a mapping f and a derivation α such that $[f(x), \alpha(y)] = [x, y]$ for all $x, y \in \mathcal{I}$, then $\mathcal{R}$ contains a nonzero central ideal of $\mathcal{R}$. In 2002, Ashraf and Rehman [2] replaced commutator by anti-commutator and proved that if $\mathcal{R}$ is a 2-torsion free prime ring, $\mathcal{I}$ is a nonzero ideal of $\mathcal{R}$ and α is a nonzero derivation of $\mathcal{R}$ such that $\alpha(x) \circ \alpha(y) = x \circ y$ for all $x, y \in \mathcal{I}$, then $\mathcal{R}$ is commutative. Inspired by the above mentioned results, we continue our line of investigation by examining the identities involving both commutator and anti-commutator namely (i) $\alpha(x \circ_m y) = [\alpha(x), \alpha(y)]_m$ and (ii) $\alpha([x, y]_m) = \alpha(x) \circ_m \alpha(y)$ for all x, y in an ideal $\mathcal{I}$ of $\mathcal{R}$, where m is a positive integer. We also examine the case when $\mathcal{R}$ is semiprime.

2 The results in prime rings

We begin our discussion with the following fact which is very crucial in developing the proof of our main results.

Fact 2.1 **([4, Lemma 7.1])** *Let* ${}_{\mathcal{D}}\mathcal{M}$ *be a left vector space over a division ring* $\mathcal{D}$ *with* $dim_{\mathcal{D}}\mathcal{M} \geq 2$ *and* $\mathcal{T} \in End(\mathcal{M})$. *If* x *and* $\mathcal{T}x$ *are* $\mathcal{D}$*-dependent for every* $x \in \mathcal{M}$, *then there exists* $\lambda \in \mathcal{D}$ *such that* $\mathcal{T}x = \lambda x$ *for all* $x \in \mathcal{M}$.

Now we are well equipped to prove our main results.

Theorem 2.2 *Let* $\mathcal{R}$ *be a prime ring of characteristic different from* 2, $\mathcal{I}$ *be a nonzero ideal of* $\mathcal{R}$ *and* $m \geq 1$ *be the fixed positive integer. If* $\mathcal{R}$ *admits a nonzero derivation* α *such that* $\alpha(x \circ_m y) = [\alpha(x), \alpha(y)]_m$ *for all* $x, y \in \mathcal{I}$, *then* $\mathcal{R}$ *is commutative.*

Proof. By the given hypothesis we can write

$$\begin{aligned}\sum_{k=1}^{m}\binom{m}{k}\Big(\sum_{i+j=k-1} y^i\alpha(y)y^j\Big)xy^{m-k}+\sum_{k=0}^{m}\binom{m}{k}y^k\alpha(x)y^{m-k}\\ +\sum_{k=0}^{m-1}\binom{m}{k}y^k x\Big(\sum_{r+s=m-k-1} y^r\alpha(y)y^s\Big)\\ =[\alpha(x),\alpha(y)]_m \text{ for all } x,y\in\mathcal{I}\end{aligned}$$

In the light of Kharchenko's theory [13], we divide the proof into two cases:

Case 1. If α is $\mathcal{Q}$-outer, then $\mathcal{I}$ satisfies the polynomial identity

$$\begin{aligned}\sum_{k=1}^{m}\binom{m}{k}\Big(\sum_{i+j=k-1} y^i z y^j\Big)xy^{m-k}\\ +\sum_{k=0}^{m}\binom{m}{k}y^k w y^{m-k}\\ +\sum_{k=0}^{m-1}\binom{m}{k}y^k x\Big(\sum_{r+s=m-k-1} y^r z y^s\Big)\\ =[w,z] \text{ for all } x,y,z,w\in\mathcal{I}.\end{aligned}$$

In particular if we take $z=0$, then $\mathcal{I}$ satisfies the polynomial identity

$$\sum_{k=0}^{m}\binom{m}{k}y^k w y^{m-k}=0 \quad\text{for all } y,w\in\mathcal{I}.$$

That is $w\circ_m y=0$ for all $w,y\in\mathcal{I}$. By Chuang [6, Theorem 2], this polynomial identity is also satisfied by $\mathcal{Q}$ and hence $\mathcal{R}$ as well. Note that this is a polynomial identity and hence by Lanski [14, Lemma 2], there exists a field $\mathcal{F}$ such that $\mathcal{R}\subseteq\mathcal{M}_k(\mathcal{F})$, the ring of $k\times k$ matrices over a field $\mathcal{F}$, where $k\geq 1$. Moreover, $\mathcal{R}$ and $\mathcal{M}_k(\mathcal{F})$ satisfy the same polynomial identity [14, Lemma 1], i.e., $w\circ_m y=0$ for all $w,y\in\mathcal{M}_k(\mathcal{F})$. Denote by e_{ij} the usual unit matrix unit with 1 in (i,j)- entry and 0 elsewhere. By choosing $w=e_{12},\ y=e_{11}$, we see that $w\circ_m y=e_{12}\neq 0$, a contradiction.

Case 2. If α is $\mathcal{Q}$-inner induced by an element $\mathsf{q}\in\mathcal{Q}$, i.e., $\alpha(x)=[\mathsf{q},x]$ for all $x\in\mathcal{R}$, then we have $[\mathsf{q},x\circ_m y]=[[\mathsf{q},x],[\mathsf{q},y]]_m$ for all $x,y\in\mathcal{I}$. Since $\mathsf{q}\notin\mathcal{C}$, thus the last identity is a non-trivial generalized polynomial identity (GPI) for $\mathcal{I}$. By Chuang [6, Theorem 1], $\mathcal{I}$ and $\mathcal{Q}$ satisfy same generalized polynomial identities (GPIs), i.e., $[\mathsf{q},x\circ_m y]=[[\mathsf{q},x],[\mathsf{q},y]]_m$ for all $x,y\in\mathcal{Q}$. If the center $\mathcal{C}$ of $\mathcal{Q}$ is infinite, then we have $[\mathsf{q},x\circ_m y]=[[\mathsf{q},x],[\mathsf{q},y]]_m$ for all $x,y\in\mathcal{Q}\otimes_{\mathcal{C}}\overline{\mathcal{C}}$, where $\overline{\mathcal{C}}$ is algebraic closure of $\mathcal{C}$. Since both $\mathcal{Q}$ and $\mathcal{Q}\otimes_{\mathcal{C}}\overline{\mathcal{C}}$ are prime and centrally closed [10, Theorem 2.5 and Theorem 3.5], we may replace $\mathcal{R}$ by $\mathcal{Q}$ or $\mathcal{Q} \otimes_{\mathcal{C}} \overline{\mathcal{C}}$ according as $\mathcal{C}$ is finite or infinite. Thus, we may assume that $\mathcal{R}$ is centrally closed over $\mathcal{C}$ (i.e., $\mathcal{RC} = \mathcal{R}$) which is either finite or algebraically closed and $[\mathtt{q}, x \circ_m y] = [[\mathtt{q}, x], [\mathtt{q}, y]]_m$ for all $x, y \in \mathcal{R}$. By Martindale [16, Theorem 3], $\mathcal{RC}$ (and so $\mathcal{R}$) is a primitive ring having nonzero socle $\mathcal{H}$ with $\mathcal{C}$ as the associated division ring. Hence, by Jacobson's theorem [12, p.75], $\mathcal{R}$ is isomorphic to a dense ring of linear transformations of some vector space $\mathcal{V}$ over $\mathcal{C}$ and $\mathcal{H}$ consists of the finite rank linear transformations in $\mathcal{R}$. If $\mathcal{V}$ is finite dimensional over $\mathcal{C}$, then the density of $\mathcal{R}$ on $\mathcal{V}$ implies that

$$\mathcal{R} \cong \mathcal{M}_m(\mathcal{C}), \text{ where } m = dim_{\mathcal{C}}\mathcal{V}. \tag{2.1}$$

Suppose that $dim_{\mathcal{C}}\mathcal{V} \geq 2$, otherwise we have done. Now, we want to show that $\mathtt{v}$ and $\mathtt{qv}$ are linearly $\mathcal{C}$-dependent for all $\mathtt{v} \in \mathcal{V}$. If $\mathtt{qv} = 0$, then $\{\mathtt{v}, \mathtt{qv}\}$ is linearly $\mathcal{C}$-dependent. Suppose on the contrary that $\mathtt{v}$ and $\mathtt{qv}$ are linearly $\mathcal{C}$-independent for some $\mathtt{v} \in \mathcal{V}$.

If $\mathtt{q}^2\mathtt{v} \notin Span_{\mathcal{C}}\{\mathtt{v}, \mathtt{qv}\}$, then the set $\{\mathtt{v}, \mathtt{qv}, \mathtt{q}^2\mathtt{v}\}$ is linearly $\mathcal{C}$-independent. Since $\mathtt{v}$ and $\mathtt{qv}$ are linearly $\mathcal{C}$-independent, by the density of $\mathcal{R}$, there exist $x, y \in \mathcal{R}$ such that

$$\begin{aligned} &x\mathtt{v} = \mathtt{v},\, x(\mathtt{qv}) = 0, \;\; x(\mathtt{q}^2\mathtt{v}) = 0; \\ &y\mathtt{v} = 0,\, y(\mathtt{qv}) = \text{-}\mathtt{v},\, y(\mathtt{q}^2\mathtt{v}) = 0. \end{aligned}$$

When $m = 1$, then we see that

$$0 = \Big([[\mathtt{q}, x], [\mathtt{q}, y]] - [\mathtt{q}, x \circ y]\Big)\mathtt{v} = 2\text{qv} - \mathtt{v}.$$

Moreover, when $m > 1$, we have

$$0 = \Big([[\mathtt{q}, x], [\mathtt{q}, y]]_m - [\mathtt{q}, x \circ_m y]\Big)\mathtt{v} = 2^m\mathtt{qv}.$$

In both the cases we get a contradiction as characteristic of $\mathcal{R}$ is different from 2.

If $\mathtt{q}^2\mathtt{v} \in Span_{\mathcal{C}}\{\mathtt{v}, \mathtt{qv}\}$, then $\mathtt{q}^2\mathtt{v} = \mathtt{v}\beta + \mathtt{qv}\gamma$ for some $\beta, \gamma \in \mathcal{C}$. By the density of $\mathcal{R}$, there exist $x, y \in \mathcal{R}$ such that

$$\begin{aligned} &x\mathtt{v} = \mathtt{v},\, x(\mathtt{qv}) = 0; \\ &y\mathtt{v} = 0,\, y(\mathtt{qv}) = \text{-}\mathtt{v}. \end{aligned}$$

For this, first we take $m = 1$, we see that

$$0 = \Big([[\mathtt{q}, x], [\mathtt{q}, y]] - [\mathtt{q}, x \circ y]\Big)\mathtt{v} = 2\text{qv} - \mathtt{v}\gamma - \mathtt{v}.$$

Now, when $m > 1$, we have

$$0 = \Big([[\mathtt{q}, x], [\mathtt{q}, y]]_m - [\mathtt{q}, x \circ_m y]\Big)\mathtt{v} = 2^m \mathtt{qv} - 2^{\mathtt{m-1}}\mathtt{v}\gamma.$$

Using the similar arguement as mentioned above again we get a contradiction in both the cases. So, we conclude that $\{\mathtt{v}, \mathtt{qv}\}$ is linearly $\mathcal{C}$-dependent for all $\mathtt{v} \in \mathcal{V}$. Thus, by Fact 2.1, there exists $\lambda \in \mathcal{C}$ such that $\mathtt{qv} = \mathtt{v}\lambda$ for any $\mathtt{v} \in \mathcal{V}$.

For $\mathtt{r} \in \mathcal{R}, \mathtt{v} \in \mathcal{V}$, we can write, $\mathtt{qv} = \mathtt{v}\lambda$, $\mathtt{r}(\mathtt{qv}) = \mathtt{r}(\mathtt{v}\lambda)$, and also $\mathtt{q}(\mathtt{rv}) = (\mathtt{rv})\lambda$. Thus $0 = [\mathtt{q}, \mathtt{r}]\mathtt{v}$ for any $\mathtt{v} \in \mathcal{V}$, i.e., $[\mathtt{q}, \mathtt{r}]\mathcal{V} = 0$. Since $\mathcal{V}$ is a left faithful irreducible $\mathcal{R}$-module, we have $[\mathtt{q}, \mathtt{r}] = 0$ for all $\mathtt{r} \in \mathcal{R}$, i.e., $\mathtt{q} \in Z(\mathcal{R})$ and $\alpha = 0$, a contradiction. This completes the proof. □

Theorem 2.3 *Let $\mathcal{R}$ be a prime ring of characteristic different from* 2*, $\mathcal{I}$ be a nonzero ideal of $\mathcal{R}$ and $m \geq 1$ be the fixed positive integer. If $\mathcal{R}$ admits a nonzero derivation α such that $\alpha([x, y]_m) = \alpha(x) \circ_m \alpha(y)$ for all $x, y \in \mathcal{I}$, then $\mathcal{R}$ is commutative.*

Proof. By given hypothesis, we can write

$$\begin{aligned}
&\sum_{k=1}^{m}(-1)^k \binom{m}{k}\Big(\sum_{i+j=k-1} y^i \alpha(y) y^j\Big) x y^{m-k} \\
&\quad + \sum_{k=0}^{m}(-1)^k \binom{m}{k} y^k \alpha(x) y^{m-k} \\
&\quad + \sum_{k=0}^{m-1}(-1)^k \binom{m}{k} y^k x\Big(\sum_{r+s=m-k-1} y^r \alpha(y) y^s\Big) \\
&\quad = \alpha(x) \circ_m \alpha(y) \text{ for all } x, y \in \mathcal{I}.
\end{aligned}$$

In the light of Kharchenko's theory [13], we divide the proof into two cases:

Case 1. If α is $\mathcal{Q}$-outer, then $\mathcal{I}$ satisfies the polynomial identity

$$\begin{aligned}
&\sum_{k=1}^{m}(-1)^k \binom{m}{k}\Big(\sum_{i+j=k-1} y^i z y^j\Big) x y^{m-k} \\
&\quad + \sum_{k=0}^{m}(-1)^k \binom{m}{k} y^k w y^{m-k} \\
&\quad + \sum_{k=0}^{m-1}(-1)^k \binom{m}{k} y^k x\Big(\sum_{r+s=m-k-1} y^r z y^s\Big) \\
&\quad = w \circ_m z \text{ for all } x, y, z, w \in \mathcal{I}.
\end{aligned}$$

In particular if we take $z = 0$, then $\mathcal{I}$ satisfies the polynomial identity

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}y^k w y^{m-k} = 0 \text{ for all } y, w \in \mathcal{I}.$$

i.e., $[w, y]_m = 0$ for all $w, y \in \mathcal{I}$, which can be written as $[\mathcal{I}_w(y), y]_{m-1} = 0$ for all $w, y \in \mathcal{I}$, where $\mathcal{I}_w(y)$ is an inner derivation determined by w. By Lanski [14, Theorem 1], either R is commutative or $\mathcal{I}_w = 0$ i.e., $\mathcal{I} \subseteq Z(\mathcal{R})$ in which case $\mathcal{R}$ is also commutative by Mayne [17, Lemma 3].

Case 2. If α is $\mathcal{Q}$-inner induced by an element $\mathtt{q} \in \mathcal{Q}$, i.e., $\alpha(x) = [\mathtt{q}, x]$ for all $x \in \mathcal{R}$, then using the similar techniques with necessary variations as used in the proof of Theorem 2.2 to get the relation (2.1), we find that if $\mathcal{V}$ is finite dimensional over $\mathcal{C}$, then the density of $\mathcal{R}$ on $\mathcal{V}$ implies that $\mathcal{R} \cong \mathcal{M}_m(\mathcal{C})$, where $m = dim_{\mathcal{C}}\mathcal{V}$.

Suppose that $dim_{\mathcal{C}}\mathcal{V} \geq 3$ such that $\mathtt{v}$ and $\mathtt{qv}$ are linearly $\mathcal{C}$-independent for some $\mathtt{v} \in \mathcal{V}$. By density of $\mathcal{R}$, there exists $\mathtt{u} \in \mathcal{V}$ such that $\mathtt{v}, \mathtt{qv}$ and $\mathtt{u}$ are linearly $\mathcal{C}$-independent and $x, y \in \mathcal{R}$ such that

$$x\mathtt{v} = 0,\ x\mathtt{qv} = -\mathtt{u},\ x\mathtt{u} = \mathtt{v},\quad x\mathtt{qu} = 0$$
$$y\mathtt{v} = 0,\ y\mathtt{qv} = -\mathtt{v},\ y\mathtt{u} = 0,\ y\mathtt{qu} = -\mathtt{u}.$$

Thus, we find that $\Big([\mathtt{q}, x] \circ_m [\mathtt{q}, y] - [\mathtt{q}, [x, y]_m]\Big)\mathtt{v} = 2^m\mathtt{u} \neq 0$, a contradiction.

Hence, we conclude that $\{\mathtt{v}, \mathtt{qv}\}$ is linearly $\mathcal{C}$-dependent for all $\mathtt{v} \in \mathcal{V}$. Thus, by Fact 2.1, there exists $\lambda \in \mathcal{C}$ such that $\mathtt{qv} = \mathtt{v}\lambda$ for any $\mathtt{v} \in \mathcal{V}$.

For $\mathtt{r} \in \mathcal{R}, \mathtt{v} \in \mathcal{V}$, we can write, $\mathtt{qv} = \mathtt{v}\lambda$, $\mathtt{r}(\mathtt{qv}) = \mathtt{r}(\mathtt{v}\lambda)$, and also $\mathtt{q}(\mathtt{rv}) = (\mathtt{rv})\lambda$. Thus $0 = [\mathtt{q}, \mathtt{r}]\mathtt{v}$ for any $\mathtt{v} \in \mathcal{V}$, i.e., $[\mathtt{q}, \mathtt{r}]\mathcal{V} = 0$. Since $\mathcal{V}$ is a left faithful irreducible $\mathcal{R}$-module, we have $[\mathtt{q}, \mathtt{r}] = 0$ for all $\mathtt{r} \in \mathcal{R}$, i.e., $\mathtt{q} \in Z(\mathcal{R})$ and $\alpha = 0$, a contradiction. Now suppose that $dim_{\mathcal{C}}\mathcal{V} \leq 2$. In this case R is a simple GPI-ring with 1 and so it is a central simple algebra finite dimensional over its center. By Lanski[14, Lemma 2], it follows that there exists a suitable field $\mathcal{F}$ such that $\mathcal{R} \subseteq \mathcal{M}_m(\mathcal{F})$ the ring of $m \times m$ matrices over $\mathcal{F}$ and moreover, $\mathcal{M}_m(\mathcal{F})$ satisfy the same GPI as $\mathcal{R}$. Assume $m \geq 3$, then by the same argument as above we get a contradiction. Obviously if $m = 1$, then $\mathcal{R}$ is commutative. Thus we may assume that $m = 2$, i.e., $\mathcal{R} \subseteq \mathcal{M}_2(\mathcal{F})$, where $\mathcal{M}_2(\mathcal{F})$ satisfies $[\mathtt{q}, x] \circ_m [\mathtt{q}, y] - [\mathtt{q}, [x, y]_m] = 0$. Denote by e_{ij} the usual unit matrix with 1 at (i, j)-entry and zero elsewhere. By putting $x = y = e_{12}$ and $\mathtt{q} = \begin{pmatrix} \mathtt{q}_{11} & \mathtt{q}_{12} \\ \mathtt{q}_{21} & \mathtt{q}_{22} \end{pmatrix}$ in the above identity and then right multiplying by e_{12}, one can easily get $(e_{12}\mathtt{q})^{m+1}e_{12} = 0$. It follows easily that $\begin{pmatrix} 0 & \mathtt{q}_{21}^{m+1} \\ 0 & 0 \end{pmatrix} = 0$ implies that $\mathtt{q}_{21} = 0$. Similarly we can get $\mathtt{q}_{12} = 0$. Thus in all, we see that $\mathtt{q}$ is a diagonal matrix in $\mathcal{M}_2(\mathcal{F})$. Let $\xi \in Aut(\mathcal{M}_2(\mathcal{F}))$. Since $[\xi(\mathtt{q}), \xi(x)] \circ_m [\xi(\mathtt{q}), \xi(y)] - [\xi(\mathtt{q}), [\xi(x), \xi(y)]_m] = 0$, $\xi(\mathtt{q})$ must be a diagonal matrix in $\mathcal{M}_2(\mathcal{F})$. In particular, let $\xi(x) = (1 - e_{ij})x(1 + e_{ij})$ for $i \neq j$. Then $\xi(\mathtt{q}) = \mathtt{q} + (\mathtt{q}_{ii} - \mathtt{q}_{jj})e_{ij}$, that is $\mathtt{q}_{ii} = \mathtt{q}_{jj}$ for $i \neq j$. This implies that $\mathtt{q}$ is central in $\mathcal{M}_2(\mathcal{F})$, which leads to $\alpha = 0$, a contradiction. This completes the proof of the theorem. □

3 Example

The following example demonstrates that R to be prime is essential in the hypothesis.

Example 3.1 Let S be any ring.

(*i*) Let $\mathcal{R} = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in S \right\}$ and $\mathcal{I} = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in S \right\}$. We define a map $\alpha : \mathcal{R} \to \mathcal{R}$ by $\alpha(x) = e_{11}x - xe_{11}$. Then it is easy to see that α is a nonzero derivation and $\mathcal{I}$ is a nonzero ideal of $\mathcal{R}$ such that for positive integer m, α satisfies the properties, $\alpha(x \circ_m y) = [\alpha(x), \alpha(y)]_m$ and $\alpha([x, y]_m) = \alpha(x) \circ_m \alpha(y)$ for $x, y \in \mathcal{I}$. However, R is not commutative.

(*ii*) Let $\mathcal{R} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in S \right\}$ and $\mathcal{I} = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in S \right\}$. Define a map $\alpha : \mathcal{R} \to \mathcal{R}$ by $\alpha(x) = [x, e_{11} + e_{12}]$. Then it is easy to see that α is a nonzero derivation and $\mathcal{I}$ is a nonzero ideal of $\mathcal{R}$. It is straightforward to check that α satisfies the properties $\alpha(x \circ_m y) = [\alpha(x), \alpha(y)]_m$ and $\alpha([x, y]_m) = \alpha(x) \circ_m \alpha(y)$ for $x, y \in \mathcal{I}$, but $\mathcal{R}$ is not commutative.

4 The results in semiprime rings

From now on, $\mathcal{R}$ is a semiprime ring, $\mathcal{U}$ is the left Utumi quotient ring of $\mathcal{R}$. For developing the proof of the main theorem, we require the following facts.

Fact 4.1 ([3, Proposition 2.5.1]) *Any derivation of a semiprime ring $\mathcal{R}$ can be uniquely extended to $\mathcal{U}$, and so any derivation of $\mathcal{R}$ can be defined on the whole $\mathcal{U}$.*

Fact 4.2 ([7, p.38]) *If $\mathcal{R}$ is a semiprime ring, then its left Utumi quotient ring is also semiprime. The extended centroid $\mathcal{C}$ of a semiprime ring coincides with the center of its left Utumi quotient ring.*

Fact 4.3 ([7, p.42]) *Let $\mathcal{B}$ be the set of all the idempotents in $\mathcal{C}$, the extended centroid of $\mathcal{R}$. Suppose that $\mathcal{R}$ is orthogonally complete $\mathcal{B}$-algebra. Then for any maximal ideal $\mathcal{P}$ of $\mathcal{B}$, $\mathcal{PR}$ forms a minimal prime ideal of $\mathcal{R}$, which is invariant under any derivation of $\mathcal{R}$.*

For a complete and detailed description of the theory of generalized polynomial identities involving derivations, we refer the reader to [3, Chapter 7]. Now we prove the following result.

Theorem 4.4 *Let $\mathcal{R}$ be a semiprime ring of characteristic different from 2 with center $Z(\mathcal{R})$. If $\mathcal{R}$ admits a nonzero derivation α such that $\alpha(x \circ_m y) = [\alpha(x), \alpha(y)]_m$ for all $x, y \in \mathcal{R}$, where $m \geq 1$ is a fixed positive integer, then there exists a central idempotent element $e \in \mathcal{U}$ such that on the direct sum decomposition $\mathcal{U} = e\mathcal{U} \oplus (1-e)\mathcal{U}$, α vanishes identically on $e\mathcal{U}$ and the ring $(1-e)\mathcal{U}$ is commutative.*

Proof. We have given that $\alpha(x \circ_m y) = [\alpha(x), \alpha(y)]_m$ for all $x, y \in \mathcal{R}$. By Fact 4.2, $Z(\mathcal{U}) = \mathcal{C}$, the extended centroid of $\mathcal{R}$, and by Fact 4.1, the derivation α can be uniquely extended on $\mathcal{U}$. By Lee [15, Theorem 3], $\mathcal{R}$ and $\mathcal{U}$ satisfy the same differential identities. Thus, $\alpha(x \circ_m y) = [\alpha(x), \alpha(y)]_m = 0$ for all $x, y \in \mathcal{U}$. Let $\mathcal{B}$ be the complete Boolean algebra of idempotents in $\mathcal{C}$ and $\mathcal{M}$ be any maximal ideal of $\mathcal{B}$. Therefore, by Chuang [7, p.42], $\mathcal{U}$ is orthogonally complete $\mathcal{B}$-algebra, and by Fact 4.3, $\mathcal{MU}$ is a prime ideal of $\mathcal{U}$, which is α-invariant. Let $\overline{\alpha}$ be the derivation induced by α on $\overline{\mathcal{U}} = \mathcal{U}/\mathcal{MU}$, i.e., $\overline{\alpha}(\overline{u}) = \overline{\alpha(u)}$ for all $u \in \mathcal{U}$. Then for all $\overline{\alpha}(\overline{x} \circ_m \overline{y}) = [\overline{\alpha}(\overline{x}), \overline{\alpha}(\overline{y})]_m = 0$. It is obvious that $\overline{\mathcal{U}}$ is prime. Therefore, by Theorem 2.2, we have either $\overline{\mathcal{U}}$ is commutative or $\overline{\alpha} = 0$, i.e., either $\alpha(\mathcal{U}) \subseteq \mathcal{MU}$ or $[\mathcal{U}, \mathcal{U}] \subset \mathcal{MU}$. Hence, $\alpha(\mathcal{U})[\mathcal{U}, \mathcal{U}] \subseteq \mathcal{MU}$, where $\mathcal{MU}$ runs over all minimal prime ideals of $\mathcal{U}$. Since $\bigcap_{\mathcal{M}} \mathcal{MU} = 0$, we obtain $\alpha(\mathcal{U})[\mathcal{U}, \mathcal{U}] = 0$.

Now using the theory of orthogonal completion for semiprime rings [3, Chapter 3], it is clear that there exists a central idempotent element $e \in \mathcal{U}$ such that on the direct sum decomposition $\mathcal{U} = e\mathcal{U} \oplus (1-e)\mathcal{U}$, α vanishes identically on $e\mathcal{U}$ and the ring $(1-e)\mathcal{U}$ is commutative. This completes the proof. □

Using similar arguments as used above, with necessary variations, one can easily prove the following. We omit the details of the proof just to avoid the repetition.

Theorem 4.5 *Let $\mathcal{R}$ be a semiprime ring of characteristic different from 2 with center $Z(\mathcal{R})$. If $\mathcal{R}$ admits a nonzero derivation α such that $\alpha([x, y]_m) = \alpha(x) \circ_m \alpha(y)$ for all $x, y \in \mathcal{R}$, where $m \geq 1$ is a fixed positive integer, then there exists a central idempotent element $e \in \mathcal{U}$ such that on the direct sum decomposition $\mathcal{U} = e\mathcal{U} \oplus (1-e)\mathcal{U}$, α vanishes identically on $e\mathcal{U}$ and the ring $(1-e)\mathcal{U}$ is commutative.*

Acknowledgements The authors are thankful to the referee for his/her valuable suggestions and critical comments.

References

1. Argaç, N.; Inceboz, H.G. – *Derivations of prime and semiprime rings*, J. Korean Math. Soc., **46** (2009), no. 5, 997-1005.
2. Ashraf, M.; Rehman, N. – *On commutativity of rings with derivations*, Results Math., **42** (2002), no. 1-2, 3-8.

3. Beidar, K.I.; Martindale III, W.S.; Mikhalev, A.V. – *Rings with generalized identities*, Pure and Applied Mathematics, Marcel Dekker 196, New York (1996), 522 pp.
4. Beidar, K.I.; Brešar, M. – *Extended Jacobson density theorem for rings with derivations and automorphisms*, Israel J. Math., **122** (2001), 317-346.
5. Bell, H.E.; Daif, M.N. – *On commutativity and strong commutativity-preserving maps*, Cand. Math. Bull., **37** (1994), no. 4, 443-447.
6. Chuang, C.L. – *GPIs having coefficients in Utumi quotient rings*, Proc. Amer. Math. Soc., **103** (1988), no. 3, 723-728.
7. Chuang, C.L. – *Hypercentral derivations*, J. Algebra **166** (1994), no. 1, 39-71.
8. Daif, M.N.; Bell, H.E. – *Remarks on derivations on semiprime rings*, Internt. J. Math. Math. Sci., **15** (1992), no. 1, 205-206.
9. Deng, Q.; Ashraf, M. – *On strong commutativity preserving mappings.* Results Math., **30** (1996), no. 3-4, 259-263.
10. Erickson, T.S.; Martindale III, W.S.; Osborn J.M. – *Prime nonassociative algebras*, Pacific. J. Math., **60** (1975), no. 1, 49-63.
11. Herstein, I.N. – *A note on derivations*, Canad. Math. Bull., **21** (1978), no. 3, 369-370.
12. Jacobson, N. – *Structure of rings*, American Mathematical Society, Colloquium Publications **37** (1956), 263 pp.
13. Kharchenko, V.K. – *Differential identities of semiprime rings* (Russian), Algebra i Logika **18** (1979), no. 1, 86-119.
14. Lanski, C. – *An Engel condition with derivation*, Proc. Amer. Math. Soc., **118** (1993), no. 3, 731-734.
15. Lee, T.K. – *Semiprime rings with differential identities*, Bull. Inst. Math. Acad. Sinica, **20** (1992), no. 1, 27-38.
16. Martindale III, W.S. – *Prime rings satisfying a generalized polynomial identity*, J. Algebra **12** (1969), 576-584.
17. Mayne, J.H. – *Centralizing mappings of prime rings*, Canad. Math. Bull. **27** (1984), no. 1, 122-126.

Received: 24.I.2017 / Revised: 26.III.2017 / Accepted: 30.VI.2017

Authors

Mohammad Ashraf (Corresponding author),
Department of Mathematics,
Aligarh Muslim University,
Aligarh-202002 India,
E-mail: mashraf80@hotmail.com

Sajad Ahmad Pary,
Department of Mathematics,
Aligarh Muslim University,
Aligarh-202002 India,
E-mail: paryamu@gmail.com

Mohd Arif Raza,
Department of Mathematics,
Faculty of Science & Arts-Rabigh,
King Abdulaziz University, Kingdom of Saudi Arabia,
E-mail: arifraza03@gmail.com